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On the f-biharmonic Maps and Submanifolds

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Abstract. In this paper, we prove that every f-biharmonic map from a complete Riemannian manifold into a Riemannian manifold with non-positive sectional curvature,satisfying some condition, is f-harmonic. Also we present some properties for the f-biharmonicity of submanifolds of $\mathbb{S}^n$, and we give the classification of f-biharmonic curves in 3-dimensional sphere.

1. Introduction

Let $f : (M^m, g) \to (0, +\infty)$ be a smooth function. An f-harmonic map is a map $\varphi : (M^m, g) \to (N^n, h)$ between two Riemannian manifolds that is a critical point of the f-energy

$$E_f(\varphi; D) = \frac{1}{2}\int_D f|d\varphi|^2\, v_g,$$

for any compact domain D, where v_g is the volume element (see [2],[5],[10]).

The f-tension field of φ is given by

$$\tau_f(\varphi) = \operatorname{trace}_g \nabla f\, d\varphi = f\,\tau(\varphi) + d\varphi(\operatorname{grad}^M f).$$

The f-tension field of φ vanishes ($\tau_f(\varphi) = 0$) means that φ is a harmonic map.

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For any compact domain $D \subseteq M$, the f-bienergy is defined by

$$E_{2,f}(\varphi; D) = \frac{1}{2}\int_D |\tau_f(\varphi)|^2 \, v_g, \tag{1.1}$$

φ is called f-biharmonic if φ is a critical point of the f-bienergy functional for any compact domain D. The Euler-Lagrange equation associated to the f-bienergy functional is given by

$$\begin{aligned} \tau_{2,f}(\varphi) \;\equiv\; & -f \operatorname{trace}_g R^N(\tau_f(\varphi), d\varphi)d\varphi \\ & - \operatorname{trace}_g \left(\nabla^{\varphi} f(\nabla^{\varphi}\tau_f(\varphi)) - f\nabla^{\varphi}_{\nabla^M}\tau_f(\varphi)\right) = 0, \end{aligned} \tag{1.2}$$

for an orthonormal frame $\{e_1, ..., e_m\}$, we have

$$\operatorname{trace}_g R^N(\tau_f(\varphi), d\varphi)d\varphi = \sum_{i=1}^{m} R^N(\tau_f(\varphi), d\varphi(e_i))d\varphi(e_i),$$

$$\operatorname{trace}_g \left(\nabla^{\varphi} f\nabla^{\varphi}\tau_f(\varphi) - f\nabla^{\varphi}_{\nabla^M}\tau_f(\varphi)\right) = \sum_{i=1}^{m} \left\{\nabla^{\varphi}_{e_i} f(\nabla^{\varphi}_{e_i}\tau_f(\varphi)) - f\nabla^{\varphi}_{\nabla^M_{e_i} e_i}\tau_f(\varphi)\right\}.$$

$\tau_{2,f}(\varphi)$ is called the f-bitension field of φ. The solutions of the equation (1.2) are the biharmonic maps ([5],[10]).

The f-harmonic and f-biharmonic concept is a natural generalization of harmonic maps (Eells and Sampson [7]), p-harmonic and exponentially harmonic maps (Eells [6]) and biharmonic maps (Jiang [8]). In mathematical physics, f-harmonic maps, relate to the equations of the motion of a continuous system of spins ([4]) and the gradient Ricci-soliton structure ([11]).

In this paper, we prove that every f-biharmonic map from a complete Riemannian manifold into a Riemannian manifold with non-positive sectional curvature, satisfies some condition, is f-harmonic (Theorem 2.3). Also we present some properties for the f-biharmonicity of submanifolds of $\mathbb{S}^n$ (Theorem 3.1), and we give the classification of f-biharmonic curves in 3-dimensional sphere (Theorem 4.1).

2. Some Results on f-harmonic and f-biharmonic Maps

Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth immersed map between two Riemannian manifolds. Then $\varphi^{-1}(TN)$ is decomposed into direct sum $\mathcal{T}(M) \oplus \mathcal{N}(M)$ of tangent fiber $\mathcal{T}(M)$ and normal fiber $\mathcal{N}(M)$. If $f \in C^\infty(M)$ be a smooth positive function, then

$$(\nabla^{\varphi}_X f d\varphi)(Y) = \nabla^{\varphi}_X(f d\varphi(Y)) - f d\varphi(\nabla^M_X Y) = B_f(X, Y) \tag{2.1}$$

for all vector fields $X, Y \in \Gamma(TM)$. For any orthonormal frame $\{e_1, ..., e_m\}$ on M, we have

$$\tau_f(\varphi) = \sum_{i=1}^{m} B_f(e_i, e_i) \in \mathcal{N}(M). \tag{2.2}$$

Proposition 2.1. *For any vector fields $\xi \in \mathcal{N}(M)$ and $X, Y \in \Gamma(TM)$, we have*

$$h(B_f(X,Y),\xi) = fh(A_\xi(X), d\varphi(Y)) \tag{2.3}$$

where A_ξ denote the shape operator with respect to ξ defined by:

$$\nabla^N_X \xi = -A_\xi(X) + \nabla^\perp_X \xi. \tag{2.4}$$

Proof. Since $\xi \in \mathcal{N}(M)$ and $d\varphi(Y) \in \mathcal{T}(M)$, then we have:

$$h(\xi, fd\varphi(Y)) = 0.$$

Whence

$$\begin{aligned}
0 &= \nabla_X h(\xi, fd\varphi(Y)) \\
&= h(\nabla^N_X \xi, fd\varphi(Y)) + h(\xi, \nabla^\varphi_X fd\varphi(Y)) \\
&= h(\nabla^N_X \xi, fd\varphi(Y)) + h(\xi, \nabla^\varphi_X fd\varphi(Y) + fd\varphi(\nabla^M_X Y)) \\
&= h(-A_\xi(X) + \nabla^\perp_X \xi, fd\varphi(Y)) + h(\xi, B_f(X,Y)) \\
&= -h(A_\xi(X), fd\varphi(Y)) + h(\xi, B_f(X,Y))
\end{aligned}$$

□

Proposition 2.2. *If $\nabla^N_X \tau_f(\varphi) = 0$ for any vector field $X \in \Gamma(TM)$, then $\tau_f(\varphi) = 0$.*

Proof. From the formula (2.4) with $\xi = \tau_f(\varphi)$, we have $A_{\tau_f(\varphi)}(X) = 0$.

By the Proposition 2.1, we obtain:

$$h(B_f(X,Y), \tau_f(\varphi)) = fh(A_{\tau_f(\varphi)}(X), d\varphi(Y)) = 0,$$

and

$$|\tau_f(\varphi)|^2 = \sum_{i=1}^m h(B_f(e_i, e_i), \tau_f(\varphi) = 0,$$

where $X, Y \in \Gamma(TM)$ and $\{e_1, ..., e_m\}$ is an orthonormal frame on M. □

Theorem 2.3. *Let (M^m, g) be a complete Riemannian manifold with infinite volume and (N^n, h) be a Riemannian manifold with non-positive sectional curvature. Let f be a smooth positive function on M. If $\varphi : (M^m, g) \to (N^n, h)$ is an f-biharmonic map with finite f-bienergy, satisfying:*

$$\operatorname{trace}_g \nabla^\varphi f \nabla^\varphi - f \operatorname{trace}_g \nabla^\varphi \nabla^\varphi \leq 0 \tag{2.5}$$

then φ is f-harmonic.

The inequality (2.5) means that for any vector field $X \in \Gamma(\varphi^{-1}(TN))$ and any orthonormal frame $(e_i)_i$ on M, we have:

$$\sum_{i=1}^{m} h(\nabla^{\varphi}_{e_i} f \nabla^{\varphi}_{e_i} X, X) - f h(\nabla^{\varphi}_{e_i} \nabla^{\varphi}_{e_i} X, X) = \sum_{i=1}^{m} h(e_i(f) \nabla^{\varphi}_{e_i} X, X) \leq 0.$$

Proof. Assume that $\varphi : (M^m, g) \to (N^n, h)$ is f-biharmonic. Fix a point $x \in M$ and let $\{e_1, ..., e_m\}$ be an orthonormal frame with respect to g on M, such that $\nabla^M_{e_i} e_j = 0$, at x for all $i, j = 1, ..., m$.

From formula (1.2), we obtain:

$$-f \sum_{i=1}^{m} R^N(\tau_f(\varphi), d\varphi(e_i)) d\varphi(e_i) - \sum_{i=1}^{m} \nabla^{\varphi}_{e_i} f \nabla^{\varphi}_{e_i} \tau_f(\varphi) = 0,$$

then, we have

$$h\big(f \sum_{i=1}^{m} R^N(\tau_f(\varphi), d\varphi(e_i)) d\varphi(e_i), \tau_f(\varphi)\big) + h\big(\sum_{i=1}^{m} \nabla^{\varphi}_{e_i} f \nabla^{\varphi}_{e_i} \tau_f(\varphi), \tau_f(\varphi)\big) = 0,$$

and

$$\begin{aligned} -h\big(f \sum_{i=1}^{m} \nabla^{\varphi}_{e_i} \nabla^{\varphi}_{e_i} \tau_f(\varphi), \tau_f(\varphi)\big) &= -h\big(f \sum_{i=1}^{m} \nabla^{\varphi}_{e_i} \nabla^{\varphi}_{e_i} \tau_f(\varphi), \tau_f(\varphi)\big) \\ &\quad + h\big(f \sum_{i=1}^{m} R^N(\tau_f(\varphi), d\varphi(e_i)) d\varphi(e_i), \tau_f(\varphi)\big) \\ &\quad + h\big(\sum_{i=1}^{m} \nabla^{\varphi}_{e_i} f \nabla^{\varphi}_{e_i} \tau_f(\varphi), \tau_f(\varphi)\big). \end{aligned} \tag{2.6}$$

Since, $\mathrm{trace}_g\, h(\nabla^{\varphi} f \nabla^{\varphi} \tau_f(\varphi), \tau_f(\varphi)) - \mathrm{trace}_g\, h(f \nabla^{\varphi} \nabla^{\varphi} \tau_f(\varphi), \tau_f(\varphi)) \leq 0$, and the sectional curvature of N is non-positive, from (2.6) we deduce

$$-h\big(\sum_{i=1}^{m} \nabla^{\varphi}_{e_i} \nabla^{\varphi}_{e_i} \tau_f(\varphi), \tau_f(\varphi)\big) \leq 0. \tag{2.7}$$

Let ρ be a smooth function with compact support on M, by (2.7) we have

$$-h\big(\sum_{i=1}^{m} \nabla^{\varphi}_{e_i} \nabla^{\varphi}_{e_i} \tau_f(\varphi), \rho^2\, \tau_f(\varphi)\big) \leq 0,$$

is equivalent to

$$(2.8)\qquad -\sum_{i=1}^{m} e_i\Big(h\big(\nabla^{\varphi}_{e_i}\tau_f(\varphi), \rho^2\,\tau_f(\varphi)\big)\Big) + \sum_{i=1}^{m} h\big(\nabla^{\varphi}_{e_i}\tau_f(\varphi), \nabla^{\varphi}_{e_i}\rho^2\,\tau_f(\varphi)\big) \leq 0.$$

by the Stoke's theorem, we have

$$(2.9)\qquad \int_M \sum_{i=1}^{m} e_i\Big(h\big(\nabla^{\varphi}_{e_i}\tau_f(\varphi), \rho^2\,\tau_f(\varphi)\big)\Big) v_g = 0.$$

From formulae (2.8) and (2.9), we obtain

$$(2.10)\qquad \int_M \sum_{i=1}^{m} \rho^2\,|\nabla^{\varphi}_{e_i}\tau_f(\varphi)|^2\,v_g + \int_M \sum_{i=1}^{m} 2\,\rho\,e_i(\rho)\,h\big(\nabla^{\varphi}_{e_i}\tau_f(\varphi), \tau_f(\varphi)\big)\,v_g \leq 0.$$

By the Young's inequality we have

$$(2.11)\qquad -2h\big(\rho\nabla^{\varphi}_{e_i}\tau_f(\varphi), e_i(\rho)\tau_f(\varphi)\big) \leq \epsilon\,\rho^2\,|\nabla^{\varphi}_{e_i}\tau_f(\varphi)|^2 + \frac{1}{\epsilon}\,e_i(\rho)^2\,|\tau_f(\varphi)|^2.$$

From (2.10) and (2.11) we deduce the inequality

$$(2.12)\qquad \begin{aligned} \int_M \sum_{i=1}^{m} \rho^2\,|\nabla^{\varphi}_{e_i}\tau_f(\varphi)|^2\,v_g \quad \leq \quad & \epsilon \int_M \sum_{i=1}^{m} \rho^2\,|\nabla^{\varphi}_{e_i}\tau_f(\varphi)|^2\,v_g \\ & + \frac{1}{\epsilon} \int_M \sum_{i=1}^{m} e_i(\rho)^2\,|\tau_f(\varphi)|^2\,v_g. \end{aligned}$$

Let $\epsilon = \frac{1}{2}$, by (2.12) we have

$$(2.13)\qquad \frac{1}{2}\int_M \sum_{i=1}^{m} \rho^2\,|\nabla^{\varphi}_{e_i}\tau_f(\varphi)|^2\,v_g \quad \leq \quad 2\int_M \sum_{i=1}^{m} e_i(\rho)^2\,|\tau_f(\varphi)|^2\,v_g.$$

consider the smooth function $\rho = \rho_R$, such that, $\rho \leq 1$ on M, $\rho = 1$ on the ball $B(x, R)$, $\rho = 0$ on $M\backslash B(x, 2R)$ and $|\operatorname{grad}^M \rho| \leq \frac{2}{R}$. Then

$$(2.14)\qquad \frac{1}{2}\int_M \rho^2 \sum_{i=1}^{m} |\nabla^{\varphi}_{e_i}\tau_f(\varphi)|^2\,v_g \quad \leq \quad \frac{8}{R^2}\int_M |\tau_f(\varphi)|^2\,v_g.$$

Since $E_{2,f}(\varphi) = \frac{1}{2}\int_M |\tau_f(\varphi)|^2\,v_g < \infty$, when $R \to \infty$, we obtain

$$\frac{1}{2}\int_M \sum_{i=1}^{m} |\nabla^{\varphi}_{e_i}\tau_f(\varphi)|^2\,v_g = 0,$$

therefore

$$\nabla^{\varphi}_{e_i}\tau_f(\varphi) = 0,$$

for all $i = 1, ..., m$, using the Proporsition 2.2, we deduce $\tau_f(\varphi) = 0$. □

3. f-biharmonic Submanifolds in Spheres

Let M be a submanifold of $\mathbb{S}^n$ of dimension m, $\mathbf{i} : M \to \mathbb{S}^n$ be the canonical inclusion and $f \in C^\infty(M)$ be a smooth positive function. We denote by B the second fundamental form of the submanifold M, by A the shape operator, by H the mean curvature vector field of M.

Theorem 3.1. *The map* $\mathbf{i}$ *is* f*-biharmonic if and only if*

$$\begin{aligned}
&(m-1)\, f\ \mathrm{grad}^M f + 3\, m\, f\, A_H(\mathrm{grad}^M f) + \frac{1}{2}\,\mathrm{grad}^M(|\,\mathrm{grad}^M f|^2) \\
&-\frac{m^2}{2}\, f^2\ \mathrm{grad}^M(|H|^2) + 2\, m\, f^2 \sum_{i=1}^m A_{(\nabla^{\mathbb{S}^n}_{e_i} H)^\perp}(e_i) + f\ \mathrm{Ricci}^M(\mathrm{grad}^M f) \\
&+f\ \mathrm{grad}^M(\Delta^M f) + f\sum_{i=1}^m A_{B(e_i,\,\mathrm{grad}^M f)}(e_i) = 0,
\end{aligned}$$

$$\begin{aligned}
&m^2 f^2 H + m\,|\,\mathrm{grad}^M f|^2 H + 3\, m\, f\,(\nabla^{\mathbb{S}^n}_{\mathrm{grad}^M f} H)^\perp + B(\mathrm{grad}^M f, \mathrm{grad}^M f) \\
&+m\, f\,(\Delta^M f)\, H + m\, f^2 \sum_{i=1}^m B(e_i, A_H(e_i)) + m\, f^2\,(\Delta^\perp H) \\
&+f\sum_{j=1}^m B(e_i, \nabla^M_{e_i}\mathrm{grad}^M f) + f\sum_{i=1}^m \left(\nabla^{\mathbb{S}^n}_{e_i} B(e_i, \mathrm{grad}^M f)\right)^\perp = 0.
\end{aligned}$$

where $\{e_1, ..., e_m\}$ be an orthonormal frame.

To prove the Theorem 3.1, we need the following lemma:

Lemma 3.2. *Let* $\Delta^\perp$ *the Laplacian in the normal bundle of* M*, then*

$$\begin{aligned}
\mathrm{trace}\,(\nabla^{\mathbb{S}^n})^2 H &= -\frac{m}{2}\ \mathrm{grad}^M(|H|^2) + 2\sum_{i=1}^m A_{(\nabla^{\mathbb{S}^n}_{e_i} H)^\perp}(e_i) \\
&\quad + \sum_{i=1}^m B(e_i, A_H(e_i)) + \Delta^\perp H.
\end{aligned}$$

Proof. Let $\{e_1, ..., e_m\}$ be an orthonormal frame such that $\nabla^M_{e_i} e_j = 0$ at $x \in M$ for all $i, j = 1, ..., m$. Then calculating at x

$$
\begin{aligned}
\sum_{i=1}^{m} \nabla^{\mathbb{S}^n}_{e_i} \nabla^{\mathbb{S}^n}_{e_i} H &= \sum_{i=1}^{m} \nabla^{\mathbb{S}^n}_{e_i} \big(A_H(e_i) + (\nabla^{\mathbb{S}^n}_{e_i} H)^{\perp} \big) \\
&= \sum_{i=1}^{m} \nabla^{M}_{e_i} A_H(e_i) + \sum_{i=1}^{m} B(e_i, A_H(e_i)) \\
&\quad + \sum_{i=1}^{m} A_{(\nabla^{\mathbb{S}^n}_{e_i} H)^{\perp}}(e_i) + \sum_{i=1}^{m} \big(\nabla^{\mathbb{S}^n}_{e_i} (\nabla^{\mathbb{S}^n}_{e_i} H)^{\perp} \big)^{\perp}. \qquad (3.1)
\end{aligned}
$$

Since $< A_H(X), Y >= - < B(X,Y), H >$ for all $X, Y \in \Gamma(TM)$, we get

$$
\begin{aligned}
\sum_{i=1}^{m} \nabla^{M}_{e_i} A_H(e_i) &= \sum_{i,j=1}^{m} \big\langle \nabla^{M}_{e_i} A_H(e_i), e_j \big\rangle \, e_j \\
&= \sum_{i,j=1}^{m} e_i \big(\big\langle A_H(e_i), e_j \big\rangle \big) \, e_j \\
&= - \sum_{i,j=1}^{m} e_i \big(\big\langle B(e_i, e_j), H \big\rangle \big) \, e_j \\
&= - \sum_{i,j=1}^{m} e_i \big(\big\langle \nabla^{\mathbb{S}^n}_{e_j} e_i, H \big\rangle \big) \, e_j.
\end{aligned}
$$

Since $\nabla^{\mathbb{S}^n}_X \nabla^{\mathbb{S}^n}_Y = R^{\mathbb{S}^n}(X,Y)Z + \nabla^{\mathbb{S}^n}_Y \nabla^{\mathbb{S}^n}_X + \nabla^{\mathbb{S}^n}_{[X,Y]} Z$ for all $X, Y, Z \in \Gamma(T\mathbb{S}^n)$,

$$
\begin{aligned}
\sum_{i=1}^{m} \nabla^{M}_{e_i} A_H(e_i) &= - \sum_{i,j=1}^{m} \big\langle \nabla^{\mathbb{S}^n}_{e_i} \nabla^{\mathbb{S}^n}_{e_j} e_i, H \big\rangle \, e_j - \sum_{i,j=1}^{m} \big\langle \nabla^{\mathbb{S}^n}_{e_j} e_i, \nabla^{\mathbb{S}^n}_{e_i} H \big\rangle \, e_j \\
&= - \sum_{i,j=1}^{m} \big\langle R^{\mathbb{S}^n}(e_i, e_j) e_i, H \big\rangle \, e_j - \sum_{i,j=1}^{m} \big\langle \nabla^{\mathbb{S}^n}_{e_j} \nabla^{\mathbb{S}^n}_{e_i} e_i, H \big\rangle \, e_j \\
&\quad - \sum_{i,j=1}^{m} \big\langle B(e_i, e_j), (\nabla^{\mathbb{S}^n}_{e_i} H)^{\perp} \big\rangle \, e_j.
\end{aligned}
$$

Since $R^{\mathbb{S}^n}(X,Y)Z = <Y,Z>X - <X,Z>Y$ for all $X,Y,Z \in \Gamma(T\mathbb{S}^n)$,

$$\begin{aligned}\sum_{i=1}^{m} \nabla^M_{e_i} A_H(e_i) &= -\sum_{i,j=1}^{m} e_j\big(\langle \nabla^{\mathbb{S}^n}_{e_i} e_i, H\rangle\big)\, e_j + \sum_{i,j=1}^{m} \langle \nabla^{\mathbb{S}^n}_{e_i} e_i, \nabla^{\mathbb{S}^n}_{e_j} H\rangle\, e_j \\ &\quad + \sum_{i,j=1}^{m} \langle A_{(\nabla^{\mathbb{S}^n}_{e_i} H)^\perp}(e_i), e_j\rangle\, e_j \\ &= -m\sum_{j=1}^{m} e_j\big(\langle H,H\rangle\big)\, e_j + m\sum_{j=1}^{m} \langle H, \nabla^{\mathbb{S}^n}_{e_j} H\rangle\, e_j \\ &\quad + \sum_{i=1}^{m} A_{(\nabla^{\mathbb{S}^n}_{e_i} H)^\perp}(e_i) \\ &= -\frac{m}{2}\sum_{j=1}^{m} e_j\big(\langle H,H\rangle\big)\, e_j + \sum_{i=1}^{m} A_{(\nabla^{\mathbb{S}^n}_{e_i} H)^\perp}(e_i). \qquad (3.2)\end{aligned}$$

By (3.1) and (3.2) the Lemma 3.2. follows. □

Proof of Theorem 3.1. The f-tension field of $\mathbf{i}$ is given by

$$\begin{aligned}\tau_f(\mathbf{i}) &= f\,\tau(\mathbf{i}) + d\mathbf{i}(\operatorname{grad}^M f) \\ &= m\,f\,H + \operatorname{grad}^M f.\end{aligned}$$

Let $\{e_1, ..., e_m\}$ be an orthonormal frame such that $\nabla^M_{e_i} e_j = 0$ at $x \in M$ for all $i,j = 1,...,m$. Then calculating at x

$$\begin{aligned}\sum_{i=1}^{m} R^{\mathbb{S}^n}(\tau_f(\mathbf{i}), d\mathbf{i}(e_i))d\mathbf{i}(e_i) &= m\,f\sum_{i=1}^{m} R^{\mathbb{S}^n}(H,e_i)e_i \\ &\quad + \sum_{i=1}^{m} R^{\mathbb{S}^n}(\operatorname{grad}^M f, e_i)e_i.\end{aligned}$$

Since $<H, e_i> = 0$ and $R^{\mathbb{S}^n}(X,Y)Z = <Y,Z>X - <X,Z>Y$, we get

$$\sum_{i=1}^{m} R^{\mathbb{S}^n}(\tau_f(\mathbf{i}), d\mathbf{i}(e_i))d\mathbf{i}(e_i) = m^2\, f\, H + (m-1)\operatorname{grad}^M f. \qquad (3.3)$$

We compute

$$\sum_{i=1}^{m} \nabla^{\mathbf{i}}_{e_i} f\, \nabla^{\mathbf{i}}_{e_i} \tau_f(\mathbf{i}) = \nabla^{\mathbf{i}}_{\operatorname{grad}^M f}\tau_f(\mathbf{i}) + f\sum_{i=1}^{m} \nabla^{\mathbf{i}}_{e_i}\nabla^{\mathbf{i}}_{e_i}\tau_f(\mathbf{i}). \qquad (3.4)$$

The first term on the left-hand side of (3.4) is

$$
\begin{aligned}
\nabla^{\mathbf{i}}_{\mathrm{grad}^M f}\tau_f(\mathbf{i}) &= m\nabla^{\mathbf{i}}_{\mathrm{grad}^M f} f\,H + \nabla^{\mathbf{i}}_{\mathrm{grad}^M f}\,\mathrm{grad}^M f\\
&= m\,|\,\mathrm{grad}^M f|^2 H + m\,f\,\nabla^{\mathbb{S}^n}_{\mathrm{grad}^M f} H + \nabla^{\mathbb{S}^n}_{\mathrm{grad}^M f}\,\mathrm{grad}^M f\\
&= m\,|\,\mathrm{grad}^M f|^2 H + m\,f\,A_H(\mathrm{grad}^M f) + m\,f\,\big(\nabla^{\mathbb{S}^n}_{\mathrm{grad}^M f} H\big)^{\perp}\\
&\quad +\nabla^M_{\mathrm{grad}^M f}\,\mathrm{grad}^M f + B(\mathrm{grad}^M f, \mathrm{grad}^M f)\\
&= m\,|\,\mathrm{grad}^M f|^2 H + m\,f\,A_H(\mathrm{grad}^M f) + m\,f\,\big(\nabla^{\mathbb{S}^n}_{\mathrm{grad}^M f} H\big)^{\perp}\\
&\quad +\frac{1}{2}\,\mathrm{grad}^M\big(|\,\mathrm{grad}^M f|^2\big) + B(\mathrm{grad}^M f, \mathrm{grad}^M f). \qquad (3.5)
\end{aligned}
$$

The second term on the left-hand side of (3.4) is

$$
\begin{aligned}
f\sum_{i=1}^m \nabla^{\mathbf{i}}_{e_i}\nabla^{\mathbf{i}}_{e_i}\tau_f(\mathbf{i}) &= m\,f\sum_{i=1}^m \nabla^{\mathbf{i}}_{e_i}\nabla^{\mathbf{i}}_{e_i} f\,H + f\sum_{i=1}^m \nabla^{\mathbf{i}}_{e_i}\nabla^{\mathbf{i}}_{e_i}\,\mathrm{grad}^M f\\
&= m\,f\sum_{i=1}^m \nabla^{\mathbf{i}}_{e_i}\big(e_i(f)\,H + f\,\nabla^{\mathbf{i}}_{e_i} H\big)\\
&\quad +f\sum_{i=1}^m \nabla^{\mathbf{i}}_{e_i}\big(\nabla^M_{e_i}\,\mathrm{grad}^M f + B(e_i, \mathrm{grad}^M f)\big)\\
&= m\,f\,(\Delta^M f)\,H + 2\,m\,f\,\nabla^{\mathbb{S}^n}_{\mathrm{grad}^M f} H\\
&\quad +m\,f^2\sum_{i=1}^m \nabla^{\mathbf{i}}_{e_i}\nabla^{\mathbf{i}}_{e_i} H + f\sum_{i=1}^m \nabla^M_{e_i}\nabla^M_{e_i}\,\mathrm{grad}^M f\\
&\quad +f\sum_{i=1}^m B(e_i, \nabla^M_{e_i}\,\mathrm{grad}^M f) + f\sum_{i=1}^m A_{B(e_i,\mathrm{grad}^M f)}(e_i)\\
&\quad +f\sum_{i=1}^m \big(\nabla^{\mathbb{S}^n}_{e_i} B(e_i, \mathrm{grad}^M f)\big)^{\perp}. \qquad (3.6)
\end{aligned}
$$

By the Lemma 3.2 we have

$$
\begin{aligned}
\sum_{i=1}^m \nabla^{\mathbf{i}}_{e_i}\nabla^{\mathbf{i}}_{e_i} H &= -\frac{m}{2}\,\mathrm{grad}^M(|H|^2) + 2\sum_{i=1}^m A_{(\nabla^{\mathbb{S}^n}_{e_i} H)^{\perp}}(e_i)\\
&\quad +\sum_{i=1}^m B(e_i, A_H(e_i)) + \Delta^{\perp} H. \qquad (3.7)
\end{aligned}
$$

From the equation

$$
\mathrm{trace}(\nabla^M)^2\,\mathrm{grad}^M f = \mathrm{Ricci}^M(\mathrm{grad}^M f) + \mathrm{grad}^M(\Delta^M f),
$$

and the formulae (3.3), (3.4), (3.5), (3.6) and (3.7), the Theorem 3.1 follows. □

Example 3.3. We consider

$$M=\mathbb{S}^m\Big(\frac{1}{\sqrt{2}}\Big)\times\Big\{\frac{1}{\sqrt{2}}\Big\}=\Big\{\Big(x_1,...,x_{m+1},\frac{1}{\sqrt{2}}\Big)\in\mathbb{R}^{m+2}\,|\,\sum_{i=1}^{m+1}x_i^2=\frac{1}{2}\Big\},$$

a parallel hypersphere of $\mathbb{S}^{m+1}$. The second fundamental form of M is

$$B(X,Y)=\nabla d\mathbf{i}(X,Y)=<X,Y>H,$$

where $H=-\eta$ and $\eta=\big(x_1,...,x_{m+1},-\frac{1}{\sqrt{2}}\big)$ ([1], [9]).

From direct calculations we have

$$A_H X=-X,\ (\nabla_X^{\mathbb{S}^{m+1}}H)^\perp=0\text{ for all }X\in\Gamma(TM)\text{ and }\Delta^\perp H=0.$$

Let $f\in C^\infty(M)$ be a smooth positive function and $\{e_1,...,e_m\}$ be an orthonormal frame, then

$$\sum_{i=1}^m A_{B(e_i,\mathrm{grad}^M f)}(e_i)=-\,\mathrm{grad}^M f,\quad \sum_{i=1}^m B(e_i,A_H(e_i))=-m\,H$$

$$\sum_{i=1}^m\big(\nabla_{e_i}^{\mathbb{S}^{m+1}}B(e_i,\mathrm{grad}^M f)\big)^\perp=(\Delta^M f)\,H.$$

According to Theorem 3.1, the inclusion map $\mathbf{i}$ is f-biharmonic if and only if

$$\begin{cases}-2(m+1)\,f\ \mathrm{grad}^M f+\frac{1}{2}\,\mathrm{grad}^M(|\,\mathrm{grad}^M f|^2)\\ +f\ \mathrm{Ricci}^M(\mathrm{grad}^M f)+f\ \mathrm{grad}^M(\Delta^M f)=0,\\ (m+1)\,|\,\mathrm{grad}^M f|^2+(m+2)\,f\,(\Delta^M f)=0.\end{cases}$$

4. f-biharmonic Curves in $\mathbb{S}^3$

Theorem 4.1. *Let $\gamma: I\subset\mathbb{R}\to\mathbb{S}^3$ be a differentiable curve parametrized by arc length, and let $f: I\to(0,\infty)$ be a smooth function. Then, the curve γ is f-biharmonic if and only if*

$$\begin{cases}f\,f'''+f'\,f''-4\,k^2\,f\,f'-3\,k\,k'\,f^2=0,\\ 3\,k\,f\,f''+4\,k'\,f\,f'+2\,k\,(f')^2+k''\,f^2-k^3\,f^2-k\,\tau^2\,f^2+k\,f^2=0,\\ 4\,k\,\tau\,f'+2\,k'\,\tau\,f+k\,\tau'\,f=0,\end{cases}$$

where k is the geodesic curvature and τ is the geodesic torsion of γ.

Proof. Let $\{T, N, B\}$ be an orthonormal frame field tangent to $\mathbb{S}^3$ along γ, where $T = d\gamma(d/dt)$ is the unit vector field tangent to γ, N is the unit normal vector field in the direction of $\nabla^{\mathbb{S}^3}_T T$ and B is chosen so that $\{T, N, B\}$ is a positive oriented basis. Then we have the following Frenet equations

$$\begin{cases} \nabla^{\mathbb{S}^3}_T T = k\,N, \\ \nabla^{\mathbb{S}^3}_T N = -k\,T + \tau\,B, \\ \nabla^{\mathbb{S}^3}_T B = -\tau\,N. \end{cases} \tag{4.1}$$

The tension field of the curve γ is given by

$$\tau(\gamma) = \nabla^{\gamma}_{\frac{d}{dt}} d\gamma(\frac{d}{dt}) = \nabla^{\mathbb{S}^3}_T T. \tag{4.2}$$

By (4.1) and (4.2), the f-tension field of the curve γ is given by

$$\tau_f(\gamma) = f'\,T + k\,f\,N. \tag{4.3}$$

The curve γ is f-biharmonic if and only if

$$f\,R^{\mathbb{S}^3}(\tau_f(\gamma), d\gamma(\frac{d}{dt}))d\gamma(\frac{d}{dt}) + \nabla^{\gamma}_{\frac{d}{dt}} f \nabla^{\gamma}_{\frac{d}{dt}} \tau_f(\gamma) = 0. \tag{4.4}$$

By (4.3), the first term on the left-hand side of (4.4) is

$$f\,R^{\mathbb{S}^3}(\tau_f(\gamma), d\gamma(\frac{d}{dt}))d\gamma(\frac{d}{dt}) \quad = \quad k\,f^2\,N, \tag{4.5}$$

here $R^{\mathbb{S}^3}(X,Y)Z = <Y,Z> X - <X,Z> Y$ for all $X, Y, Z \in \Gamma(T\mathbb{S}^3)$.

By (4.1) and (4.3), the second term on the left-hand side of (4.4) is

$$\begin{aligned} \nabla^{\gamma}_{\frac{d}{dt}} f \nabla^{\gamma}_{\frac{d}{dt}} \tau_f(\gamma) \quad = \quad & \big(f\,f''' + f'\,f'' - 4\,k^2\,f\,f' - 3\,k\,k'\,f^2\big)\,T \\ & + \big(3\,k\,f\,f'' + 4\,k'\,f\,f' + 2\,k\,(f')^2 \\ & + k''\,f^2 - k^3\,f^2 - k\,\tau^2\,f^2\big)\,N \\ & + \big(4\,k\,\tau\,f\,f' + 2\,k'\,\tau\,f^2 + k\,\tau'\,f^2\big)\,B. \end{aligned} \tag{4.6}$$

The Theorem 4.1 follows from (4.5) and (4.6). □

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