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MULTIPLE POSITIVE SOLUTIONS OF NONLINEAR BOUNDARY VALUE PROBLEM WITH FINITE FRACTIONAL DIFFERENCE

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Abstract. In this paper, we consider a discrete fractional nonlinear boundary value problem in which nonlinear term f is involved with the fractional order difference. We transform the fractional boundary value problem into boundary value problem of integer order difference equation. By using a generalization of Leggett-Williams fixed-point theorem due to Avery and Peterson, we provide sufficient conditions for the existence of at least three positive solutions.

1. Introduction

Let $\mathbb{R}$ and $\mathbb{Z}$ be the sets of real numbers and integers, respectively. For $a, b \in \mathbb{R}, N \in \mathbb{Z}^+$ with $b = a + N$, define $[a, b]_{\mathbb{N}_a} = \{a, a+1, \cdots, b\}$. Assume that T is a given positive integer with $T > 2$. We consider the fractional difference boundary value problem(briefly FBVP)

$$\begin{aligned}(1.1)\quad & \Delta(\varphi_p(\Delta^{\nu}_{\nu-1}x(t))) \\ & + q(t)f(t+\nu-1, x(t+\nu-1), {}_{t+\nu-\beta}\Delta^{\beta}_{\nu-1}x(t)) = 0, t \in [0, T]_{\mathbb{N}_0},\end{aligned}$$

$$(1.2)\qquad [\triangle^{\nu}_{\nu-1}x(t)]_{t=T+1} = 0, x(\nu-1) - B_0([\triangle^{\nu}_{\nu-1}x(t)]_{t=\xi}) = 0$$

where $\nu, \beta \in (0, 1)$ with $0 < \beta < \nu, \varphi_p(s) = |s|^{p-2}s, p > 1, \xi \in [0, T]_{\mathbb{N}_0}, \Delta^{\nu}_{\nu-1}$ is a fractional difference operator (which will be explained in more detail later). We give the following assumptions:

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(H1) $f(t+\nu-1,.,.) : [\nu-1, T+\nu-1]_{\mathbb{N}_{\nu-1}} \times [0,+\infty) \times \mathbb{R} \longrightarrow [0,+\infty)$ is continuous;

(H2) $q(t)$ is nonnegative on $[0,T]_{\mathbb{N}_0}, q(t) \equiv 0$ does not hold on $[0,T]_{\mathbb{N}_0}$ and $\sum_{t=0}^{T} q(t) < \infty$. Here $\varphi_q = \varphi_p^{-1}, 1/p + 1/q = 1$;

(H3) $B_0(y)$ is a nondecreasing and continuous odd function on $(-\infty, +\infty)$. And there exists $m > 0$ such that $0 \leq B_0(y) \leq my$ for all $y \geq 0$.

Fixed-point theorems and their applications to nonlinear problems have a long history, some of which is documented in Zeidler's book ([1]). There seems to be increasing interest in multiple fixed-point theorems and their applications to boundary value problems for ordinary differential equations or finite difference equations. Such applications can be found in the papers ([2-6]). An interest in triple solutions has evolved from the Leggett-Williams multiple fixed-point theorem ([7]). And lately, two triple fixed-point theorems by Avery ([8]) and Avery and Peterson ([9]) have been applied to obtain triple solutions of certain boundary value problems for ordinary differential equations as well as for their discrete analogues. On the other hand, fractional differential and difference 'operators' are found themselves in concrete applications, and hence attention has to be paid to associated fractional difference and differential equations under various boundary or side conditions. For example, a recent paper by Atici and Eloe ([10]) explores some of the theories of a discrete conjugate FBVP. Similarly, in ([11]), a discrete right-focal FBVP is analyzed. Other recent advances in the theory of the discrete fractional calculus may be found in ([12-35]). In particular, an interesting recent paper by Atici and Sengül ([20]) addressed the use of fractional difference equations in tumor growth modeling. Thus, it seems that there exists some promise in using fractional difference equations as mathematical models for describing physical problems in more accurate manners.

In order to handle the existence problem for FBVP, various methods (among which are some standard fixed point theorems) can be used. For example, in ([14-17]), authors investigated the existence to some boundary value problems by fixed point theorems on a cone. In ([18]), we established the existence conditions for a boundary value problem by using the coincidence degree theory. In ([19]), authors given the existence of multiple solutions for a fractional difference boundary value problem with parameter by establishing the corresponding variational framework and using the mountain pass theorem, linking theorem, and Clark theorem in critical point theory. To the best of our knowledge, Leggett-Williams fixed-point theorem has not be used in discrete fractional boundary value problems. The aim of this paper is to establish the existence conditions for boundary value problem (1.1), (1.2). The proof relies on the Leggett-Williams fixed-point theorem.

For convenience, throughout this paper, we make the convention that $\sum_{i=j}^{m} x(i) = 0$, for $m < j$.

2. Preliminaries

We first collect some basic lemmas for manipulating discrete fractional operators. These and other related results can be found in the references ([3,12,20,21]).

First, for any integer β, we let $\mathbb{N}_\beta = \{\beta, \beta+1, \beta+2, ...\}$. We define $t^{(\nu)} := \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$, for any t and ν for which the right-hand side is defined. We also appeal to the convention that if $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{(\nu)} = 0$.

Definition 2.1. ([20]) The ν-th fractional sum of f for $\nu > 0$ is defined by

$$\Delta_a^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} f(s),$$

for $t \in \mathbb{N}_{a+\nu}$. We also define the ν-th fractional difference for $\nu > 0$ by $\Delta_a^\nu f(t) := \Delta^N \Delta_a^{\nu-N} f(t)$ where $t \in \mathbb{N}_{a+N-\nu}$ and $N \in \mathbb{N}$ is chosen so that $0 \leq N-1 < \nu \leq N$.

Definition 2.2. ([20]) Let f be any real-valued function and $\nu \in (0,1)$.The discrete fractional difference operator is defined as

$$\Delta_a^\nu f(t) = \Delta \Delta_a^{-(1-\nu)} f(t) = \frac{1}{\Gamma(1-\nu)} \Delta \sum_{s=a}^{t+\nu-1} (t-s-1)^{(-\nu)} f(s),$$

for $t \equiv a + \nu - 1 (mod 1)$.

Definition 2.3. ([1]) Let $\mathbb{X}$ be a real Banach space. A nonempty closed convex set $P \subset \mathbb{X}$ is called a cone of $\mathbb{X}$ if it satisfies the following conditions:

(1)$x \in P, \lambda \geq 0$ implies $\lambda x \in P$; (2)$x \in P, -x \in P$ implies $x = 0$.

Every cone $P \subset \mathbb{X}$ induces a partial ordering "$\leq$" on $\mathbb{X}$ defined by $x \leq y$ if and only if $y - x \in P$.

Definition 2.4. ([1]) Given a cone P in a real Banach space $\mathbb{X}$, a functional $\psi : P \longrightarrow \mathbb{R}$ is said to be increasing on P, provided that $\psi(x) \leq \psi(y)$ for all $x, y \in P$ with $x \leq y$.

Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P. Then for positive real numbers a, b, c and d, we define the following convex sets:

$P(\gamma, d) = \{x \in \ P | \gamma(x) < d\}$,

$P(\gamma, \alpha, b, d) = \{x \in P | b \leq \alpha(x), \gamma(x) \leq d\}$,

$P(\gamma, \theta, \alpha, b, c, d) = \{x \in P | b \leq \alpha(x), \theta(x) \leq c, \gamma(x) \leq d\}$ and a closed set $R(\gamma, \psi, a, d) = \{x \in P | a \leq \psi(x), \gamma(x) \leq d\}$.

The following fixed-point theorem due to Avery and Peterson is fundamental in the proof of our main results.

Lemma 2.5. ([3]) *Let $\mathbb{X}$ be a Banach space and let* $\mathrm{P} \subset \mathbb{X}$ *be a cone. Let γ and θ be nonnegative continuous convex functionals on* P, *α be a nonnegative continuous concave functional on* P, *and ψ be a nonnegative continuous functional on* P *satisfying $\psi(\lambda x) \leq \lambda \psi(x)$ for $0 \leq \lambda \leq 1$, such that for some positive numbers M and $d, \alpha(x) \leq \psi(x)$ and $||x|| \leq M\gamma(x)$ for all $x \in \overline{\mathrm{P}(\gamma, d)}$. Suppose $Q : \overline{\mathrm{P}(\gamma, d)} \longrightarrow \overline{\mathrm{P}(\gamma, d)}$ is completely continuous and there exist positive numbers a, b, c with $a < b$ such that*

- (S1) *$\{x \in \mathrm{P}(\gamma, \theta, \alpha, b, c, d) | \alpha(x)T > b\} \neq \emptyset$ and $\alpha(Qx) > b$ for $x \in \mathrm{P}(\gamma, \theta, \alpha, b, c, d)$;*
- (S2) *$\alpha(Qx) > b$ for $x \in \mathrm{P}(\gamma, \alpha, b, d)$ with $\theta(Qx) > c$;*
- (S3) *$0 \notin R(\gamma, \psi, a, d)$ and $\psi(Qx) < a$ for $x \in R(\gamma, \psi, a, d)$ with $\psi(x) = a$.*

Then Q has at least three fixed-points $x_1, x_2, x_3 \in \overline{\mathrm{P}(\gamma, d)}$ such that

$\gamma(x_i) \leq d$ for $i = 1, 2, 3$,

$b < \alpha(x_1), \psi(x_3) < a, a < \psi(x_2)$ with $\alpha(x_2) < b$.

Lemma 2.6. ([21]) *Let $y : \mathbb{N}_a \longrightarrow \mathbb{R}$ and $\nu > 0$ with $N - 1 < \nu \leq N$, then*

$$\Delta_a^\nu y(t) = \sum_{k=0}^{t-a+\nu} (-1)^k \begin{pmatrix} \nu \\ k \end{pmatrix} y(t + \nu - k)$$

for $t \in \mathbb{N}_{a+N-\nu}$, where $\begin{pmatrix} \nu \\ k \end{pmatrix} = \frac{\Gamma(\nu+1)}{\Gamma(k+1)\Gamma(\nu-k+1)}$.

LEMMA 2.7. ([21]) *$f : \mathbb{N}_a \longrightarrow \mathbb{R}$ be given and suppose $k \in \mathbb{N}_0$ and $\nu > 0$. Then for $t \in \mathbb{N}_{a+\nu}$,*

$$\Delta_a^{-\nu}\Delta^k f(t) = \Delta_a^{k-\nu} f(t) - \sum_{j=0}^{k-1} \frac{\Delta^j f(a)}{\Gamma(\nu - k + j + 1)}(t-a)^{(\nu-k+j)}.$$

Moreover, if $\mu > 0$ with $M - 1 \leq \mu \leq M$, then for $t \in \mathbb{N}_{a+M-\mu+\nu}$,

$$\begin{aligned}&\Delta_{a+M-\mu}^{-\mu}\Delta_a^{\mu} f(t)\\ &= \Delta_a^{\mu-\nu} f(t) - \sum_{j=0}^{M-1} \frac{\Delta_a^{j-M+\mu} f(a+M-\mu)}{\Gamma(\nu - k + j + 1)}(t-a-M+\mu)^{(\nu-M+j)}.\end{aligned}$$

LEMMA 2.8. ([21]) *Let $f : \mathbb{N}_a \longrightarrow \mathbb{R}$ be given and suppose $\mu, \nu > 0$ with $N - 1 < \mu \leq N$. Then for $t \in \mathbb{N}_{a+\nu+N-\mu}$,*

$$\Delta_{a+\mu}^{\mu}\Delta_a^{-\nu} f(t) = \Delta_a^{\mu-\nu} f(t).$$

LEMMA 2.9. ([21]) *Let $a \in \mathbb{R}$ and $\mu > 0$ be given. Then*

$$\Delta(t-a)^{(\mu)} = \mu(t-a)^{(\mu-1)},$$

for any t for which both sides are well-defined. Furthermore, for $\nu > 0$,

$$\Delta_{a+\mu}^{-\nu}(t-a)^{(\mu)} = \mu^{(-\nu)}(t-a)^{(\mu+\nu)}, t \in \mathbb{N}_{a+\mu+\nu},$$

$$\Delta_{a+\mu}^{\nu}(t-a)^{(\mu)} = \mu^{(\nu)}(t-a)^{(\mu-\nu)}, t \in \mathbb{N}_{a+\mu+N-\nu}.$$

3. Triple positive solutions

In this section, we impose growth conditions on f to obtain the triple positive solutions for FBVP (1.1)(1.2).

We first note that by the transformation $y(t) = \Delta_{\nu-1}^{-(1-\nu)} x(t)$, (1.1)(1.2) are equivalent to the following (3.1)(3.2)

(3.1)
$$\Delta(\varphi_p(\Delta y(t)))+q(t)f(t+\nu-1, x(t+\nu-1),_{t+\nu-\beta} \Delta_{\nu-1}^{\beta} x(t)))0, t \in [0,T]_{\mathbb{N}_0},$$

(3.2)
$$\Delta y(T+1) = 0, y(0) - B_0(\Delta y(\xi)) = 0,$$

where $x(t) = \Delta_0^{(1-\nu)} y(t),_{t+\nu-\beta} \Delta_{\nu-1}^{\beta} x(t) =_{t+\nu-\beta} \Delta_{\nu-1}^{\beta} \Delta_0^{(1-\nu)} y(t)$.

From $\Delta\varphi_p(\Delta y(t)) \leq 0$ and (3.2), we may easily see $y(t) \geq 0, \Delta y(t) \geq 0, \Delta^2 y(t) \leq 0$.

By $y(t) = \Delta_{\nu-1}^{-(1-\nu)} x(t) = \frac{1}{\Gamma(1-\nu)} \sum_{s=\nu-1}^{t+\nu-1} (t-s-1)^{(-\nu)} x(s)$, we have $y(t) \geq 0$ if $x(t) \geq 0$.

On the other hand, from $y(t) = \Delta_{\nu-1}^{-(1-\nu)} x(t)$ and Lemma 2.6, 2.8, we have

$$x(t) = \Delta_0^{1-\nu} y(t) = \sum_{k=0}^{t+1-\nu} (-1)^k \begin{pmatrix} 1-\nu \\ k \end{pmatrix} y(t+1-\nu-k).$$

$$\begin{aligned}
x(\nu-1) &= y(0) \geq 0, \\
y(1) &\geq x(\nu) = y(1) - (1-\nu)y(0) \geq \nu y(1), \\
y(2) &\geq x(\nu+1) = y(2) - (1-\nu)y(1) + \frac{(1-\nu)(-\nu)}{2!} y(0) \geq \frac{\nu(1+\nu)}{2!} y(2), \\
&\vdots \\
y(t) &\geq x(t+\nu-1) = y(t) - (1-\nu)y(t-1) + \cdots \\
&\quad + \frac{(1-\nu)(-\nu)\cdots(-\nu-t+1)}{t!} y(0) \\
&\geq \frac{\nu(1+\nu)\cdots(t+\nu-1)}{t!} y(t),
\end{aligned}$$

$$\frac{(T+\nu-1)^{(\nu-1)}}{\Gamma(\nu)} y(t) \leq x(t+\nu-1) \leq y(t), t \in [0,T]_{\mathbb{N}_0}. \tag{3.3}$$

Therefore, the problem (1.1)(1.2) has positive solutions if and only if the problem (3.1)(3.2) have positive solutions.

Summing (3.1) from t to T, one gets

$$\varphi_p(\Delta y(T+1)) - \varphi_p(\Delta y(t)) = -\Sigma_{i=t}^T q(i) f(i+\nu-1, x(i+\nu-1),_{i+\nu-\beta} \Delta_{\nu-1}^\beta x(i)).$$

Thus,

$$\Delta y(t) = \varphi_q \Big(\Sigma_{i=t}^T q(i) f(i+\nu-1, x(i+\nu-1),_{i+\nu-\beta} \Delta_{\nu-1}^\beta x(i)) \Big). \tag{3.4}$$

Again summing (3.4) from 0 to $t-1$, it follows that

$$y(t) - y(0) = \sum_{j=0}^{t-1} \varphi_q \Big(\Sigma_{i=j}^T q(j) f(j+\nu-1, x(j+\nu-1),_{i+\nu-\beta} \Delta_{\nu-1}^\beta x(j)) \Big)$$

Since $\Delta y(\xi) = \varphi_q \Big(\Sigma_{i=\xi}^T q(i) f(i+\nu-1, x(i+\nu-1),_{i+\nu-\beta} \Delta_{\nu-1}^\beta x(i)) \Big)$,
and $y(0) = B_0(\Delta y(\xi))$,
one gets

$$y(t) = B_0 \Big(\varphi_q \Big(\Sigma_{i=\xi}^T q(i) f(i+\nu-1, x(i+\nu-1),_{i+\nu-\beta} \Delta_{\nu-1}^\beta x(i)) \Big) \Big)$$

$$+\sum_{j=0}^{t-1}\varphi_q\Big(\Sigma_{i=j}^{T}q(i)f(i+\nu-1,x(i+\nu-1),_{i+\nu-\beta}\Delta_{\nu-1}^{\beta}x(i))\Big). \tag{3.5}$$

Next, we let $B=\{y:[0,T+2]_{\mathbb{N}_0}\longrightarrow\mathbb{R}\}$ be endowed with the norm $||y||=\max\{||y||_\infty,||\Delta y||_\infty\}$, where$||y||_\infty=\max_{t\in[0,T+2]_{\mathbb{N}_0}}|y(t)|$. Then B is a Banach space. Choose the cone $P\subset B$ defined by

$$P=\{y\in B: y(t)\geq 0\,\text{for}\,t\in[0,T+2]_{\mathbb{N}_0}\,\text{and}\,\Delta^2y(t)\leq 0,\Delta y(t)\geq 0$$

$\text{for}\,t\in[0,T]_{\mathbb{N}_0},\Delta y(T+1)=0\}$.

Let the nonnegative continuous concave functional α, the nonnegative continuous convex functional θ,γ, and the nonnegative continuous functional ψ be defined on the cone P by

$$\begin{aligned}\gamma(y)&=\max_{t\in[h,T+1]_{\mathbb{N}_0}}|\Delta y(t)|,\alpha(y)\\&=\min_{t\in[0,T+2]_{\mathbb{N}_0}}|y(t)|,\psi(y)=\theta(y)=\max_{t\in[0,T+2]_{\mathbb{N}_0}}|y(t)|\end{aligned} \tag{3.6}$$

where $h=[\frac{T+2}{2}]$, and $[x]$ is the greatest integer not greater than x. Clearly, $\gamma(y)=\Delta y(0),\psi(y)=\theta(y)=y(T+2),\alpha(y)=y(h)$.

For $t\in[0,T+2]_{\mathbb{N}_0}$, defined an operator $Q:P\longrightarrow B$ by

$$\begin{aligned}Qy(t)&=B_0\Big(\varphi_q\Big(\Sigma_{i=\xi}^{T}q(i)f(i+\nu-1,x(i+\nu-1),_{i+\nu-\beta}\Delta_{\nu-1}^{\beta}x(i))\Big)\Big)\\&+\sum_{i=0}^{t-1}\Big(\varphi_q\Big(\Sigma_{j=i}^{T}q(j)f(j+\nu-1,x(j+\nu-1),_{j+\nu-\beta}\Delta_{\nu-1}^{\beta}x(j))\Big)\Big).\end{aligned} \tag{3.7}$$

Lemma 3.1. *Let T is defined by above equation. If $y\in\mathrm{P}$, then*

(i) $\Delta(Qy)(t)\geq 0$ *for* $t\in[0,T+1]_{\mathbb{N}_0},(Qy)(t)\geq 0$ *for* $t\in[0,T+2]_{\mathbb{N}_0}$,
(ii) $\Delta(Qy)(T+1))=0,(Qy)(0)-B_0(\Delta Qy)(\xi))=0$,
(iii) $Q:\mathrm{P}\longrightarrow\mathrm{P}$ *is completely continuous.*
(iv) *Finding positive solutions of FBVP (3.1)(3.2) is equivalent to find fixed-points of the operator Q on* P.
(v) *If* $y\in\mathrm{P}$, *then* $y(t)\geq\frac{t}{T+2}||y||_\infty=\frac{t}{T+2}y(T+2)$, *for* $t\in[0,T+2]_{\mathbb{N}_0}$.

The proof is simple and omitted. By Lemma 3.1 and (3.6), for all $y\in\overline{P(\gamma,d)}\subset P$, the functionals defined above satisfy:

$$\frac{h}{T+2}\theta(y)\leq\alpha(y)\leq\theta(y)=\psi(y). \tag{3.8}$$

Furthermore, since

$$\frac{y(T+2)-y(0)}{T+2} \leq \Delta y(0),$$

we have

$$||y|| = \max\{y(T+2), \Delta y(0)\} \leq (T+2+m)\Delta y(0). \tag{3.9}$$

Therefore, $\alpha(y) \leq \psi(y)$ and $||y|| \leq M\gamma(y)$ are satisfied. Denote

$$M_1 = \sum_{i=0}^{T} q(i), M_2 = \phi_q \sum_{i=\xi}^{T} q(i), M_3 = \sum_{j=0}^{T} \phi_q\big(\sum_{i=j}^{T} q(i)\big),$$
$$N = \max\{M_2 m, M_3\}.$$

THEOREM 3.2. *Suppose $(H1)-(H3)$ hold. In addition, assume that there exist numbers a, b, d with $0 < a < b \leq \frac{h}{T+2}d$ such that the following conditions are satisfied:*

(H4) $f(t+\nu-1, u, v) \leq \frac{1}{M_1}\phi_p(d)$ *for* $(t, u, v) \in [0, T]_{\mathbb{N}_0} \times [0, (T+2+m)d] \times [0, (T+1+m(\gamma-\beta))d]$,

(H5) $f(t+\nu-1, u, v) > \phi_p(\frac{(T+2)b}{hM_3})$ *for* $(t, u, v) \in [h, T]_{\mathbb{N}_h} \times [\nu^T b, \frac{T+2}{h}b] \times [0, (T+1+m(\gamma-\beta))d]$,

(H6) $f(t+\nu-1, u, v) < \phi_p(\frac{a}{2N})$ *for* $(t, u, v) \in [0, T]_{\mathbb{N}_0} \times [0, a] \times [0, (T+1+m(\gamma-\beta))d]$.

Then FBVP (1.1)(1.2) has at least three positive solutions x_1, x_2 and x_3 satisfying

$$x_1(t) = \Delta_0^{1-\nu} y_1(t), x_2(t) = \Delta_0^{1-\nu} y_2(t), x_3(t) = \Delta_0^{1-\nu} y_3(t), \tag{3.10}$$

$$\max_{t\in[0,T+1]_{\mathbb{N}_0}} |\Delta y_i(t)| \leq d, \textit{for} i = 1, 2, 3, \tag{3.11}$$

$$b < \min_{t\in[h,T+2]_{\mathbb{N}_h}} |y_1(t)|, \max_{t\in[0,T+2]_{\mathbb{N}_0}} |y_3(t)| < a, \tag{3.12}$$

$$a < \max_{t\in[0,T+2]_{\mathbb{N}_0}} |y_2(t)|, \textit{with} \min_{t\in[h,T+2]_{\mathbb{N}_h}} |y_2(t)| < b. \tag{3.13}$$

Proof. By the definition of operator Q and its properties $(i)-(v)$, it suffices to show that the conditions of Lemma 2.5 with respect to Q.

If $y \in \overline{P(\gamma, d)}$, then $\gamma(y) = \max_{t\in[0,T+1]_{\mathbb{N}_0}} |\Delta y(t)| \leq d$. From (3.9), we have

$$\max_{t\in[0,T+2]_{\mathbb{N}_0}} |y(t)| \leq (T+2+m)d.$$

By Lemma 2.7-2.9, we may get

$$\Delta_0^{(1-\nu)}y(t) = \Delta\Delta_0^{(-\nu)}y(t) = \Delta_0^{(-\nu)}\Delta y(t) + \frac{y(0)}{\Gamma(\nu)}t^{(\nu-1)},$$

and

$$\begin{aligned}
{}_{t+\nu-\beta}\Delta_{\nu-1}^{\beta}x(t) &= {}_{t+\nu-\beta}\Delta_{\nu-1}^{\beta}\Delta_0^{1-\nu}y(t) \\
&= {}_{t+\nu-\beta}\Delta_{\nu-1}^{\beta}\Delta_0^{-\nu}\Delta y(t) + {}_{t+\nu-\beta}\,\Delta_{\nu-1}^{\beta}\frac{y(0)}{\Gamma(\nu)}t^{(\nu-1)} \\
&= {}_{t+\nu-\beta}\Delta_0^{\beta-\nu}\Delta y(t) + {}_{t+\nu-\beta}\,\Delta_{\nu-1}^{\beta}\frac{y(0)}{\Gamma(\nu)}t^{(\nu-1)} \\
&= \frac{1}{\Gamma(\nu-\beta)}\sum_{s=0}^{t}(t+\nu-\beta-s-1)^{(\nu-\beta-1)}\Delta y(s) \\
&\quad + \frac{y(0)}{\Gamma(\nu)}(\nu-1)^{(\beta)}(t+\nu-\beta)^{(\nu-1-\beta)} \\
&= \frac{1}{\Gamma(\nu-\beta)}\sum_{s=\beta-\nu}^{t-(\nu-\beta)}(t-s-1)^{(\nu-\beta-1)}\Delta y(s) \\
&\quad + \frac{1}{\Gamma(\nu-\beta)}(t+\nu-\beta)^{(\nu-1-\beta)}B_0(\Delta y(\xi)) \\
&\le d\sum_{s=0}^{t}\frac{(t+\nu-\beta-s-1)^{(\nu-\beta-1)}}{\Gamma(\nu-\beta)} \\
&\quad + md\frac{1}{\Gamma(\nu-\beta)}(t+\nu-\beta)^{(\nu-1-\beta)}.
\end{aligned}$$

$$0 \le_{t+\nu-\beta} \Delta_{\nu-1}^{\beta}x(t) \le (T+1)d + m(\nu-\beta)d.$$

On the other hand, assumption $(H4)$ and Lemma 2.1 imply that

$$\begin{aligned}
\gamma(Qy) &= \max_{t\in[0,T+1]_{\mathbb{N}_0}}|\Delta(Qy)(t)| = |\Delta(Qy)(0)| \\
&= \varphi_q\Big(\Sigma_{i=0}^{T}q(i)f(i+\nu-1, x(i+\nu-1), {}_{i+\nu-\beta}\,\Delta_{\beta-1}^{\beta}x(i))\Big) \\
&\le \varphi_q\Big(\frac{1}{M_1}\varphi_p(d)\Sigma_{i=0}^{T}q(i)\Big) = d.
\end{aligned}$$

Hence, $Q:\overline{P(\gamma,d)}\longrightarrow\overline{P(\gamma,d)}$.

To check condition $(S1)$ of Lemma 2.5, we choose $y_0(t)$ as follow for $t \in [0, T+2]_{\mathbb{N}_0}$,

$$y_0(t) = \begin{cases} -\frac{2b}{(T+2)^2}(t-(T+2))^2 + \frac{T+2}{h}b, & \text{t is even;} \\ -\frac{2b}{(T+3)^2}(t-(T+2))^2 + \frac{T+2}{h}b, & \text{t is odd.} \end{cases}$$

It is easy to verified that $y_0 \in P(\gamma, \theta, \alpha, b, \frac{T+2}{h}b, d)$ and $\alpha(y_0) = \min_{t\in[h,T+2]_{\mathbb{N}_0}} |y_0(t)| = y_0(h) > b$. So $\{y \in P(\gamma, \theta, \alpha, b, \frac{T+2}{h}b, d) | \alpha(y) > b\} \neq \emptyset$.

Hence, if $y \in P(\gamma, \theta, \alpha, b, \frac{T+2}{h}b, d)$, then $b \leq y(t) \leq \frac{T+2}{h}b, |\Delta y(t)| \leq d$ for $t \in [h, T+2]_{\mathbb{N}_h}$.

Then by (3.3), we have $\nu^T b \leq x(t+\nu-1) \leq \frac{T+2}{h}b$. From assumption $(H4)(H5)$ and Lemma 3.1, we have

$$\begin{aligned}
\alpha(Qy) &= \min_{t\in[h,T+2]_{\mathbb{N}_h}} |(Qy)(t)| \geq \frac{h}{T+2} \max_{t\in[0,T+2]_{\mathbb{N}_0}} |(Qy)(t)| \\
&= \frac{h}{T+2} B_0(\varphi_q(\Sigma_{i=\xi}^T q(i) f(i+\nu-1, x(i+\nu-1),_{i+\nu-\beta} \Delta_{\nu-1}^{\beta} x(i))) \\
&\quad + \Sigma_{i=0}^T \varphi_q(q(i) f(i+\nu-1, x(i+\nu-1),_{i+\nu-\beta} \Delta_{\nu-1}^{\beta} x(i))) \\
&> \frac{h}{T+2}\left(\varphi_q \varphi_p(\frac{(T+2)b}{M_3 h}) \Sigma_{i=j}^T q(i)\right) = \frac{b}{M_3} M_3 = b.
\end{aligned}$$

This show that condition $(S1)$ of Lemma 2.5 is satisfied.

Secondly, from (3.8), we have

$$\alpha(Qy) \geq \frac{h}{T+2} \theta(Qy) > b$$

for all $y \in P(\gamma, \alpha, b, d)$ with $\theta(Qy) > \frac{T+2}{h}b$. Thus, condition $(S2)$ of Lemma 2.5 is satisfied.

We finally show that $(S3)$ of Lemma 2.5 holds. Clearly, as $\psi(0) = 0 < a$, we have $0 \notin R(\gamma, \psi, a, d)$. Suppose that $y \in R(\gamma, \psi, a, d)$ with $\psi(y) = a$. Then, by $(H6)$, we

$$\begin{aligned}
\psi(Qy) &= \max_{t\in[0,T+2]_{\mathbb{N}_0}} |(Qy)(t)| \\
&= B_0(\varphi_q(\Sigma_{i=\xi}^T q(i) f(i+\nu-1, x(i+\nu-1),_{i+\nu-\beta} \Delta_{\nu-1}^{\beta} x(i)))) \\
&\quad + \Sigma_{i=0}^T \varphi_q(q(i) f(i+\nu-1, x(i+\nu-1),_{i+\nu-\beta} \Delta_{\nu-1}^{\beta} x(i))) \\
&\leq m\varphi_q(\Sigma_{i=\xi}^T q(i) f(i+\nu-1, x(i+\nu-1),_{i+\nu-\beta} \Delta_{\nu-1}^{\beta} x(i)))
\end{aligned}$$

$$
\begin{aligned}
& +\varphi_q(\varphi_p(\frac{a}{2M_3}))\Sigma_{i=0}^{T}\varphi_q(q(i)) \\
< \ & m\varphi_q(\varphi_p(\frac{a}{2M_2 m}))\varphi_q(\Sigma_{i=\xi}^{T}q(i)) + \varphi_q(\varphi_p(\frac{a}{2M_3}))\Sigma_{i=0}^{T}\varphi_q(q(i)) \\
= \ & a.
\end{aligned}
$$

So, condition $(S3)$ of Lemma 2.5 is satisfied. Therefore, an application of Lemma 2.5 implies that the FBVP (3.1)(3.2) has at least three positive solutions y_1, y_2 and y_3 satisfying (3.10)-(3.13), i.e., the FBVP (1.1)(1.2) has at least three positive solutions x_1, x_2 and x_3 satisfying (3.10). $\square$

EXAMPLE 3.3. *Consider the boundary value problem*

(3.14)
$$\Delta(\varphi_p(\Delta_{-0.05}^{0.95}x(t)))+f(t+\nu-1, x(t+\nu-1),_{t+0.5}\Delta_{-0.55}^{0.45}x(t))) = 0, t \in [0,10]_{\mathbb{N}_0},$$

(3.15) $$[\triangle_{-0.5}^{0.95}x(t)]_{t=11} = 0, x(-0.5) - [\triangle_{-0.5}^{0.95}x(t)]_{t=1} = 0$$

Compared to (1.1), $q(t) = 1, p = 3, q = \frac{3}{2}, \xi = 1, T = 10, B_0(v) = v$ *and* $f(t+\nu-1, u, v)$

$$
= \begin{cases}
|\sin(t+\nu-1)| + 50 + \frac{v}{1500}, & 0 \le u \le 500; \\
|\sin(t+\nu-1)| + 50 + \frac{1}{11.5}(u-500)^2 + \frac{v}{1500}, & 500 \le u \le 1000; \\
|\sin(t+\nu-1)| + 50 + \frac{1}{11.5}(500)^2 + \frac{v}{1500}, & u \ge 1000.
\end{cases}
$$

Then, FBVP (3.14)(3.15) has at least three positive solutions.

Proof. Choose $a = 500, b = 1800, d = 5000, p = 3, q = \frac{3}{2}, m = 1$, by computation, we know $M_1 = 11, M_2 = \sqrt{10}, M_3 = \sum_{j=0}^{10}(11 - j)^{\frac{1}{2}} \approx 25.78, N = M_3$.

It is easy to see that $0 < a < b < 2d$. And f satisfies that

$$f(t+\nu-1, u, v) \le 51 + \frac{250000}{11.5} + \frac{500}{1500} \approx 21870 < 22727.27 \approx \frac{1}{M_1}\varphi_p(d),$$

for$(t, u, v) \in [0, 10]_{\mathbb{N}_0} \times [0, 6500] \times [0, 57500]$,

$$f(t+\nu-1, u, v) \ge 50 + \frac{500^2}{11.5} \approx 21789 > 13542 \approx (\frac{3600}{25.78})^2 = \varphi_p(\frac{(T+2)b}{hM_3}),$$

for$(t, u, v) \in [6, 10]_{\mathbb{N}_0} \times [(0.95)^{10} \times 1800, 3600] \times [0, 57500]$,

$$f(t+\nu-1, u, v) < 51 + 40 \le (\frac{500}{2 \times 25.78})^2 < 94.04 \approx \varphi_p(\frac{a}{2N}),$$

for$(t, u, v) \in [0, 10]_{\mathbb{N}_0} \times [0, 500] \times [0, 57500]$.

Then FBVP (3.14)(3.15) has at least three positive solutions satisfying

$$x_1(t) = \Delta_0^{0.05} y_1(t), x_2(t) = \Delta_0^{0.05} y_2(t), x_3(t) = \Delta_0^{0.05} y_3(t),$$
$$\max_{t\in[0,T+1]_{\mathbb{N}_0}} |\Delta y_i(t)| \leq 5000, \text{for } i = 1, 2, 3,$$
$$1800 < \min_{t\in[6,12]_{\mathbb{N}_6}} |y_1(t)|, \max_{t\in[0,12]_{\mathbb{N}_0}} |y_3(t)| < 500,$$
$$500 < \max_{t\in[0,12]_{\mathbb{N}_0}} |y_2(t)|, \text{with} \min_{t\in[6,12]_{\mathbb{N}_6}} |y_2(t)| < 1800.$$

□

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