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Weak Baire Spaces

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Abstract. In this paper, we study Baire property of a family of spaces which contains properly the family of all topological spaces and generalize the existing results. Also, we study the images and inverse images of such spaces.

1. Introduction

Let X be a nonempty set and a subfamily $\mathcal{W}$ of $\wp(X)$ is called a *weak structure* [3] on X if $\emptyset \in \mathcal{W}$. Each member of $\mathcal{W}$ is said to be a $\mathcal{W}-$open set and complement of a $\mathcal{W}-$open set is said to be a $\mathcal{W}-$closed set. The pair $(X, \mathcal{W})$ is called a *weak space*. For a subset A of X, $c_{\mathcal{W}}(A) = \cap\{B : A \subset B, X - B \in \mathcal{W}\}$ is called the $\mathcal{W}-$*closure* of A and $i_{\mathcal{W}}(A) = \cup\{U : U \subset A, U \in \mathcal{W}\}$ is called the $\mathcal{W}-$*interior* of A. A subfamily μ of the power set $\wp(X)$ of a nonempty set X is called a *generalized topology* on X [2] if $\emptyset \in \mu$ and μ is closed under arbitrary union. The pair (X, μ) is called a *generalized space*. Elements of μ are called $\mu-$open sets and complement of a $\mu-$open set is a $\mu-$closed set. A subfamily $\mathcal{M}$ of the power set $\wp(X)$ of a nonempty set X is called an *m-structure* on X [1] if $\mathcal{M}$ contains $\emptyset$ and X, and $\mathcal{M}$ is closed under the finite intersection. The pair $(X, \mathcal{M})$ is called an *m-space*. A weak space $(X, \mathcal{W})$ is said to have the property [$\mathcal{I}$] [4] if the intersection of finite elements of $\mathcal{W}$ belongs to $\mathcal{W}$. If $Y \subset X$, then $\{M \cap Y \mid M \in \mathcal{W}\}$ is a weak structure on Y and it is denoted by $\mathcal{W}_Y$. The pair $(Y, \mathcal{W}_Y)$ is called a *weak subspace* of $(X, \mathcal{W})$. A subset A is said to be $\mathcal{W}-dense$ (resp., $\mathcal{W}-rare$) if $c_{\mathcal{W}}(A) = X$ (resp., $i_{\mathcal{W}}c_{\mathcal{W}}(A) = \emptyset$). A subset A of X is said to be $\mathcal{W}^\star-open$ if $i_{\mathcal{W}}(A) = A$ and A is called $\mathcal{W}^\star-closed$ if $c_{\mathcal{W}}(A) = A$. If $P_{\mathcal{W}}$ is the family of all $\mathcal{W}^\star-$open sets of a weak structure, then one can easily prove that $P_{\mathcal{W}}$ is a generalized topology. A subset A in a weak space $(X, \mathcal{W})$ is called a $\mathcal{W} - first\, category\ set$ if A can be written as

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a countable union of $\mathcal{W}-$rare sets. Any set not of $\mathcal{W}-first\,category\,set$ is said to be a $\mathcal{W}-second\,category\,set$. A function $f:(X,\mathcal{W}_X)\to(Y,\mathcal{W}_Y)$ is said to be $\mathcal{W}-open$ if $f(U)\in\mathcal{W}_Y$ for every $U\in\mathcal{W}_X$. f is said to be *almost* $\mathcal{W}-open$ if $f(i_{\mathcal{W}_X}(A))\subset i_{\mathcal{W}_Y}(f(A))$ for all $A\subset X$. Clearly, every $\mathcal{W}-$open function is almost $\mathcal{W}-$open. A function $f:(X,\mathcal{W}_X)\to(Y,\mathcal{W}_Y)$ is said to be $\mathcal{W}-continuous$ if for each $x\in X$ and each $V\in\mathcal{W}_Y$ containing $f(x)$, there exists $U\in\mathcal{W}_X$ containing x such that $f(U)\subset V$. We will make use of the following lemmas either directly or indirectly.

Lemma 1.1. [3, Lemma 2.3] *Let $\mathcal{W}$ be a weak structure on a nonempty set X and $A\subset X$. Then $x\in c_{\mathcal{W}}(A)$ if and only if $U\cap A\neq\emptyset$ for every $U\in\mathcal{W}$ containing x.* □

Lemma 1.2. [3, Theorem 2.1] *Let $\mathcal{W}$ be a weak structure on a nonempty set X. For subsets A and B of X, the following hold.*
(a) *If $A\subset B$, then* $c_{\mathcal{W}}(A)\subset c_{\mathcal{W}}(B)$ and $i_{\mathcal{W}}(A)\subset i_{\mathcal{W}}(B)$.
(b) $A\subset c_{\mathcal{W}}(A)$ and $i_{\mathcal{W}}(A)\subset A$.
(c) $c_{\mathcal{W}}(X-A)=X-i_{\mathcal{W}}(A)$ and $i_{\mathcal{W}}(X-A)=X-c_{\mathcal{W}}(A)$.
(d) $c_{\mathcal{W}}(c_{\mathcal{W}}(A))=c_{\mathcal{W}}(A)$ and $i_{\mathcal{W}}(i_{\mathcal{W}}(A))=i_{\mathcal{W}}(A)$. □

Lemma 1.3. [3, Proposition 2.4] *Let $\mathcal{W}$ be a weak structure on a nonempty set X and $A\subset X$. If $X-A\in\mathcal{W}$, then $c_{\mathcal{W}}(A)=A$* and *if $A\in\mathcal{W}$, then $i_{\mathcal{W}}(A)=A$.* □

Lemma 1.4.[4, Theorem 2.5] *Let $(X,\mathcal{W})$ be a weak space having the* [$\mathcal{I}$] *property and $A\subset X$. Then the following hold.*
(a) $c_{\mathcal{W}}(A\cup B)=c_{\mathcal{W}}(A)\cup c_{\mathcal{W}}(B)$.
(b) $i_{\mathcal{W}}(A\cap B)=i_{\mathcal{W}}(A)\cap i_{\mathcal{W}}(B)$.
(c) $c_{\mathcal{W}}(A)\cap G\subset c_{\mathcal{W}}(A\cap G)$ *for every $G\in\mathcal{W}$ and $A\subset X$.*
(d) $c_{\mathcal{W}}(G)=c_{\mathcal{W}}(A\cap G)$ *for every $G\in\mathcal{W}$ and A is $\mathcal{W}-$dense.* □

Lemma 1.5. *Let $(X,\mathcal{W}_X)$ and $(Y,\mathcal{W}_Y)$ be two weak spaces. For a function $f:(X,\mathcal{W}_X)\to(Y,\mathcal{W}_Y)$, the following are equivalent.*
(a) *f is $\mathcal{W}-$continuous.*
(b) $f^{-1}(V)=i_{\mathcal{W}_X}(f^{-1}(V))$ *for every $V\in\mathcal{W}_Y$.*
(c) $f(c_{\mathcal{W}_X}(A))\subset c_{\mathcal{W}_Y}(f(A))$ *for every subset A of X.*
(d) $c_{\mathcal{W}_X}(f^{-1}(B))\subset f^{-1}(c_{\mathcal{W}_Y}(B))$ *for every subset B of Y.*
(e) $f^{-1}(i_{\mathcal{W}_Y}(B))\subset i_{\mathcal{W}_X}(f^{-1}(B))$ *for every subset B of Y.*
(f) $c_{\mathcal{W}_X}(f^{-1}(B))=f^{-1}(B)$ *for every subset B of Y such that $Y-B\in\mathcal{W}_Y$.*

Proof. (a) ⇒ (b). Let $V\in\mathcal{W}_Y$ and $x\in f^{-1}(V)$. Then $f(x)\in V$. There exists $U\in\mathcal{W}_X$ containing x such that $f(U)\subset V$. Thus, $x\in U\subset f^{-1}(V)$ and so $x\in i_{\mathcal{W}_X}(f^{-1}(V))$. Hence $f^{-1}(V)\subset i_{\mathcal{W}_X}(f^{-1}(V))$. By Lemma 1.2, $i_{\mathcal{W}_X}(f^{-1}(V))\subset f^{-1}(V)$. Therefore, $f^{-1}(V)=i_{\mathcal{W}_X}(f^{-1}(V))$.

(b) ⇒ (c). Let A be a subset of X. Let $x\in c_{\mathcal{W}_X}(A)$ and $V\in\mathcal{W}_Y$ such that $f(x)\in V$. Then $x\in f^{-1}(V)=i_{\mathcal{W}_X}(f^{-1}(V))$. There exists $U\in\mathcal{W}_X$ such that $x\in U\subset f^{-1}(V)$. Since $x\in c_{\mathcal{W}_X}(A)$, $U\cap A\neq\emptyset$ and so $f(U\cap A)\neq\emptyset$. Now $V\cap f(A)\supset f(U)\cap f(A)\supset f(U\cap A)$ implies that $V\cap f(A)\neq\emptyset$. Since $V\in\mathcal{W}_Y$ containing $f(x)$, $f(x)\in c_{\mathcal{W}_Y}(f(A))$ and hence $f(c_{\mathcal{W}_X}(A))\subset c_{\mathcal{W}_Y}(f(A))$.

(c) ⇒ (d). Let B be any subset of Y. Then $f(c_{\mathcal{W}_X}(f^{-1}(B))) \subset c_{\mathcal{W}_Y}(f(f^{-1}(B))) \subset c_{\mathcal{W}_Y}(B)$. Therefore, $c_{\mathcal{W}_X}(f^{-1}(B)) \subset f^{-1}(c_{\mathcal{W}_Y}(B))$.

(d) ⇒ (e). Let B be a subset of Y. Then $X - i_{\mathcal{W}_X}(f^{-1}(B)) = c_{\mathcal{W}_X} - (f^{-1}(Y - B)) \subset f^{-1}(c_{\mathcal{W}_Y}(Y - B)) = f^{-1}(Y - i_{\mathcal{W}_Y}(B)) = X - f^{-1}(i_{\mathcal{W}_Y}(B))$. Therefore, $f^{-1}(i_{\mathcal{W}_Y}(B)) \subset i_{\mathcal{W}_X}(f^{-1}(B))$.

(e) ⇒ (f). Let B be a subset of Y such that $Y - B \in \mathcal{W}_Y$. By (e), we have $X - f^{-1}(B) = f^{-1}(i_{\mathcal{W}_Y}(Y - B)) \subset i_{\mathcal{W}_X}(f^{-1}(Y - B)) = i_{\mathcal{W}_X}(X - f^{-1}(B)) = X - c_{\mathcal{W}_X}(f^{-1}(B))$. Therefore, $c_{\mathcal{W}_X}(f^{-1}(B)) \subset f^{-1}(B) \subset c_{\mathcal{W}_X}(f^{-1}(B))$. Thus, $c_{\mathcal{W}_X}(f^{-1}(B)) = f^{-1}(B)$.

(f) ⇒ (a). Let $x \in X$ and $V \in \mathcal{W}_Y$ containing $f(x)$. By (f), $X - f^{-1}(V) = f^{-1}(Y - V) = c_{\mathcal{W}_X}(f^{-1}(Y - V)) = c_{\mathcal{W}_X}(X - f^{-1}(V)) = X - i_{\mathcal{W}_X}(f^{-1}(V))$. Hence $x \in f^{-1}(V) = i_{\mathcal{W}_X}(f^{-1}(V))$. Therefore, there exists $U \in \mathcal{W}_X$ such that $x \in U \subset f^{-1}(V)$. Thus, U is a $\mathcal{W}_X-$ open set containing x such that $f(U) \subset V$. Hence f is $\mathcal{W}-$continuous. □

2. Weak Baire Spaces

A space $(X, \mathcal{W})$ is called a $\mathcal{W}-$*Baire space* if the intersection of each countable family of $\mathcal{W}-$dense $\mathcal{W}^\star-$open sets is $\mathcal{W}-$dense in X. The following Theorem 2.1 gives the properties of the closure and interior operators of weak subspaces.

Theorem 2.1. *Let $(X, \mathcal{W})$ be a weak space and $(Y, \mathcal{W}_Y)$ be a weak subspace of X. If $A \subset Y$, then the following hold.*
(a) $c_{\mathcal{W}_Y}(A) = c_{\mathcal{W}}(A) \cap Y$.
(b) $i_{\mathcal{W}}(A) \cap Y \subset i_{\mathcal{W}_Y}(A)$.

Proof. (a) Let $y \in c_{\mathcal{W}_Y}(A)$. Then $U \cap A \neq \emptyset$ for every $\mathcal{W}_Y-$open set U containing y which implies that $(B \cap Y) \cap A \neq \emptyset$ where $U = B \cap Y$ and B is $\mathcal{W}-$open in X which in turn implies that $B \cap A \neq \emptyset$ and so $y \in c_{\mathcal{W}}(A)$ which implies that $y \in c_{\mathcal{W}}(A) \cap Y$. Therefore, $c_{\mathcal{W}_Y}(A) \subset c_{\mathcal{W}}(A) \cap Y$. Suppose $y \notin c_{\mathcal{W}_Y}(A)$. Then by Lemma 1.1, there exists $V \in \mathcal{W}_Y$ containing y such that $V \cap A = \emptyset$. Since $V \in \mathcal{W}_Y, V = U \cap Y$ where $U \in \mathcal{W}$. Now $V \cap A = \emptyset$ implies that $(U \cap Y) \cap A = \emptyset$ which in turn implies that $U \cap (Y \cap A) = \emptyset$ and so $U \cap A = \emptyset$. Since $y \in U$, $y \notin c_{\mathcal{W}}(A)$ and so $y \notin c_{\mathcal{W}}(A) \cap Y$. Therefore, $c_{\mathcal{W}}(A) \cap Y \subset c_{\mathcal{W}_Y}(A)$. Hence $c_{\mathcal{W}}(A) \cap Y = c_{\mathcal{W}_Y}(A)$.
(b) Let $y \in i_{\mathcal{W}}(A) \cap Y$. Then $y \in i_{\mathcal{W}}(A)$ which implies that there exists a $\mathcal{W}-$open set U containing y such that $U \subset A$. Since U is $\mathcal{W}-$open, $U \cap Y$ is $\mathcal{W}_Y-$open. Also, $y \in U \cap Y \subset U \subset A$. Thus, there exists a $\mathcal{W}_Y-$open set $U \cap Y$ of y in Y such that $U \cap Y \subset A$. Therefore, $y \in i_{\mathcal{W}_Y}(A)$. Hence $i_{\mathcal{W}}(A) \cap Y \subset i_{\mathcal{W}_Y}(A)$. □

Theorem 2.2. *Let $(X, \mathcal{W})$ be a weak space and $(Y, \mathcal{W}_Y)$ be a weak subspace of $(X, \mathcal{W})$ having the $[\mathcal{I}]-$property. If A is a $\mathcal{W}^\star-$open set, then $A \cap Y$ is a $\mathcal{W}_Y^\star-$open set in Y.*

Proof. Let A be a $\mathcal{W}^\star-$open set in X. Then $i_{\mathcal{W}}(A) = A$. Let $x \in A \cap Y$. Now $x \in A$ implies that $x \in i_{\mathcal{W}}(A)$ which implies that $x \in i_{\mathcal{W}}(A) \cap Y$ and so $x \in i_{\mathcal{W}_Y}(A)$, by Theorem 2.1. Also, it follows that $x \in i_{\mathcal{W}_Y}(Y)$. Therefore, $x \in i_{\mathcal{W}_Y}(A) \cap i_{\mathcal{W}_Y}(Y)$.

Since $\mathcal{W}$ has the $[\mathcal{I}]$ property, $\mathcal{W}_Y$ has the $[\mathcal{I}]$ property and so $x \in i_{\mathcal{W}_Y}(A \cap Y)$, by Lemma 1.4(b). Therefore, $A \cap Y \subset i_{\mathcal{W}_Y}(A \cap Y)$. But always, $i_{\mathcal{W}_Y}(A \cap Y) \subset A \cap Y$. Hence $A \cap Y$ is a $\mathcal{W}^\star-$open set in Y. □

The following Lemma 2.3 is essential to state the properties of first category sets in weak spaces and also to characterize weak spaces which are Baire.

Lemma 2.3. *Let $(X, \mathcal{W})$ be a weak space and $A \subset X$. Then the following hold.*
(a) *If A is a $\mathcal{W}-$rare set, then $X - A$ is a $\mathcal{W}-$dense set.*
(b) *If A is a $\mathcal{W}-$dense, $\mathcal{W}^\star-$open set, then $X - A$ is a $\mathcal{W}-$rare set.*

Proof.(a) Suppose A is $\mathcal{W}-$rare. Then $i_{\mathcal{W}}c_{\mathcal{W}}(A) = \emptyset$. Now $c_{\mathcal{W}}(X - A) = X - i_{\mathcal{W}}(A) \supset X - i_{\mathcal{W}}c_{\mathcal{W}}(A) = X$ and so $X - A$ is $\mathcal{W}-$dense.
(b) Suppose A is a $\mathcal{W}-$dense $\mathcal{W}^\star-$open set. Then $c_{\mathcal{W}}(A) = X$ and $i_{\mathcal{W}}(A) = A$. Now $i_{\mathcal{W}}c_{\mathcal{W}}(X-A) = i_{\mathcal{W}}(X-i_{\mathcal{W}}(A)) = i_{\mathcal{W}}(X-A) = X-c_{\mathcal{W}}(A) = \emptyset$ and so $X-A$ is $\mathcal{W}-$rare. □

The following Example 2.4 shows that the converse of Lemma 2.3(a) need not be true and the condition $\mathcal{W}^\star-$open on A cannot be dropped in Lemma 2.3(b).

Example 2.4. *Let $X = \{a, b, c, d\}$ and $\mathcal{W} = \{\emptyset, \{a\}, \{d\}, \{b, c\}, \{a, c\}\}$. Then $(X, \mathcal{W})$ is a weak space. If $A = \{a, b, d\}$, then $c_{\mathcal{W}}(A) = X$ but $i_{\mathcal{W}}c_{\mathcal{W}}(X - A) = i_{\mathcal{W}}(\{b, c\}) = \{b, c\}$. Thus, A is $\mathcal{W}-$dense but $X - A$ is not $\mathcal{W}-$rare. Also, $i_{\mathcal{W}}(A) = \{a, d\} \neq A$ and so A is not $\mathcal{W}^\star-$open.* □

The following Theorem 2.5 gives the property of $\mathcal{W}-$second category sets. Theorem 2.7 below shows that every $\mathcal{W}-$Baire space is of $\mathcal{W}-$second category.

Theorem 2.5. *Let $(X, \mathcal{W})$ be a weak space. Then the following hold.*
(a) *X is of $\mathcal{W}-$second category, if the intersection of every countable family of $\mathcal{W}-$dense sets in X is nonempty.*
(b) *If X is of $\mathcal{W}-$second category, then the intersection of every countable family of $\mathcal{W}-$dense $\mathcal{W}^\star-$open sets in X is nonempty.*

Proof. Suppose that the intersection of every countable family of $\mathcal{W}-$dense sets in X is nonempty. If $X = \cup\{A_n \mid n \in \mathbf{N}\}$, where each A_n is a $\mathcal{W}-$rare set and $\mathbf{N}$ is the set of all natural numbers, then $\emptyset = X - \cup\{A_n \mid n \in \mathbf{N}\} = \cap\{X - A_n \mid n \in \mathbf{N}\}$, a countable intersection of $\mathcal{W}-$dense sets, by Lemma 2.3(a), a contradiction to the hypothesis. Thus, $X \neq \cup\{A_n \mid n \in \mathbf{N}\}$ for any sequence of $\mathcal{W}-$rare sets A_n. Hence X is of $\mathcal{W}-$second category.
(b) Let $\{G_i \mid i \in \mathbf{N}\}$ be a countable family of $\mathcal{W}-$dense $\mathcal{W}^\star-$open sets in a weak space $(X, \mathcal{W})$. Then $\{X - G_i \mid i \in \mathbf{N}\}$ is a countable family of $\mathcal{W}-$rare sets, by Lemma 2.3(b) and so $\cup\{X - G_i \mid i \in \mathbf{N}\}$ is of $\mathcal{W}-$first category. Since X is of $\mathcal{W}-$second category, $X \neq \cup\{X - G_i \mid i \in \mathbf{N}\}$ and so $X \neq X - \cap\{G_i \mid i \in \mathbf{N}\}$. Therefore, $\cap\{G_i \mid i \in \mathbf{N}\} \neq \emptyset$. □

Corollary 2.6. *Let $(X, \mathcal{W})$ be a weak space. Then X is of $\mathcal{W}-$second category in itself if and only if the intersection of every countable family of $\mathcal{W}-$dense $\mathcal{W}^\star-$open sets in X is nonempty.* □

Theorem 2.7. *Every $\mathcal{W}-$Baire space $(X, \mathcal{W})$, $\mathcal{W} \neq \{\emptyset\}$, is of $\mathcal{W}-$second category.*

Proof. Let $\{A_n \mid n \in \mathbf{N}\}$ be a countable collection of $\mathcal{W}-$dense $\mathcal{W}^\star-$open sets in X. Since X is a $\mathcal{W}-$Baire space, $c_\mathcal{W}(\cap\{A_n \mid n \in \mathbf{N}\}) = X$ and so $\cap\{A_n \mid n \in \mathbf{N}\} \neq \emptyset$. Therefore, X is of $\mathcal{W}-$second category, by Theorem 2.5(a). □

The following Theorem 2.8 characterizes $\mathcal{W}-$Baire spaces in terms of $\mathcal{W}^\star-$closed sets. Theorem 2.9 below shows that every $\mathcal{W}-$open subspace of a $\mathcal{W}-$Baire space is a $\mathcal{W}-$Baire.

Theorem 2.8. *A weak space $(X, \mathcal{W})$ is a $\mathcal{W}-$Baire space if and only if any countable collection $\{A_n\}$ of $\mathcal{W}^\star-$closed sets, each of them has empty $\mathcal{W}-$interior, then their union $\cup_n A_n$ also has empty $\mathcal{W}-$interior.*

Proof. Let $\{A_n \mid n \in \mathbf{N}\}$ be a countable collection of $\mathcal{W}^\star-$closed subsets of X having empty $\mathcal{W}-$interior in X. If $B_n = X - A_n$, then for each $n \in \mathbf{N}$, B_n is $\mathcal{W}^\star-$open and $c_\mathcal{W}(B_n) = c_\mathcal{W}(X - A_n) = X - i_\mathcal{W}(A_n) = X$ so that B_n is $\mathcal{W}-$dense. Thus, $\{B_n \mid n \in \mathbf{N}\}$ is a countable family of $\mathcal{W}-$dense $\mathcal{W}^\star-$open sets of X. Since X is $\mathcal{W}-$Baire, $\cap\{B_n \mid n \in \mathbf{N}\}$ is $\mathcal{W}-$dense and so $c_\mathcal{W}(\cap\{B_n \mid n \in \mathbf{N}\}) = X$ which implies that $X - c_\mathcal{W}(\cap\{B_n \mid n \in \mathbf{N}\}) = \emptyset$ and so $i_\mathcal{W}(X - \cap\{B_n \mid n \in \mathbf{N}\}) = \emptyset$ which implies that $i_\mathcal{W}(\cup\{X - B_n \mid n \in \mathbf{N}\}) = \emptyset$ and so $i_\mathcal{W}(\cup\{A_n \mid n \in \mathbf{N}\}) = \emptyset$. Conversely, suppose the condition holds. Let $\{U_n \mid n \in \mathbf{N}\}$ be a countable family of $\mathcal{W}-$dense $\mathcal{W}^\star-$open sets in X. Then $\{X - U_n \mid n \in \mathbf{N}\}$ is a countable family of $\mathcal{W}^\star-$closed sets with empty $\mathcal{W}-$interior. By hypothesis, $i_\mathcal{W}(\cup\{X - U_n \mid n \in \mathbf{N}\}) = \emptyset$. Now $c_\mathcal{W}(\cap\{U_n \mid n \in \mathbf{N}\}) = c_\mathcal{W}(\cap\{X - (X - U_n) \mid n \in \mathbf{N}\}) = c_\mathcal{W}(X - \cup\{X - U_n \mid n \in \mathbf{N}\}) = X - i_\mathcal{W}(\cup\{X - U_n \mid n \in \mathbf{N}\}) = X - \emptyset = X$. Hence $\cap\{U_n \mid n \in \mathbf{N}\}$ is $\mathcal{W}-$dense in X and so X is $\mathcal{W}-$Baire. □

Theorem 2.9. *Let $(X, \mathcal{W})$ be a $\mathcal{W}-$Baire space having the $[\mathcal{I}]-$property. Then every nonempty $\mathcal{W}-$open subspace Y of X is itself a $\mathcal{W}-$Baire space.*

Proof. Let $\{A_n\}$ be a countable collection of $\mathcal{W}^\star-$closed subsets of Y having empty $\mathcal{W}-$interior in Y. Then the set $c_\mathcal{W}(A_n)$ is $\mathcal{W}^\star-$closed subset of X that has empty $\mathcal{W}-$interior in X. For, if U is a nonempty $\mathcal{W}-$open set in X contained in $c_\mathcal{W}(A_n)$, then U must intersect A_n and $U \cap Y \subset c_\mathcal{W}(A_n) \cap Y = c_{\mathcal{W}_Y}(A_n) = A_n$, by Theorem 2.1. That is, $U \cap Y$ is a nonempty $\mathcal{W}-$open subset of Y contained in A_n, which is a contradiction to the fact that $i_{\mathcal{W}_Y}(A_n) = \emptyset$ for all n. Suppose $W \subset \cup_n A_n$ where W is a nonempty $\mathcal{W}-$open set in Y. Now $W \subset \cup_n A_n$ implies that $W \subset \cup_n c_\mathcal{W}(A_n)$. Since W is $\mathcal{W}-$open in Y, there is a $\mathcal{W}-$open set V in X such that $W = V \cap Y$. Since $(X, \mathcal{W})$ has the $[\mathcal{I}]-$property, by Lemma 1.4, W is $\mathcal{W}^\star-$open in X and so $i_\mathcal{W}(\cup_n c_\mathcal{W}(A_n)) \neq \emptyset$, which is a contradiction to the fact that X is a $\mathcal{W}-$Baire space, by Theorem 2.8. Therefore, $i_{\mathcal{W}_Y}(\cup_n A_n) = \emptyset$. Hence Y is a $\mathcal{W}-$Baire space. □

The following Example 2.10 shows that the condition $\mathcal{W}-$open cannot be dropped in Theorem 2.9.

Example 2.10. *Consider the weak space* $(X, \mathcal{W})$ *where* $X = \mathbf{N}$, *the set of all natural numbers and* $\mathcal{W} = \{\emptyset, \{1\}, \{2\}, X\}$. *Then* $(X, \mathcal{W})$ *is a* $\mathcal{W}-$*Baire space. If* $A = \{3, 4, \ldots, n, \ldots\}$, *then* A *is not* $\mathcal{W}-$*open. But* $A = \cup\{A_n \mid n \in \mathbf{N}\}$ *where* $A_n = \{n+2\}$, $n \in \mathbf{N}$. *Now* $i_{\mathcal{W}_A} c_{\mathcal{W}_A}(A_n) = i_{\mathcal{W}_A}(\{3, 4, \ldots, n, \ldots\}) = \emptyset$ *and so* A_n *is* $\mathcal{W}-$*rare in* A *for every* $n \in \mathbf{N}$. *Thus,* A *is a set of* $\mathcal{W}-$*first category so that* A *is not of* $\mathcal{W}-$*second category. Hence* A *is not* $\mathcal{W}-$*Baire, by Theorem 2.7.*

The following Theorem 2.11 and Theorem 2.12 characterize $\mathcal{W}-$Baire spaces in terms of $\mathcal{W}-$open sets.

Theorem 2.11. *Let* $(X, \mathcal{W})$ *be a weak space having the* $[\mathcal{I}]-$*property. Then* X *is a* $\mathcal{W}-$*Baire space if and only if every point of* X *has a* $\mathcal{W}-$*open set which is a* $\mathcal{W}-$*Baire space.*

Proof. Let $\{G_n \mid n \in \mathbf{N}\}$ be a countable collection of $\mathcal{W}-$dense $\mathcal{W}^\star-$open sets in X. Let $x \in X$ and U be a $\mathcal{W}-$open subset of X containing x. By Lemma 1.4(d), for each n, $c_\mathcal{W}(U) = c_\mathcal{W}(U \cap G_n)$ and $c_{\mathcal{W}_U}(U \cap G_n) = U \cap c_\mathcal{W}(U \cap G_n)$ by Theorem 2.1(a). Therefore, for each n, $c_{\mathcal{W}_U}(U \cap G_n) = U \cap c_\mathcal{W}(U) = U$ and so $U \cap G_n$ is $\mathcal{W}-$dense in U. Now $i_{\mathcal{W}_U}(U \cap G_n) \supset U \cap i_\mathcal{W}(U \cap G_n) = U \cap i_\mathcal{W}(U) \cap i_\mathcal{W}(G_n) = U \cap G_n$ and so $i_{\mathcal{W}_U}(U \cap G_n) = U \cap G_n$ so that $U \cap G_n$ is $\mathcal{W}^\star-$open. Also, by hypothesis, U is a $\mathcal{W}-$Baire space and so U is of $\mathcal{W}-$second category by Theorem 2.7. Thus, $\cap\{U \cap G_n \mid n \in \mathbf{N}\} \neq \emptyset$, by Theorem 2.5(b) and so $U \cap (\cap\{G_n \mid n \in \mathbf{N}\}) \neq \emptyset$. Therefore, $x \in c_\mathcal{W}(\cap\{G_n \mid n \in \mathbf{N}\})$ which implies that $\cap\{G_n \mid n \in \mathbf{N}\}$ is $\mathcal{W}-$dense in X. Hence $(X, \mathcal{W})$ is a $\mathcal{W}-$Baire space. Conversely, suppose X is a $\mathcal{W}-$Baire space. Let $x \in X$ and U be a $\mathcal{W}-$open set containing x. Then by Theorem 2.9, U is a $\mathcal{W}-$Baire space. Therefore, every point has a $\mathcal{W}-$open set which is a $\mathcal{W}-$Baire space.

Theorem 2.12. *Let* $(X, \mathcal{W})$ *be a weak space having the* $[\mathcal{I}]-$*property with* $\mathcal{W} \neq \{\emptyset\}$. *Then* X *is a* $\mathcal{W}-$*Baire space if and only if every nonempty* $\mathcal{W}-$*open set is of* $\mathcal{W}-$*second category.*

Proof. Let $\{G_n \mid n \in \mathbf{N}\}$ be a countable collection of $\mathcal{W}-$dense $\mathcal{W}^\star-$open sets in X. Let $x \in X$ and $x \notin c_\mathcal{W}(\cap\{G_n \mid n \in \mathbf{N}\})$. Then there exists a $\mathcal{W}-$open set U containing x such that $U \cap (\cap\{G_n \mid n \in \mathbf{N}\}) = \emptyset$ which implies that $\cap\{U \cap G_n \mid n \in \mathbf{N}\} = \emptyset$. Now U is $\mathcal{W}-$open and G_n is $\mathcal{W}-$dense in X for each n implies that $U \cap G_n$ is $\mathcal{W}-$dense in U, by Theorem 2.1(a). Also, $i_{\mathcal{W}_U}(U \cap G_n) \supset U \cap i_\mathcal{W}(U \cap G_n) = U \cap i_\mathcal{W}(U) \cap i_\mathcal{W}(G_n) = U \cap G_n$ and so $U \cap G_n$ is $\mathcal{W}^\star-$open. Hence by hypothesis, U is of $\mathcal{W}-$second category and so $\cap\{U \cap G_n \mid n \in \mathbf{N}\} \neq \emptyset$, a contradiction. Therefore, $x \in c_\mathcal{W}(\cap\{G_n \mid n \in \mathbf{N}\})$ and so $\cap\{G_n \mid n \in \mathbf{N}\}$ is $\mathcal{W}-$dense. Hence X is a $\mathcal{W}-$Baire space. Conversely, suppose X is a $\mathcal{W}-$Baire space and G is a nonempty $\mathcal{W}-$open set in X. Then by Theorem 2.9, G is a $\mathcal{W}-$Baire space and hence G is of $\mathcal{W}-$second category, by Theorem 2.7. □

3. Images of Weak Baire Spaces

Let $(X, \mathcal{W}_X)$ and $(Y, \mathcal{W}_Y)$ be two weak spaces. A mapping $f : X \to Y$ is said to be *feebly* $\mathcal{W}-$ *continuous* if $i_{\mathcal{W}_X}(f^{-1}(B)) \neq \emptyset$ for every $B \subset Y$ with $i_{\mathcal{W}_Y}(B) \neq \emptyset$ and f is said to be *feebly* $\mathcal{W}-$ *open* if $i_{\mathcal{W}_Y}(f(A)) \neq \emptyset$ for every $A \subset X$ with $i_{\mathcal{W}_X}(A) \neq \emptyset$. Let (X, m_X) and (Y, m_Y) be two minimal spaces. The following Theorem 3.1 characterizes feebly $\mathcal{W}-$continuous functions. Theorem 3.2 below gives a characterization of feebly $\mathcal{W}-$open functions.

Theorem 3.1. *Let $(X, \mathcal{W}_X)$ and $(Y, \mathcal{W}_Y)$ be two weak spaces. A mapping $f : (X, \mathcal{W}_X) \to (Y, \mathcal{W}_Y)$ is feebly $\mathcal{W}-$continuous if and only if A is $\mathcal{W}-$dense in X implies $f(A)$ is $\mathcal{W}-$dense in Y.*

Proof. Suppose that A is $\mathcal{W}-$dense in X implies $f(A)$ is $\mathcal{W}-$dense in Y. Let B be a nonempty subset of Y such that $i_{\mathcal{W}_Y}(B) \neq \emptyset$. Suppose that $i_{\mathcal{W}_X}(f^{-1}(B)) = \emptyset$. Then $X = X - i_{\mathcal{W}_X}(f^{-1}(B)) = c_{\mathcal{W}_X}(X - f^{-1}(B))$ and so $X - f^{-1}(B)$ is $\mathcal{W}-$dense in X. By hypothesis, $f(X - f^{-1}(B))$ is $\mathcal{W}-$dense in Y and so $Y = c_{\mathcal{W}_Y}(f(X - f^{-1}(B))) = c_{\mathcal{W}_Y}(f(f^{-1}(Y) - f^{-1}(B))) = c_{\mathcal{W}_Y}(f(f^{-1}(Y - B))) \subset c_{\mathcal{W}_Y}(Y - B) = Y - i_{\mathcal{W}_Y}(B)$ which implies that $i_{\mathcal{W}_Y}(B) = \emptyset$, a contradiction. Therefore, $i_{\mathcal{W}_X}(f^{-1}(B)) \neq \emptyset$ which implies that f is feebly $\mathcal{W}-$continuous. Conversely, suppose f is feebly $\mathcal{W}-$continuous and A is $\mathcal{W}-$dense in X. Since A is $\mathcal{W}-$dense in X, $X - c_{\mathcal{W}_X}(A) = \emptyset$ which implies that $i_{\mathcal{W}_X}(X - A) = \emptyset$ which in turn implies that $i_{\mathcal{W}_X}(X - f^{-1}(f(A))) = \emptyset$ and so $i_{\mathcal{W}_X}(f^{-1}(Y) - f^{-1}f(A)) = \emptyset$. Thus, $i_{\mathcal{W}_X}(f^{-1}(Y - f(A))) = \emptyset$ and so by hypothesis, $i_{\mathcal{W}_Y}(Y - f(A)) = \emptyset$. Hence $Y - c_{\mathcal{W}_Y}(f(A)) = \emptyset$ and so $f(A)$ is $\mathcal{W}-$dense in Y. □

Theorem 3.2. *Let $(X, \mathcal{W}_X)$ and $(Y, \mathcal{W}_Y)$ be two weak spaces. A mapping $f : (X, \mathcal{W}_X) \to (Y, \mathcal{W}_Y)$ is feebly $\mathcal{W}-$open if and only if A is $\mathcal{W}-$dense in Y implies $f^{-1}(A)$ is $\mathcal{W}-$dense in X.*

Proof. Suppose that A is $\mathcal{W}-$dense in Y implies $f^{-1}(A)$ is $\mathcal{W}-$dense in X. Let $B \subset X$ such that $i_{\mathcal{W}_X}(B) \neq \emptyset$. If $i_{\mathcal{W}_Y}(f(B)) = \emptyset$, then $Y - i_{\mathcal{W}_Y}(f(B)) = Y$ which implies that $c_{\mathcal{W}_Y}(Y - f(B)) = Y$ and so by hypothesis $c_{\mathcal{W}_X}(f^{-1}(Y - f(B))) = X$. Now $c_{\mathcal{W}_X}(f^{-1}(Y - f(B))) = X$ implies that $c_{\mathcal{W}_X}(f^{-1}(Y) - f^{-1}(f(B))) = X$ which in turn implies that $c_{\mathcal{W}_X}(X - B) = X$ so that $X - i_{\mathcal{W}_X}(B) = X$ and hence $i_{\mathcal{W}_X}(B) = \emptyset$, a contradiction. Therefore, $i_{\mathcal{W}_Y}(f(B)) \neq \emptyset$. Conversely, suppose that f is feebly $\mathcal{W}-$open and A is $\mathcal{W}-$dense in Y. Then $c_{\mathcal{W}_Y}(A) = Y$ and so $i_{\mathcal{W}_Y}(Y - A) = \emptyset$ which implies that $i_{\mathcal{W}_Y}(f(f^{-1}(Y - A))) = \emptyset$ and so $i_{\mathcal{W}_X}(f^{-1}(Y - A)) = \emptyset$, by assumption. Now $i_{\mathcal{W}_X}(f^{-1}(Y - A)) = i_{\mathcal{W}_X}(f^{-1}(Y) - f^{-1}(A)) = i_{\mathcal{W}_X}(X - f^{-1}(A)) = X - c_{\mathcal{W}_X}(f^{-1}(A))$ and so $c_{\mathcal{W}_X}(f^{-1}(A)) = X$. Therefore, $f^{-1}(A)$ is $\mathcal{W}-$dense in X. □

The following Theorem 3.3 shows that every $\mathcal{W}-$continuous onto function is feebly $\mathcal{W}-$continuous. But the converse of Theorem 3.3 is not true as shown by Example 3.4.

Theorem 3.3. *Every $\mathcal{W}-$continuous function defined on a space $(X, \mathcal{W}_X)$ onto a space $(Y, \mathcal{W}_Y)$ is feebly $\mathcal{W}-$continuous.*

Proof. Let $A \subset Y$ and $i_{\mathcal{W}_Y}(A) \neq \emptyset$. Since f is $\mathcal{W}-$continuous, by Lemma 1.5, $f^{-1}(i_{\mathcal{W}_Y}(A)) \subset i_{\mathcal{W}_X}(f^{-1}(A))$. Since f is onto, $f^{-1}(i_{\mathcal{W}_Y}(A)) \neq \emptyset$ and so $i_{\mathcal{W}_X}(f^{-1}(A)) \neq \emptyset$. Therefore, f is feebly $\mathcal{W}-$continuous. □

Example 3.4. *Consider the weak spaces* $(X, \mathcal{W}_X)$ *and* $(Y, \mathcal{W}_Y)$ *where* $X = \{a, b, c, d\}$, $Y = \{0, 1, 2\}$, $\mathcal{W}_X = \{\emptyset, \{a\}, \{c\}, \{c, d\}\}$ *and* $\mathcal{W}_Y = \{\emptyset, \{0\}, \{1\}\}$. *Define* $f : (X, \mathcal{W}_X) \to (Y, \mathcal{W}_Y)$ *by* $f(a) = 0, f(b) = 0, f(c) = 1, f(d) = 2$. *Here the function* f *is feebly* $\mathcal{W}-$*continuous. But* $f^{-1}(i_{\mathcal{W}_Y}(\{0\})) \not\subset i_{\mathcal{W}_X}(f^{-1}(\{0\}))$. *Therefore,* f *is not* $\mathcal{W}-$*continuous, by Lemma 1.5.*

Theorem 3.5 shows that every almost $\mathcal{W}-$open function is feebly $\mathcal{W}-$open. Example 3.6 below shows that the converse of Theorem 3.5 need not be true in general.

Theorem 3.5. *Let* $(X, \mathcal{W}_X)$ *and* $(Y, \mathcal{W}_Y)$ *be two weak spaces. Then every almost* $\mathcal{W}-$*open function* $f : (X, \mathcal{W}_X) \to (Y, \mathcal{W}_Y)$ *is feebly* $\mathcal{W}-$*open.*

Proof. Suppose that $A \subset X$ and $i_{\mathcal{W}_X}(A) \neq \emptyset$. Since f is almost $\mathcal{W}-$open, $f(i_{\mathcal{W}_X}(A)) \subset i_{\mathcal{W}_Y}(f(A))$. Since $i_{\mathcal{W}_X}(A) \neq \emptyset$, $f(i_{\mathcal{W}_X}(A)) \neq \emptyset$ so that $i_{\mathcal{W}_Y}(f(A)) \neq \emptyset$. Therefore, f is feebly $\mathcal{W}-$open. □

Example 3.6. *Consider the spaces defined in Example 3.1. Here* f *is feebly* $\mathcal{W}-$*open. Also,* $\{c, d\}$ *is* $\mathcal{W}-$*open in X, but* $f(\{c, d\}) = \{1, 2\}$ *is not* $\mathcal{W}-$*open in Y. Therefore,* f *is not* $\mathcal{W}-$*almost open.*

Theorem 3.7. *Let* $(X, \mathcal{W}_X)$ *and* $(Y, \mathcal{W}_Y)$ *be two weak spaces and* $f : (X, \mathcal{W}_X) \to (Y, \mathcal{W}_Y)$ *be feebly* $\mathcal{W}-$*continuous such that for each* $\mathcal{W}-$*rare set E in Y,* $f^{-1}(E)$ *is* $\mathcal{W}-$*rare in X. If X is a* $\mathcal{W}-$*Baire space, then Y is a* $\mathcal{W}-$*Baire space.*

Proof. Suppose that X is a $\mathcal{W}-$Baire space. Let $\{U_n \mid n \in \mathbf{N}\}$ be a sequence of $\mathcal{W}-$dense $\mathcal{W}^\star-$open subsets of Y. For each n, let $Z_n = i_{\mathcal{W}_X}(f^{-1}(U_n))$. Since each U_n is $\mathcal{W}-$dense $\mathcal{W}^\star-$open in Y, by Lemma 2.3(a), $Y - U_n$ is $\mathcal{W}-$rare in Y and so $f^{-1}(Y - U_n)$ is $\mathcal{W}-$rare in X. For each n, $Z_n = X - c_{\mathcal{W}_X}(X - f^{-1}(U_n)) = X - c_{\mathcal{W}_X}(f^{-1}(Y) - f^{-1}(U_n)) = X - c_{\mathcal{W}_X}(f^{-1}(Y - U_n))$ and $f^{-1}(Y - U_n)$ is $\mathcal{W}-$rare in X, Z_n is a $\mathcal{W}-$dense $\mathcal{W}^\star-$open set. Since X is a $\mathcal{W}-$Baire space, $\cap\{Z_n \mid n \in \mathbf{N}\}$ is $\mathcal{W}-$dense in X. By Theorem 3.1, since f is feebly $\mathcal{W}-$continuous, the set $f(\cap\{Z_n \mid n \in \mathbf{N}\})$ is $\mathcal{W}-$dense in Y. Now $i_{\mathcal{W}_X}(f^{-1}(U_n)) \subset f^{-1}(U_n)$ for all n implies that $\cap\{i_{\mathcal{W}_X}(f^{-1}(U_n)) \mid n \in \mathbf{N}\} \subset f^{-1}(U_n)$ for all n which implies that $f(\cap\{i_{\mathcal{W}_X}(f^{-1}(U_n)) \mid n \in \mathbf{N}\}) \subset f(f^{-1}(U_n))$ for all n which in turn implies that $f(\cap\{i_{\mathcal{W}_X}(f^{-1}(U_n)) \mid n \in \mathbf{N}\}) \subset U_n$ for all n and so $f(\cap\{i_{\mathcal{W}_X}(f^{-1}(U_n)) \mid n \in \mathbf{N}\}) \subset \cap\{U_n \mid n \in \mathbf{N}\}$. Hence $f(\cap\{Z_n \mid n \in \mathbf{N}\}) \subset \cap\{U_n \mid n \in \mathbf{N}\}$. Since $f(\cap\{Z_n \mid n \in \mathbf{N}\})$ is $\mathcal{W}-$dense in Y, $\cap\{U_n \mid n \in \mathbf{N}\}$ is $\mathcal{W}-$dense in Y. Therefore, Y is a $\mathcal{W}-$Baire space. □

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