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Some Common Fixed Point Theorems via Generalized c-Distance

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Abstract. In this paper, we introduce the concept of generalized c-distance on a cone metric space and prove some common fixed point and coincidence point results by using this notion. Our results generalize and extend several well known comparable results in the literature.

1. Introduction

In 2007, Huang and Zhang [8] had introduced the concept of cone metric spaces. Then, based on the notion of cone metric spaces, a series of articles have been dedicated to the improvement of fixed point theory. In some works, the authors used normal cones to extend some fixed point theorems. Very recently, Wang and Guo [21] introduced the concept of c-distance on a cone metric space, which is a cone version of the w-distance of Kada et.al.[11] and proved a common fixed point theorem for a pair of self mappings in cone metric spaces. In this work, we introduce the concept of generalized c-distance on a cone metric space and prove some common fixed point theorems for a pair of weakly compatible mappings in cone metric spaces by employing this new notion. It is worth mentioning that the cone under consideration is non-normal. Finally, an example is provided to show that the generalized c-distances form a bigger category than that of c-distances.

2. Preliminaries

Let E be a real Banach space and θ denote the zero element in E. A cone P is a subset of E such that

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(i) P is closed nonempty and P$\neq \{\theta\}$

(ii) $a, b \in \mathbb{R}, a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$

(iii) $P \cap (-P) = \{\theta\}$.

For a given cone $P \subseteq E$, we can define a partial ordering $\preceq$ on E with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ (equivalently, $y \succ x$) if $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in int(P)$, where $int(P)$ denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$\theta \preceq x \preceq y \textit{ implies } \|x\| \leq K\,\|y\|.$$

The least positive number satisfying the above inequality is called the normal constant of P. Rezapour and Hamlbarani [18] proved that there are no normal cones with normal constant $K < 1$ and for each $k > 1$ there are cones with normal constant $K > k$.

Definition 2.1.([8]) Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

(i) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Definition 2.2.([8]) Let (X, d) be a cone metric space. Let (x_n) be a sequence in X and $x \in X$. If for every $c \in E$ with $\theta \ll c$ there is a natural number n_0 such that for all $n > n_0$, $d(x_n, x) \ll c$, then (x_n) is said to be convergent and (x_n) converges to x, and x is the limit of (x_n). We denote this by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ $(n \to \infty)$.

Definition 2.3.([8]) Let (X, d) be a cone metric space, (x_n) be a sequence in X. If for any $c \in E$ with $\theta \ll c$, there is a natural number n_0 such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$, then (x_n) is called a Cauchy sequence in X.

Definition 2.4.([8]) Let (X, d) be a cone metric space, if every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

Lemma 2.5.([19]) *Let E be a real Banach space with a cone P. Then*

(i) *If $a \ll b$ and $b \ll c$, then $a \ll c$.*

(ii) *If $a \preceq b$ and $b \ll c$, then $a \ll c$.*

Lemma 2.6.([8]) *Let E be a real Banach space with cone P. Then one has the following.*

(i) *If $\theta \ll c$, then there exists $\delta > 0$ such that $\|b\| < \delta$ implies $b \ll c$.*

(ii) *If a_n, b_n are sequences in E such that $a_n \to a$, $b_n \to b$ and $a_n \preceq b_n$ for all $n \geq 1$, then $a \preceq b$.*

Proposition 2.7.([9]) *If E is a real Banach space with cone P and if $a \preceq \lambda a$ where $a \in P$ and $0 \leq \lambda < 1$ then $a = \theta$.*

Definition 2.8.([2]) Let T and S be self mappings of a set X. If $y = Tx = Sx$ for some x in X, then x is called a coincidence point of T and S and y is called a point of coincidence of T and S.

Definition 2.9.([10]) The mappings $T, S : X \to X$ are weakly compatible, if for every $x \in X$, the following holds:

$$T(Sx) = S(Tx) \text{ whenever } Sx = Tx.$$

Proposition 2.10.([2]) *Let S and T be weakly compatible selfmaps of a nonempty set X. If S and T have a unique point of coincidence $y = Sx = Tx$, then y is the unique common fixed point of S and T.*

Definition 2.11.([21]) Let (X, d) be a cone metric space. Then a mapping $q : X \times X \to E$ is called a c-distance on X if the following are satisfied :

(i) $\theta \preceq q(x, y)$ for all x, $y \in X$;

(ii) $q(x, z) \preceq q(x, y) + q(y, z)$ for all $x, y, z \in X$;

(iii) for all $x \in X$, if $q(x, y_n) \preceq u$ for some $u = u_x \in P$ and all $n \geq 1$, then $q(x, y) \preceq u$ whenever (y_n) is a sequence in X converging to a point $y \in X$;

(iv) for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.

Example 2.12.([21]) Let (X, d) be a cone metric space and P be a normal cone. Put $q(x, y) = d(x, y)$ for all $x, y \in X$. Then q is a c-distance.

Definition 2.13. Let (X, d) be a cone metric space and $j \in \mathbb{N}$. A function $q : X \times X \to E$ is called a generalized c-distance of order j on X if the following conditions are satisfied:

(q1) $\theta \preceq q(x, y)$, for all $x, y \in X$;

(q2) $q(x, z) \preceq \sum_{i=0}^{j} q(x_i, x_{i+1})$, for all $x, z \in X$ and for all distinct points $x_i \in X$, $i \in \{1, 2, 3, \cdots, j\}$ each of them different from $x(= x_0)$ and $z(= x_{j+1})$;

(q3) for all $x \in X$, if $q(x, y_n) \preceq u$ for some $u = u_x \in P$ and all $n \geq 1$, then $q(x, y) \preceq u$ whenever (y_n) is a sequence in X converging to a point $y \in X$;

(q4) for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.

It follows from above definition that every c-distance is a generalized c-distance of order 1. In fact, every c-distance may also be considered as a generalized c-distance of any order $j \in \mathbb{N}$. But the converse does not hold (see Example 3.11).

3. Main Results

In this section we always suppose that E is a real Banach space, P is a cone in E with $int(P) \neq \emptyset$ and $\preceq$ is the partial ordering with respect to P. We begin with the following Lemma which is crucial in the proofs of the main theorems.

Lemma 3.1. *Let (X, d) be a cone metric space and q be a generalized c-distance of order j on X. Let (x_n) and (y_n) be sequences in X. Suppose that (α_n) and (β_n) are sequences in P converging to θ, and let $x, y, z \in X$. Then the following hold :*

(i) *If $q(x_n, y_n) \preceq \alpha_n$ and $q(x_n, z) \preceq \beta_n$ for any $n \in \mathbb{N}$, then (y_n) converges to z;*

(ii) *If $q(x_n, y) \preceq \alpha_n$ and $q(x_n, z) \preceq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $q(x, y) = \theta$ and $q(x, z) = \theta$, then $y = z$;*

(iii) *If $q(x_n, x_m) \preceq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then (x_n) is a Cauchy sequence.*

Proof. (i) Let $c \in E$ with $\theta \ll c$. Then there exists $\delta > 0$ such that $\| x \| < \delta$ implies $c - x \in int\,(P)$. Since (α_n) and (β_n) are converging to θ, there exists $n_0 \in \mathbb{N}$ such that $\| \alpha_n \| < \delta$ and $\| \beta_n \| < \delta$ for all $n > n_0$. Thus $c - \alpha_n \in int\,(P)$ and $c - \beta_n \in int\,(P)$ for all $n > n_0$ and so $\alpha_n \ll c$ and $\beta_n \ll c$ for all $n > n_0$. By hypothesis, $q(x_n, y_n) \preceq \alpha_n \ll c$ and $q(x_n, z) \preceq \beta_n \ll c$ for all $n > n_0$. Now from $(q4)$ with $e = c$ it follows that $d(y_n, z) \ll c$ for all $n > n_0$. Therefore (y_n) converges to z.

Clearly, (ii) is immediate from (i).

(iii) Let $c \in E$ with $\theta \ll c$. Then by the arguments similar to that used above, there exists a positive integer n_0 such that $q(x_n, x_m) \preceq \alpha_n \ll c$ for all $m > n$ with $n > n_0$. This implies that $q(x_n, x_{n+1}) \preceq \alpha_n \ll c$ and $q(x_n, x_{m+1}) \preceq \alpha_n \ll c$ for all $m > n$ with $n > n_0$. From $(q4)$ with $e = c$ it follows that $d(x_{n+1}, x_{m+1}) \ll c$ for all $m > n$ with $n > n_0$. This shows that (x_n) is a Cauchy sequence in X. □

Theorem 3.2. *Let (X, d) be a cone metric space and q be a generalized c-distance of order j on X. Suppose the mappings $f,\, g : X \to X$ satisfy*

$$q(fx, fy) \preceq r\, q(gx, gy) \tag{3.1}$$

for all $x, y \in X$ and $0 \leq r < 1$. If $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary element of X. Since $f(X) \subseteq g(X)$, there exists an element $x_1 \in X$ such that $fx_0 = gx_1$. Proceeding in this way, a sequence (x_n) can be chosen such that $gx_n = fx_{n-1},\ n = 1, 2, 3, \cdots$.

We can suppose that $gx_n \neq gx_m$ for all distinct $n, m \in \{0, 1, 2, \cdots\}$. In fact, if $gx_n = gx_m$ for some $n, m \in \{0, 1, 2, \cdots\}$, $m \neq n$ then assuming $m > n$, we may write

$$gx_n = gx_{n+k},\ where\ k = m - n \geq 1.$$

Put $y = gx_n$. Then

$$\begin{aligned} q(y, gx_{n+1}) &= q(gx_n, gx_{n+1}) \\ &= q(gx_{n+k}, gx_{n+1}) \\ &= q(fx_{n+k-1}, fx_n) \\ &\preceq r\, q(gx_{n+k-1}, gx_n) \\ &= r\, q(gx_{n+k-1}, gx_{n+k}) \\ &\preceq r^2\, q(gx_{n+k-2}, gx_{n+k-1}) \\ &\cdot \\ &\cdot \\ &\cdot \\ &\preceq r^k\, q(gx_n, gx_{n+1}) \\ &= r^k\, q(y, gx_{n+1}). \end{aligned}$$

Since $0 \leq r < 1$, by using Proposition 2.7 it follows that $q(y, gx_{n+1}) = \theta$.

Now,

$$\begin{aligned} q(y, y) &= q(gx_n, gx_n) \\ &= q(gx_{n+k}, gx_{n+k}) \\ &\preceq r^k\, q(gx_n, gx_n) \\ &= r^k\, q(y, y). \end{aligned}$$

Again by using Proposition 2.7, $q(y, y) = \theta$. Since $q(y, gx_{n+1}) = \theta$ and $q(y, y) = \theta$, by using Lemma 3.1(ii), we have $gx_{n+1} = y$. Therefore, $fx_n = y = gx_n$ i.e., y is a point of coincidence of f and g.
Thus in the sequel of the proof we can suppose that $gx_n \neq gx_m$ for all distinct $n, m \in \{0,\ 1,\ 2,\ \cdots\}$.

For any natural number n, we have by using condition (3.1) that

(3.2) $$q(gx_n, gx_{n+1}) = q(fx_{n-1}, fx_n) \preceq r\, q(gx_{n-1}, gx_n) \preceq \cdots \preceq r^n\, q(gx_0, gx_1).$$

Let us now prove that for all $m, n \in \mathbb{N}$ with $m > n$, one has

(3.3) $$q(gx_n, gx_m) \preceq \frac{r^n}{1-r}\, M,$$

where $M = q(gx_0, gx_1) + q(gx_0, gx_2) + \cdots + q(gx_0, gx_j) \in P$.
Taking $m = n + p$ where $p = 1,\ 2,\ 3,\ \cdots$ and using (3.1), we have

(3.4) $$q(gx_n, gx_m) \preceq r^n\, q(gx_0, gx_{m-n}) = r^n\, q(gx_0, gx_p).$$

If $p \leq j$, then

$$\begin{aligned} q(gx_0, gx_p) &\preceq (1 + r + r^2 + \cdots)\, q(gx_0, gx_p) \\ &\preceq \frac{1}{1-r}\, M. \end{aligned}$$

If $p > j$, then there exists $s \in \mathbb{N}$ such that $p = sj + t$, where $0 \leq t < j$. If $t = 0$, then by using conditions (3.2) and (3.4)

$$\begin{aligned} q(gx_0, gx_p) &\preceq q(gx_0, gx_1) + q(gx_1, gx_2) + \cdots + q(gx_{j-1}, gx_j) + q(gx_j, gx_p) \\ &\preceq q(gx_0, gx_1) + rq(gx_0, gx_1) + \cdots + r^{j-1}q(gx_0, gx_1) + r^j q(gx_0, gx_{p-j}) \\ &= \sum_{\nu=0}^{j-1} r^\nu q(gx_0, gx_1) + r^j q(gx_0, gx_{p-j}). \end{aligned} \tag{3.5}$$

By repeated application of (3.5), we obtain at $(s-1)$-th step that

$$\begin{aligned} q(gx_0, gx_p) &\preceq \sum_{\nu=0}^{(s-1)j-1} r^\nu q(gx_0, gx_1) + r^{(s-1)j} q(gx_0, gx_j) \\ &\preceq (1 + r + r^2 + \cdots + r^{(s-1)j})\, M \\ &\preceq \frac{1}{1-r}\, M. \end{aligned}$$

If $t \neq 0$, then

$$\begin{aligned} q(gx_0, gx_p) &\preceq q(gx_0, gx_1) + q(gx_1, gx_2) + \cdots + q(gx_{j-1}, gx_j) + q(gx_j, gx_p) \\ &\preceq \sum_{\nu=0}^{j-1} r^\nu q(gx_0, gx_1) + r^j q(gx_0, gx_{p-j}). \end{aligned} \tag{3.6}$$

By repeated application of (3.6), we obtain at s-th step that

$$\begin{aligned} q(gx_0, gx_p) &\preceq \sum_{\nu=0}^{sj-1} r^\nu q(gx_0, gx_1) + r^{sj} q(gx_0, gx_t) \\ &\preceq (1 + r + r^2 + \cdots + r^{sj})\, M \\ &\preceq \frac{1}{1-r}\, M. \end{aligned}$$

Thus, for the case $p > j$, we have

$$q(gx_0, gx_p) \preceq \frac{1}{1-r}\, M.$$

It now follows from (3.4) that for all $m, n \in \mathbb{N}$ with $m > n$,

$$q(gx_n, gx_m) \preceq \frac{r^n}{1-r}\, M. \tag{3.7}$$

By using Lemma 3.1(iii), we conclude that (gx_n) is a Cauchy sequence in $g(X)$. Since $g(X)$ is complete, there exists an element $u \in g(X)$ such that $gx_n \to u$ as $n \to \infty$.
By (3.7) and $(q3)$, we have

$$q(gx_n, u) \preceq \frac{r^n}{1-r} M,$$

which implies that, $q(gx_n, u) \to \theta$ as $n \to \infty$.
Since $u \in g(X)$, there exists $z \in X$ such that $u = gz$.
Again, from (3.1)

$$q(gx_n, fz) = q(fx_{n-1}, fz) \preceq r\, q(gx_{n-1}, gz) = r\, q(gx_{n-1}, u) \to \theta \text{ as } n \to \infty.$$

By Lemma 3.1(ii), $q(gx_n, u) \to \theta$ and $q(gx_n, fz) \to \theta$ imply that $fz = u = gz$. Therefore, u becomes a point of coincidence of f and g.
For uniqueness, let there exists $w(\neq u) \in X$ such that $fx = gx = w$ for some $x \in X$. Then

$$q(u, w) = q(fz, fx) \preceq r\, q(gz, gx) = r\, q(u, w).$$

By applying Proposition 2.7, it follows that $q(u, w) = \theta$.
Now,

$$q(u, u) = q(fz, fz) \preceq r\, q(gz, gz) = r\, q(u, u),$$

which implies that, $q(u, u) = \theta$.
Thus, $q(u, w) = \theta$ and $q(u, u) = \theta$ imply that $u = w$. Therefore, f and g have a unique point of coincidence in X.
If f and g are weakly compatible, then by Proposition 2.10 , f and g have a unique common fixed point in X. □

Remark 3.3. We see that if u is a point of coincidence of f and g, then $q(u, u) = \theta$.

The following Corollary is an extension of Theorem 3.1[7].

Corollary 3.4. *Let (X, d) be a complete cone metric space and q be a generalized c-distance of order j on X. Suppose the mapping $f : X \to X$ satisfies*

$$q(fx, fy) \preceq r\, q(x, y)$$

for all $x, y \in X$ and $0 \leq r < 1$. Then f has a unique fixed point in X.

Proof. The proof can be obtained from Theorem 3.2 by considering $g = I$, the identity mapping. □

Corollary 3.5. *Let (X, d) be a complete cone metric space and q be a generalized c-distance of order j on X. Suppose $g : X \to X$ is an onto mapping satisfying*

$$k\, q(x, y) \preceq q(gx, gy)$$

for all $x, y \in X$, where $k > 1$ is a constant. Then g has a unique fixed point in X.

Proof. The conclusion of the Corollary follows from Theorem 3.2 by taking $f = I$. □

The following Corollary is Theorem 1[8].

Corollary 3.6. *Let (X, d) be a complete cone metric space, P a normal cone. Suppose the mapping $f : X \to X$ satisfies the contractive condition*

$$d(fx, fy) \preceq k\, d(x, y)$$

for all $x, y \in X$, where $k \in [0, 1)$ is a constant. Then f has a unique fixed point in X.

Proof. Since P is normal, it follows that d is a c-distance on X. So, we may consider d as a generalized c-distance of any order j. The desired result can be obtained from Theorem 3.2 by putting $q = d,\ g = I$. □

Theorem 3.7. *Let (X, d) be a cone metric space and q be a generalized c-distance of order j on X. Suppose the mappings $f, g : X \to X$ satisfy*

$$q(fx, fy) \preceq a_1\, q(gx, fx) + a_2\, q(gy, fy) \tag{3.8}$$

for all $x, y \in X$, and $a_1, a_2 \in [0, 1)$ with $a_1 + a_2 < 1$ and that

$$inf\,\{q(gx, y) + q(fx, y) + q(gx, fx) : x \in X\} \succ \theta \tag{3.9}$$

for all $y \in X$ with y is not a point of coincidence of f and g. If $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. As in Theorem 3.2, we can construct a sequence (x_n) such that $gx_n = fx_{n-1}$ for all $n \geq 1$.

For any natural number n, we have by using condition (3.8) that

$$\begin{aligned} q(gx_n, gx_{n+1}) &= q(fx_{n-1}, fx_n) \\ &\preceq a_1\, q(gx_{n-1}, fx_{n-1}) + a_2\, q(gx_n, fx_n) \\ &= a_1\, q(gx_{n-1}, gx_n) + a_2\, q(gx_n, gx_{n+1}) \end{aligned}$$

which implies that

$$q(gx_n, gx_{n+1}) \preceq r\, q(gx_{n-1}, gx_n) \tag{3.10}$$

where $r = \frac{a_1}{1-a_2} \in [0, 1)$.
By repeated application of (3.10), we obtain

$$q(gx_n, gx_{n+1}) \preceq r^n\, q(gx_0, gx_1). \tag{3.11}$$

Let $m, n \in \mathbb{N}$ with $m > n$. Then using (3.8), we have

$$\begin{aligned} q(gx_n, gx_m) &= q(fx_{n-1}, fx_{m-1}) \\ &\preceq a_1 q(gx_{n-1}, fx_{n-1}) + a_2 q(gx_{m-1}, fx_{m-1}) \\ &= a_1 q(gx_{n-1}, gx_n) + a_2 q(gx_{m-1}, gx_m) \\ &\preceq a_1 r^{n-1} q(gx_0, gx_1) + a_2 r^{m-1} q(gx_0, gx_1) \\ &\preceq (a_1 + a_2) r^{n-1} q(gx_0, gx_1), \end{aligned} \tag{3.12}$$

since $r^{m-1} \leq r^{n-1}$.

By using Lemma 3.1(iii), we conclude that (gx_n) is a Cauchy sequence in $g(X)$. Since $g(X)$ is complete, there exists an element $u \in g(X)$ such that $gx_n \to u$ as $n \to \infty$.
By (3.12) and ($q3$), we have

$$q(gx_n, u) \preceq (a_1 + a_2) r^{n-1} q(gx_0, gx_1). \tag{3.13}$$

Suppose that u is not a point of coincidence of f and g. Then by hypothesis, (3.11) and (3.13), we have

$$\begin{aligned} \theta &\prec inf\{q(gx, u) + q(fx, u) + q(gx, fx) : x \in X\} \\ &\preceq inf\{q(gx_n, u) + q(fx_n, u) + q(gx_n, fx_n) : n \in \mathbb{N}\} \\ &= inf\{q(gx_n, u) + q(gx_{n+1}, u) + q(gx_n, gx_{n+1}) : n \in \mathbb{N}\} \\ &\preceq inf\{(a_1 + a_2) r^{n-1} q(gx_0, gx_1) + (a_1 + a_2) r^n q(gx_0, gx_1) \\ &\quad + r^n q(gx_0, gx_1) : n \in \mathbb{N}\} \\ &= \theta, \end{aligned}$$

which is a contradiction. Therefore, u is a point of coincidence of f and g. So there exists $z \in X$ such that $fz = gz = u$.
For uniqueness, let there exists $w (\neq u) \in X$ such that $fx = gx = w$ for some $x \in X$. Then

$$\begin{aligned} q(u, u) = q(fz, fz) &\preceq a_1 q(gz, fz) + a_2 q(gz, fz) \\ &= (a_1 + a_2) q(u, u). \end{aligned}$$

Hence by Proposition 2.7, it follows that $q(u, u) = \theta$.
By the arguments similar to that used above, we have $q(w, w) = \theta$.
Now,

$$\begin{aligned} q(u, w) = q(fz, fx) &\preceq a_1 q(gz, fz) + a_2 q(gx, fx) \\ &= a_1 q(u, u) + a_2 q(w, w) \\ &= \theta, \end{aligned}$$

which gives that, $q(u, w) = \theta$.

But, $q(u,w)=\theta$ and $q(u,u)=\theta$ imply that $u=w$. Therefore, f and g have a unique point of coincidence in X.
If f and g are weakly compatible, then by Proposition 2.10 , f and g have a unique common fixed point in X. □

Corollary 3.8. *Let (X,d) be a complete cone metric space and q be a generalized c-distance of order j on X. Suppose the mapping $f: X \to X$ satisfies*

$$q(fx,fy) \preceq a_1\, q(x,fx) + a_2\, q(y,fy) \tag{3.14}$$

for all $x,y \in X$, and a_1, $a_2 \in [0,1)$ with $a_1 + a_2 < 1$ and that

$$inf\,\{q(x,y)+q(fx,y)+q(x,fx) : x \in X\} \succ \theta$$

for all $y \in X$ with $y \neq fy$. Then f has a unique fixed point in X.

Proof. The proof follows from Theorem 3.7 by taking $g = I$. □

As an application of Corollary 3.8, we have the following results.

Theorem 3.9.([8]) *Let (X,d) be a complete cone metric space, P a normal cone. Suppose the mapping $T: X \to X$ satisfies the contractive condition*

$$d(Tx,Ty) \preceq \alpha\,[d(x,Tx)+d(y,Ty)] \tag{3.15}$$

for all $x,y \in X$, where $\alpha \in [0,\frac{1}{2})$ is a constant. Then T has a unique fixed point in X.

Proof. Since P is normal, it follows that d is a c-distance on X. So, we may consider d as a generalized c-distance of any order j. The condition (3.15) can be restated as

$$d(Tx,Ty) \preceq \alpha\, d(x,Tx) + \alpha\, d(y,Ty)$$

for every $x,y \in X$, where $2\alpha \in [0,1)$ is a constant. Thus, condition (3.14) of Corollary 3.8 is satisfied.
Assume that there exists $y \in X$ with $y \neq Ty$ and

$$inf\,\{d(x,y)+d(Tx,y)+d(x,Tx) : x \in X\} = \theta.$$

Then, there exists a sequence (x_n) in X such that

$$\lim_{n\to\infty}\{d(x_n,y)+d(Tx_n,y)+d(x_n,Tx_n)\} = \theta,$$

which implies that $d(x_n,y) \to \theta,\ d(Tx_n,y) \to \theta,\ d(x_n,Tx_n) \to \theta$.
From condition (3.15), we have

$$d(Tx_n,Ty) \preceq \alpha\,[d(x_n,Tx_n)+d(y,Ty)]$$

for any $n \in \mathbb{N}$. Since P is normal, it follows that

$$d(y,Ty) \preceq \alpha\, d(y,Ty),$$

which gives that $d(y,Ty)=\theta$ i.e., $y=Ty$, a contradiction.
Hence, if $y\neq Ty$, then

$$inf\,\{d(x,y)+d(Tx,y)+d(x,Tx):x\in X\}\succ\theta.$$

So, using Corollary 3.8, we have the desired result. □

Theorem 3.10. *Let (X,d) be a complete cone metric space and q be a c-distance on X. Suppose that the mapping $f:X\to X$ is continuous and satisfies the contractive condition:*

$$q(fx,fy)\preceq a_1\,q(x,fx)+a_2\,q(y,fy) \tag{3.16}$$

for all $x,y\in X$, where $a_1,\ a_2\in[0,1)$ with $a_1+a_2<1$. Then f has a unique fixed point in X.

Proof. We treat q as a generalized c-distance of order 1. Assume that there exists $y\in X$ with $y\neq fy$ and

$$inf\,\{q(x,y)+q(fx,y)+q(x,fx):x\in X\}=\theta.$$

Then, there exists a sequence (x_n) in X such that

$$\lim_{n\to\infty}\{q(x_n,y)+q(fx_n,y)+q(x_n,fx_n)\}=\theta,$$

which implies that $q(x_n,y)\to\theta,\ q(fx_n,y)\to\theta,\ q(x_n,fx_n)\to\theta.$
Since $q(x_n,y)\to\theta$ and $q(x_n,fx_n)\to\theta$, by Lemma 3.1, we have (fx_n) converges to y.
We obtain from condition (3.16) that

$$q(fx_n,f^2x_n)\preceq a_1\,q(x_n,fx_n)+a_2\,q(fx_n,f^2x_n).$$

So, it must be the case that

$$q(fx_n,f^2x_n)\preceq\frac{a_1}{1-a_2}\,q(x_n,fx_n). \tag{3.17}$$

From $(q2)$ and (3.17), we have

$$\begin{aligned} q(x_n,f^2x_n) &\preceq q(x_n,fx_n)+q(fx_n,f^2x_n)\\ &\preceq q(x_n,fx_n)+\frac{a_1}{1-a_2}\,q(x_n,fx_n)\\ &\longrightarrow \theta. \end{aligned}$$

Again, by Lemma 3.1, (f^2x_n) converges to y.
Since f is continuous, we have

$$fy=f(\lim_{n\to\infty}fx_n)=\lim_{n\to\infty}f^2x_n=y,$$

which is a contradiction.
Hence, if $y \neq fy$, then

$$inf\,\{q(x,y)+q(fx,y)+q(x,fx) : x \in X\} \succ \theta.$$

So, by Corollary 3.8, f has a unique fixed point in X. □

We conclude with an example in favour of Theorem 3.2.

Example 3.11. Let $E = \mathbb{R}^2$, the Euclidean plane and $P = \{(x,y) \in \mathbb{R}^2 : x, y \geq 0\}$ a cone in E. Let $X = \{1,3,5,7\}$ and define $d : X \times X \to E$ by

$$d(x,y) = (\alpha \mid x-y \mid,\ \beta \mid x-y \mid)$$

for all $x, y \in X$, where α, β are positive constants. Then (X,d) is a cone metric space. Let $q : X \times X \to E$ be defined by

$$q(1,3) = q(3,1) = (8,8),\ q(1,5) = q(5,1) = q(3,5) = q(5,3) = (2,2),$$

$$q(1,7) = q(7,1) = q(3,7) = q(7,3) = q(5,7) = q(7,5) = (4,4)$$

$$and\ q(x,x) = (0,0)\ for\ every\ x \in X.$$

Then q satisfies condition $(q2)$ for $j = 2$. Also, conditions $(q1)$ and $(q3)$ are immediate. To show $(q4)$, for any $c \in E$ with $\theta \ll c$, put $e = (\frac{1}{2}, \frac{1}{2})$. Then

$$q(z,x) \ll e\ and\ q(z,y) \ll e\ imply\ d(x,y) \ll c.$$

Thus q is a generalized c-distance of order 2 on X but it is not a c-distance on X since it lacks the triangular property:

$$q(1,3) = (8,8) \not\preceq q(1,5) + q(5,3) = (2,2) + (2,2).$$

We define $f, g : X \to X$ by

$$fx = 5,\ for\ all\ x \in X$$

and

$$\begin{aligned} gx &= 5,\ for\ x \in \{1,\ 5,\ 7\} \\ &= 7,\ for\ x = 3. \end{aligned}$$

Then, for every $x, y \in X$ one has

$$q(fx,fy) \preceq r\, q(gx,gy).$$

where $r \in [0,\ 1)$ is a constant. Thus, we have all the conditions of Theorem 3.2 and 5 is the unique common fixed point of f and g in X. □

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