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# CHARACTERIZATIONS OF GAMMA DISTRIBUTION VIA SUB-INDEPENDENT RANDOM VARIABLES

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Abstract. The concept of sub-independence is based on the convolution of the distributions of the random variables. It is much weaker than that of independence, but is shown to be sufficient to yield the conclusions of important theorems and results in probability and statistics. It also provides a measure of dissociation between two random variables which is much stronger than uncorrelatedness. Inspired by the excellent work of Jin and Lee (2014), we present certain characterizations of gamma distribution based on the concept of sub-independence.

## 1. Introduction

The concept of sub-independence is stated as follows: The *rv 's* (random variables) $X$ and $Y$ with *cdf 's* (cumulative distribution function) $F_X$ and $F_Y$ are *s.i.* (sub-independent) if the *cdf* of $X+Y$ is given by

$$F_{X+Y}(z) = (F_X * F_Y)(z) = \int_{\mathbb{R}} F_X(z-y)\, dF_Y(y), \quad z \in \mathbb{R}, \tag{1.1}$$

or equivalently if and only if

$$\varphi_{X+Y}(t) = \varphi_{X,Y}(t,t) = \varphi_X(t)\,\varphi_Y(t), \quad for\ all\ \ t \in \mathbb{R}, \tag{1.2}$$

where $\varphi_X$ , $\varphi_Y$ , $\varphi_{X+Y}$ and $\varphi_{X,Y}$ are $cf$ *'s* (characteristic functions) of $X$ , $Y$ , $X+Y$ and $(X,Y)$, respectively.
The drawback of the concept of sub-independence in comparison with that of independence had been that the former does not have an equivalent definition based on the events, which some believe, to be the natural definition of independence. We have found such a definition now (see Hamedani (2013)) which is stated below for the sake of completeness.

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We shall give two separate definitions, one for the discrete case (Definition 1.1) and the other for the continuous case (Definition 1.2).

Let $(X, Y) : \Omega \to \mathbb{R}^2$ be a discrete random vector with range $\Re(X, Y) = \{(x_i, y_j) : i , j = 1, 2, ...\}$ (finitely or infinitely countable). Consider the events

$$A_i = \{\omega \in \Omega : X(\omega) = x_i\}, B_j = \{\omega \in \Omega : Y(\omega) = y_j\}$$

and

$$A^z = \{\omega \in \Omega : X(\omega) + Y(\omega) = z\}, z \in \Re(X + Y).$$

Definition 1.1. The discrete $rv$'s $X$ and $Y$ are $s.i.$ if for every $z \in \Re(X + Y)$

$$P(A^z) = \sum_{i , j ,} \sum_{x_i + y_j = z} P(A_i) P(B_j). \tag{1.3}$$

To see that (1.3) is equivalent to (1.2), suppose $X$ and $Y$ are $s.i.$ via (1.2), then

$$\sum_i \sum_j e^{it(x_i + y_j)} f(x_i, y_j) = \sum_i \sum_j e^{it(x_i + y_j)} f_X(x_i) \; f_Y(y_j),$$

where $f$, $f_X$ and $f_Y$ are probability functions of $(X, Y)$, $X$ and $Y$ respectively. Let $z \in \Re(X + Y)$, then

$$e^{itz} \sum_{i , j ,} \sum_{x_i + y_j = z} f(x_i, y_j) = e^{itz} \sum_{i , j ,} \sum_{x_i + y_j = z} f_X(x_i) \; f_Y(y_j),$$

which implies (1.3).

For the continuous case, we observe that the half-plane $H = \{(x, y) : x + y < 0\}$ can be written as a countable disjoint union of rectangles:

$$H = \cup_{i=1}^{\infty} E_i \times F_i,$$

where $E_i$ and $F_i$ are intervals. Now, let $(X, Y) : \Omega \to \mathbb{R}^2$ be a continuous random vector and for $c \in \mathbb{R}$, let

$$A_c = \{\omega \in \Omega : X(\omega) + Y(\omega) < c\}$$

and

$$A_i^{(c)} = \left\{\omega \in \Omega : X(\omega) - \frac{c}{2} \in E_i\right\}, \; B_i^{(c)} = \left\{\omega \in \Omega : Y(\omega) - \frac{c}{2} \in F_i\right\}.$$

Definition 1.2. The continuous $rv$'s $X$ and $Y$ are $s.i.$ if for every $c \in \mathbb{R}$

$$P(A_c) = \sum_{i=1}^{\infty} P\left(A_i^{(c)}\right) P\left(B_i^{(c)}\right). \tag{1.4}$$

To see that (1.4) is equivalent to (1.1), observe that ($LHS$ of (1.4))

$$P(A_c) = P(X+Y<c) = P((X,Y)\in H_c)\,, \tag{1.5}$$

where $H_c = \{(x,y) : x+y<c\}$. Now, if $X$ and $Y$ are $s.i.$ then

$$P(A_c) = (P_X \times P_Y)(H_c)$$

where $P_X$ , $P_Y$ are probability measures on $\mathbb{R}$ defined by

$$P_X(B) = P(X\in B) \quad \text{and} \quad P_Y(B) = P(Y\in B)\ ,$$

and $P_X \times P_Y$ is the product measure.

We also observe that ($RHS$ of (1.4))

$$\begin{aligned}\sum_{i=1}^{\infty} P\left(A_i^{(c)}\right) P\left(B_i^{(c)}\right) &= \sum_{i=1}^{\infty} P\left(X-\frac{c}{2}\in E_i\right) P\left(Y-\frac{c}{2}\in F_i\right)\\ &= \sum_{i=1}^{\infty} P\left(X\in E_i+\frac{c}{2}\right) P\left(Y\in F_i+\frac{c}{2}\right)\\ &= \sum_{i=1}^{\infty} P_X \times P_Y \left(E_i+\frac{c}{2}\right)\times\left(F_i+\frac{c}{2}\right).\end{aligned} \tag{1.6}$$

Now, (1.5) and (1.6) will be equal if $H_c = \cup_{i=1}^{\infty}\left\{\left(E_i+\frac{c}{2}\right)\times\left(F_i+\frac{c}{2}\right)\right\}$ , which is true since the points in $H_c$ are obtained by shifting each point in $H$ over to the right by $\frac{c}{2}$ units and then up by $\frac{c}{2}$ units.

Remark 1.3. ($i$) Note that $H$ can be written as a union of squares and triangles. The triangles are congruent to $0\le y<x$ , $0\le x<1$ which in turn can be written as a disjoint union of squares. For example, take $[0,1/2)\times[0,1/2)$ then $[1/2,3/4)\times[0,1/4)$ and so on. ($ii$) The discrete $rv\,'s$ $X$ , $Y$ and $Z$ are $s.i.$ if (1.3) holds for any pair and

$$P(A^s) = \sum_{i\ ,\ j\ ,\ k\ ,\ }\ \sum_{x_i\ +\ y_j\ +\ z_k=\ s}\ \sum P(A_i)\,P(B_j)\,P(C_k)\,. \tag{1.7}$$

For $p$ variate case we need $2^p-p-1$ equations of the above form. ($iii$) The representation (1.2) can be extended to the multivariate case as well. ($iv$) For a detailed treatment of the concept of sub-independence, we refer the interested reader to Hamedani (2013).

## 2. Characterizations

Characterizations of distributions are important to many researchers in the applied fields. An investigator will be vitally interested to know if their model fits the requirements of a particular distribution. To this end, one will depend on the characterizations of this distribution which provide conditions under which the underlying distribution is indeed that particular distribution. Various characterizations of distributions have been established in many different directions. In this work, several characterizations of gamma distribution are presented. Inspired by the excellent work of Jin and Lee (2014), we would like to establish the following characterizations of gamma distribution based on the concept of sub-independence.

PROPOSITION 2.1. *Let* $X$ *and* $Y$ *be positive* $s.i.i.d.$ *(sub-independent and identically distributed)* $rv$ *'s with an absolutely continuous* $cdf$ $F$ *and* $E\left[X^2\right] < \infty$. *The* $rv$ $X$ *has a gamma distribution if and only if there exists a* $\gamma \in (0, 1/4)$ *such that for* $t \in \mathbb{R}$,

$$\frac{(\varphi'(t))^2}{\varphi_1''(t)} = \gamma \ , \tag{2.1}$$

*where* $\varphi(t)$ *and* $\varphi_1(t)$ *are* $cf$ *'s of* $X$ *and* $X+Y$ *respectively.*

*Proof.* If $X$ has a gamma distribution, then clearly (2.1) holds. Now, assume (2.1) holds. Since $X$ and $Y$ are $s.i.$, from (1.2) we have

$$\varphi_1(t) = (\varphi(t))^2 , \quad t \in \mathbb{R}. \tag{2.2}$$

Differentiating both sides of (2.2) twice, we arrive at

$$\varphi_1''(t) = 2\left\{\varphi(t)\varphi''(t) + (\varphi'(t))^2\right\}. \tag{2.3}$$

Now, from (2.1) and (2.3) we obtain

$$\varphi(t)\varphi''(t) = \left(\frac{1-2\gamma}{2\gamma}\right)\left((\varphi'(t))^2\right),$$

or

$$\frac{\varphi''(t)}{\varphi'(t)} = \left(\frac{1-2\gamma}{2\gamma}\right)\frac{\varphi'(t)}{\varphi(t)}, \tag{2.4}$$

or

$$\varphi'(t) = iE(X)(\varphi(t))^{\frac{1-2\gamma}{2\gamma}} , \tag{2.5}$$

in which the initial conditions $\varphi(0) = 1$ and $\varphi'(0) = iE(X)$ are used.

Solving (2.5) for $\varphi(t)$, we have, in view of $0 < \gamma < 1/4$

$$\varphi(t) = \left(1 - i\left(\frac{1-4\gamma}{2\gamma}\right)E(X)\,t\right)^{-\left(\frac{2\gamma}{1-4\gamma}\right)}, \quad t \in \mathbb{R} ,$$

which is the $cf$ of a gamma distribution with parameters $\left(\frac{2\gamma}{1-4\gamma}\right)$ and $\left(\frac{1-4\gamma}{2\gamma}\right) E(X)$. □

Under the assumption of independence of the $rv$'s $X$ and $Y$, we have the following proposition.

Proposition 2.2. *Let $X$ and $Y$ be positive i.i.d. $rv$'s with an absolutely continuous cdf $F$ and $E\left[X^2\right] < \infty$. The $rv$ $X$ has a gamma distribution if and only if there exists a $\gamma \in (0, 1/4)$ such that for $t \in \mathbb{R}$,*

$$\frac{\left(E\left[X \exp\{itX\}\right]\right)^2}{E\left[(X+Y)^2 \exp\{it(X+Y)\}\right]} = \gamma. \tag{2.6}$$

*Proof.* If $X$ has a gamma distribution, then clearly (2.6) holds. Now, assume (2.6) holds and observe that

$$\begin{aligned}
&\int_0^\infty \int_0^\infty (x-y)^2 \exp\{it(X+Y)\}\, dF(x)\, dF(y) \\
&= \int_0^\infty \int_0^\infty (x+y)^2 \exp\{it(X+Y)\}\, dF(x)\, dF(y) \\
&\quad - 4\left(\int_0^\infty x \exp\{itX\}\, dF(x)\right)^2 \\
&= (1-4\gamma) \int_0^\infty \int_0^\infty (x+y)^2 \exp\{it(X+Y)\}\, dF(x)\, dF(y).
\end{aligned} \tag{2.7}$$

Now, expressing (2.7) in terms of $cf$'s, we have

$$-2\varphi(t)\varphi''(t) + 2\left(\varphi'(t)\right)^2 = -2(1-4\gamma)\left\{\varphi(t)\varphi''(t) + \left(\varphi'(t)\right)^2\right\},$$

from which we arrive at the basic differential equation (2.4). □

Proposition 2.3. *Let $X$ and $Y$ be positive s.i.i.d. $rv$'s with an absolutely continuous cdf $F$ and $E\left[X^2\right] < \infty$ and let $(X+Y)$ and $\left(\frac{X-Y}{X+Y}\right)^2$ be s.i.i.d.. The $rv$ $X$ has a gamma distribution if and only if there exists a $\gamma \in (0, 1/4)$ such that for $t \in \mathbb{R}$,*

$$\frac{\left(\varphi_1'(t)\right)^2}{\varphi_2''(t)} = \gamma, \tag{2.8}$$

*where $\varphi_1(t)$ and $\varphi_2(t)$ are $cf$'s of $(X+Y)$ and $(X+Y)+\left(\frac{X-Y}{X+Y}\right)^2$, respectively.*

*Proof.* Observe that in view of sub-independence of $(X+Y)$ and $\left(\frac{X-Y}{X+Y}\right)^2$,we have

$$\varphi_2(t) = (\varphi_1(t))^2 . \tag{2.9}$$

It is easy to see from (2.9) that if $X$ has a gamma distribution, (2.8) holds. Now, assume that (2.8) holds and note that differentiating (2.9) twice and using (2.8), we have

$$(1-2\gamma)\left(\varphi_1'(t)\right)^2 = 2\gamma\varphi_1(t)\,\varphi_1''(t),$$

or

$$\frac{\varphi_1''(t)}{\varphi_1'(t)} = \left(\frac{1-2\gamma}{2\gamma}\right)\frac{\varphi_1'(t)}{\varphi_1(t)}$$

from which we arrive at

$$\varphi_1(t) = \left(1 - i\left(\frac{1-4\gamma}{2\gamma}\right)E(X+Y)\,t\right)^{-\left(\frac{2\gamma}{1-4\gamma}\right)}, \quad t \in \mathbb{R},$$

i.e., $(X+Y)$ has a gamma distribution with parameters $\left(\frac{2\gamma}{1-4\gamma}\right)$ and $\left(\frac{1-4\gamma}{2\gamma}\right)E(X+Y)$. Since $X$ and $Y$ are *s.i.i.d.* , their common *cf* is

$$\varphi(t) = \left(1 - i\left(\frac{1-4\gamma}{2\gamma}\right)E(X+Y)\,t\right)^{-\left(\frac{\gamma}{1-4\gamma}\right)}, \quad t \in \mathbb{R},$$

i.e., $X$ has a gamma distribution with parameters $\left(\frac{\gamma}{1-4\gamma}\right)$ and $\left(\frac{1-4\gamma}{2\gamma}\right)$ $E(X+Y)$. $\square$

Under the assumption of independence of the *rv's* $X$ and $Y$ we have the following proposition.

Proposition 2.4. *Let* $X$ *and* $Y$ *be positive* *i.i.d. rv's* *with an absolutely continuous* *cdf* $F$ *and* $E\left[X^2\right] < \infty$ *and let* $(X+Y)$ *and* $\left(\frac{X-Y}{X+Y}\right)^2$ *be* *s.i.i.d.* *The* *rv* $X$ *has a gamma distribution if and only if there exists a* $\gamma \in (0,1)$ *such that for* $t \in \mathbb{R}$,

$$\frac{E\left[\left\{(X+Y) - \left(\frac{X-Y}{X+Y}\right)^2\right\}^2 \exp\left\{it\left[(X+Y) + \left(\frac{X-Y}{X+Y}\right)^2\right]\right\}\right]}{E\left[\left\{(X+Y) + \left(\frac{X-Y}{X+Y}\right)^2\right\}^2 \exp\left\{it\left[(X+Y) + \left(\frac{X-Y}{X+Y}\right)^2\right]\right\}\right]} = \gamma. \tag{2.10}$$

*Proof.* If $X$ has a gamma distribution, then clearly (2.10) holds. Now, if (2.10) holds, then we have

$$\int_0^\infty \int_0^\infty \left\{(X+Y) - \left(\frac{X-Y}{X+Y}\right)^2\right\}^2$$
$$\times \exp\left\{it\left[(X+Y) + \left(\frac{X-Y}{X+Y}\right)^2\right]\right\} dF(x)\, dF(y)$$
$$= \int_0^\infty \int_0^\infty \left\{(X+Y) + \left(\frac{X-Y}{X+Y}\right)^2\right\}^2$$
$$\times \exp\left\{it\left[(X+Y) + \left(\frac{X-Y}{X+Y}\right)^2\right]\right\} dF(x)\, dF(y)$$
$$-4\left(\int_0^\infty \int_0^\infty (X+Y)\exp\{it(X+Y)\}\, dF(x)\, dF(y)\right)^2$$
$$= \gamma \int_0^\infty \int_0^\infty \left\{(X+Y) + \left(\frac{X-Y}{X+Y}\right)^2\right\}^2$$
$$\times \exp\left\{it\left[(X+Y) + \left(\frac{X-Y}{X+Y}\right)^2\right]\right\} dF(x)\, dF(y),$$

or

(2.11)

$$(1-\gamma)\int_0^\infty \int_0^\infty \left\{(X+Y) + \left(\frac{X-Y}{X+Y}\right)^2\right\}^2$$
$$\times \exp\left\{it\left[(X+Y) + \left(\frac{X-Y}{X+Y}\right)^2\right]\right\} dF(x)\, dF(y)$$
$$= 4\left(\int_0^\infty \int_0^\infty (X+Y)\exp\{it(X+Y)\}\, dF(x)\, dF(y)\right)^2.$$

Now, expressing (2.11) in terms of $cf\,'s$, we have

$$(1-\gamma)\,\varphi_1(t)\,\varphi_1''(t) + (1-\gamma)\left(\varphi_1'(t)\right)^2 = 2\left(\varphi_1'(t)\right)^2,$$

or

$$\frac{\varphi_1''(t)}{\varphi_1'(t)} = \left(\frac{1+\gamma}{1-\gamma}\right)\frac{\varphi_1'(t)}{\varphi_1(t)},$$

from which, we obtain

$$\varphi_1(t) = \left(1 - i\left(\frac{2\gamma}{1-\gamma}\right)E(X+Y)\,t\right)^{-\left(\frac{1-\gamma}{2\gamma}\right)}, \quad t \in \mathbb{R},$$

i.e., $(X+Y)$ has a gamma distribution with parameters $\left(\frac{1-\gamma}{2\gamma}\right)$ and $\left(\frac{2\gamma}{1-\gamma}\right)E(X+Y)$. Since $X$ and $Y$ are *s.i.i.d.*, their common $cf$ is

$$\varphi(t) = \left(1 - i\left(\frac{2\gamma}{1-\gamma}\right)E(X+Y)\,t\right)^{-\left(\frac{1-\gamma}{4\gamma}\right)}, \quad t \in \mathbb{R},$$

i.e., $X$ has a gamma distribution with parameters $\left(\frac{1-\gamma}{4\gamma}\right)$ and $\left(\frac{2\gamma}{1-\gamma}\right)$ $E(X+Y)$. $\square$

Remark 2.5. Similar Propositions can be stated for $S_n = \sum_{i=1}^{n} X_i$ and $\frac{\sum_{i=1}^{m} {X_i}^2}{{S_n}^2}$, $1 \leq m < n$, where $X_i\,'s$ are (*s.i.i.d.* or *i.i.d.*) and $S_n$ and $\frac{\sum_{i=1}^{m} {X_i}^2}{{S_n}^2}$, $m = 1, 2, ..., n$ are *s.i.i.d.*

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