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Some Symmetric Properties on $(LCS)_n$-manifolds

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Abstract. We analyze the $(LCS)_n$-manifolds endowed with some symmetric properties, focusing on Ricci tensor and the 1-form γ. We study some properties of special Weakly Ricci-Symmetric $(LCS)_n$-manifolds and also shown that Weakly ϕ-Ricci Symmetric $(LCS)_n$-manifold is an η-Einstein manifold.

1. Introduction

The study of Riemannian symmetric manifolds began with the work of E. Cartan [2]. A Riemannian manifold (M^n, g) is said to be locally symmetric due to E. Cartan [2] if its curvature tensor R satisfies the relation $\nabla R = 0$, where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g.

A non-flat Riemannian manifold (M^n, g) is called a weakly symmetric manifold if its curvature tensor R of type $(0, 4)$ satisfies the condition

$$\begin{aligned}(\nabla_W R)(X,Y,Z,U) &= A(W)R(X,Y,Z,U) + B(X)R(W,Y,Z,U)\\ &+ H(Y)R(X,W,Z,U) + D(Z)R(X,Y,W,U)\\ &+ E(U)R(X,Y,Z,W),\end{aligned}$$

for all vector fields W, X, Y, Z, $U \in \chi(M^n)$, where A, B, H, D and E are 1-forms (not simultaneously zero) and ∇ denotes the operator of covariant differentiation with respect to the Riemannian metric g. The 1-forms are called the associated 1-forms of the manifold and an n-dimensional manifold of this kind is denoted by $(WS)_n$.

In 1989, Tamassy and Binh [13, 14] introduced the notion of weakly symmetric and weakly Ricci-symmetric Riemannian manifolds and studied such structures on

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Sasakian manifolds and proved that such a structure does not exist always. Weakly symmetric and weakly Ricci-symmetric structures are also studied by Shaikh and Jana [10, 11]. A Riemannian manifold (M^n, g) is called weakly Ricci symmetric if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition:

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) + B(Y)S(Z,X) + D(Z)S(Y,X),$$

where A, B and D are 1-forms (not simultaneously zero). Such an n-dimensional manifold is denoted by $(WRS)_n$.

The above relation can be written as

$$(\nabla_X Q)(Y) = A(X)Q(Y) + B(Y)Q(X) + S(Y,X)\sigma,$$

where σ is the vector field associated to the 1-form D such that $D(Z) = g(Z, \sigma)$ and Q is the Ricci operator, i.e., $g(QX, Y) = S(X, Y)$ for all X, Y.

In this paper, we study some symmetric properties of $(LCS)_n$ manifolds by presenting two new sections, preceded by a preliminaries section containing some background on $(LCS)_n$ manifolds.

The first one is devoted to the study of special weakly Ricci-symmetric $(LCS)_n$ manifolds. We begin by studying special Weakly Ricci-Symmetric $(LCS)_n$-manifolds with a cyclic parallel Ricci tensor, in this case the 1-form γ must vanish and also we show that if a special weakly Ricci-symmetric $(LCS)_n$-manifolds does not satisfy the condition of Einstein manifold, then the 1-form γ is non zero. Further it is proved that the Ricci tensor is parallel in a special weakly Ricci-symmetric $(LCS)_n$-manifold.

Finally, in the last section we study weakly ϕ-Ricci symmetric $(LCS)_n$ manifold and proved that it is an η-Einstein manifold. Moreover we have seen that ϕ-Ricci symmetric $(LCS)_n$-manifold is an Einstein manifold.

2. Preliminaries

In 2003, A.A. Shaikh [9] introduced the notion of Lorentzian concircular structure manifolds (briefly $(LCS)_n$-manifolds). An n-dimensional Lorentzian manifold M^n is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g, that is, M^n admits a smooth symmetric tensor field g of type $(0, 2)$ such that for each point $p \in M$, the tensor $g_p : T_pM^n \times T_pM^n \to R$ is a non-degenerate inner product of signature $(-, +, ..., +)$, where T_pM^n denotes the tangent vector space of M^n at p and R is the real number space. A non-zero vector $v \in T_pM^n$ is said to be timelike if it satisfies $g_p(v, v) < 0$.

Definition 2.1. In a Lorentzian manifold (M^n, g) a vector field P defined by

$$g(X, P) = A(X),$$

for any $X \in \chi(M^n)$ is said to be a concircular vector field

$$(\nabla_X A)(Y) = \alpha(g(X, Y) + \omega(X)A(Y)),$$

where α is a non zero scalar and ω is a closed 1-form.

Let M^n be an n-dimensional Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the characteristic vector field of the manifold. Then we have

$$g(\xi, \xi) = -1. \tag{2.1}$$

Since ξ is a unit concircular vector field, there exists a non-zero 1-form η such that

$$g(X, \xi) = \eta(X). \tag{2.2}$$

The equation of the following form holds

$$(\nabla_X \eta)(Y) = \alpha[g(X, Y) + \eta(X)\eta(Y)], \quad (\alpha \neq 0) \tag{2.3}$$

for all vector fields X, Y, where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfying

$$\nabla_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X), \tag{2.4}$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. If we put

$$\phi X = \frac{1}{\alpha} \nabla_X \xi, \tag{2.5}$$

then from (2.3) and (2.4), we have

$$\phi X = X + \eta(X)\xi, \tag{2.6}$$

from which it follows that ϕ is a symmetric $(1, 1)$ tensor. Thus the Lorentzian manifold M together with the unit timelike concircular vector field ξ, its associated 1-form η and $(1, 1)$ tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly $(LCS)_n$-manifold)[9]. Especially, if we take $\alpha = 1$, then we can obtain the Lorentzian para-Sasakian structure of Matsumoto [6]. In a $(LCS)_n$-manifold, the following relations hold:

$$\eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \tag{2.7}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.8}$$

$$R(X, Y)Z = (\alpha^2 - \rho)[g(Y, Z)X - g(X, Z)Y], \tag{2.9}$$

$$\eta(R(X, Y)Z) = (\alpha^2 - \rho)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \tag{2.10}$$

$$(\nabla_X \phi)(Y) = \alpha[g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X], \tag{2.11}$$

$$S(X, \xi) = (n - 1)(\alpha^2 - \rho)\eta(X), \tag{2.12}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n - 1)(\alpha^2 - \rho)\eta(X)\eta(Y), \tag{2.13}$$

$$Q\xi = (n - 1)(\alpha^2 - \rho)\xi, \tag{2.14}$$

for any vector fields X, Y, Z, where R, S denotes respectively the curvature tensor and the Ricci tensor of the manifold.

Definition 2.2. An $(LCS)_n$-manifold M^n is said to be η-Einstein if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{2.15}$$

for any vector fields X and Y, where a and b are some functions.

3. On Special Weakly Ricci-Symmetric $(LCS)_n$-manifolds

An n-dimensional Riemannian manifold (M^n, g) is called a special Weakly Ricci-Symmetric manifold $(SWRS)_n$ if

$$(\nabla_X S)(Y,Z) = 2\gamma(X)S(Y,Z) + \gamma(Y)S(X,Z) + \gamma(Z)S(Y,X), \tag{3.1}$$

for any vector fields X,Y on M^n, where γ is a 1-form defined by

$$\gamma(X) = g(X,W), \tag{3.2}$$

where W is the associated vector field.

Theorem 3.1. *In a special weakly Ricci-symmetric $(LCS)_n$-manifold, the Ricci tensor is parallel.*

Proof. Putting $Z = \xi$ in (3.1), we get

$$(\nabla_X S)(Y,\xi) = 2\gamma(X)S(Y,\xi) + \gamma(Y)S(X,\xi) + \gamma(\xi)S(Y,X). \tag{3.3}$$

The left hand side of above equation can be written as

$$(\nabla_X S)(Y,\xi) = \nabla_X S(Y,\xi) - S(\nabla_X Y,\xi) - S(Y,\nabla_X \xi). \tag{3.4}$$

By using (2.13), (3.2) and (3.4), the equation (3.3) can be written as

$$\begin{aligned} &\nabla_X S(Y,\xi) - S(\nabla_X Y,\xi) - S(Y,\nabla_X \xi) \\ &= 2(n-1)(\alpha^2-\rho)\gamma(X)\eta(Y) + (n-1)(\alpha^2-\rho)\gamma(Y)\eta(X) + \eta(W)S(Y,X). \end{aligned} \tag{3.5}$$

Taking $Y = \xi$ in (3.5) and using (2.10), (2.12), (2.13) and (3.2), we get

$$\begin{aligned} -2(n-1)(\alpha^2-\rho)\gamma(X) &+ (n-1)(\alpha^2-\rho)\eta(W)\eta(X) \\ &+ (n-1)(\alpha^2-\rho)\eta(W)\eta(X) = 0. \end{aligned} \tag{3.6}$$

Putting $X = \xi$ in (3.6), we obtain

$$\eta(W) = 0. \tag{3.7}$$

Using (3.7) in (3.6) gives

$$\gamma(X) = 0, \tag{3.8}$$

for any vector fields X on M^n.
Hence in view of (3.8), we obtain from (3.1) that

$$\nabla_X S = 0. \qquad \square$$

Theorem 3.2. *Let (M^n, g) be a special Weakly Ricci-Symmetric $(LCS)_n$-manifold with a cyclic parallel Ricci tensor. Then the 1-form γ must vanish.*

Proof. Taking the cyclic sum of (3.1), we get

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) \\ = 4(\gamma(X)S(Y,Z) + \gamma(Y)S(X,Z) + \gamma(Z)S(X,Y)). \tag{3.9}$$

Let M^n admit a cyclic parallel Ricci tensor. Then (3.9) reduces to

$$\gamma(X)S(Y,Z) + \gamma(Y)S(X,Z) + \gamma(Z)S(X,Y) = 0. \tag{3.10}$$

Taking $Z = \xi$ in (3.10) and using (2.7) and (3.2), we get

$$(n-1)(\alpha^2-\rho)\gamma(X)\eta(Y) + (n-1)(\alpha^2-\rho)\gamma(Y)\eta(X) \\ + \eta(W)S(X,Y) = 0. \tag{3.11}$$

Now putting $Y = \xi$ in (3.11) and using (2.7), (2.12) and (3.2), we get

$$-(n-1)(\alpha^2-\rho)\gamma(X) + (n-1)(\alpha^2-\rho)\eta(W)\eta(X) \\ + (n-1)(\alpha^2-\rho)\eta(W)\eta(X) = 0. \tag{3.12}$$

Further, taking $X = \xi$ in (3.12) and using (2.7) and (3.2), we obtain

$$\eta(W) = 0. \tag{3.13}$$

So by the use of (3.13) in (3.12), we have $\gamma(X) = 0$, for any vector field X on M^n.
This completes the theorem. □

Theorem 3.3. *If a special Weakly Ricci-symmetric $(LCS)_n$-manifold is not an Einstein manifold, then 1-form $\gamma \neq 0$.*

Proof. For an Einstein manifold $(\nabla_X S)(Y,Z) = 0$ and $S(Y,Z) = kg(Y,Z)$,
By (3.1), we have

$$2\gamma(X)S(Y,Z) + \gamma(Y)S(X,Z) + \gamma(Z)S(Y,X) = 0. \tag{3.14}$$

Taking $Z = \xi$ in (3.14) and using (2.13) and (3.2), we get

$$2(n-1)(\alpha^2-\rho)\gamma(X)\eta(Y) + (n-1)(\alpha^2-\rho)\eta(X)\gamma(Y) \\ + \eta(W)S(Y,X) = 0. \tag{3.15}$$

Taking $X = \xi$ in (3.15) and using (2.12) (2.13) and (3.2), we get

$$\begin{aligned} 2(n-1)(\alpha^2-\rho)\eta(W)\eta(Y) &- (n-1)(\alpha^2-\rho)\gamma(Y) \\ &+ (n-1)(\alpha^2-\rho)\eta(W)\eta(Y) = 0. \end{aligned} \tag{3.16}$$

Again taking $Y = \xi$ in (3.16) and by virtue of (2.12) and (3.2), we get

$$\eta(W) = 0. \tag{3.17}$$

Using (3.17) in (3.16), we get

$$\gamma(Y) = 0,$$

for any vector field Y on M^n. □

4. Weakly ϕ-Ricci Symmetric $(LCS)_n$-manifolds

Definition 4.3. A $(LCS)_n$-manifold is said to be weakly ϕ-Ricci Symmetric if the Ricci operator satisfies

$$\phi^2((\nabla_X Q)(Y)) = A(X)\phi^2(QY) + B(Y)\phi^2(QX) + S(Y,X)\phi^2(\sigma). \tag{4.1}$$

Especially, if the 1-forms $A = B = D = 0$, then (4.1) turns into the notion of ϕ-Ricci symmetric introduced by Shukla and Shukla [12].

Let us take $(LCS)_n$-manifold, which is weakly ϕ-Ricci symmetric. Then from (2.8), equation (4.1) becomes

$$\begin{aligned} &(\nabla_X Q)(Y) + \eta((\nabla_X Q)(Y))\xi = A(X)[QY + \eta(QY)\xi] \\ &+B(Y)[QX + \eta(QX)\xi] + S(Y,X)[\sigma + \eta(\sigma)\xi], \end{aligned}$$

from which it follows that

$$\begin{aligned} &g(\nabla_X Q(Y), Z) - S(\nabla_X Y, Z) + \eta((\nabla_X Q)(Y))\eta(Z) = A(X)[S(Y,Z) \\ &+\eta(QY)\eta(Z)] + B(Y)[S(X,Z) + \eta(QX)\eta(Z)] + S(Y,X)[\sigma + \eta(\sigma)\xi]. \end{aligned} \tag{4.2}$$

Putting $Y = \xi$ in (4.2) and using (2.9), (2.12) and (2.14), we get

$$\begin{aligned} (\alpha + B(\xi))S(X,Z) &= -(n-1)(\alpha^2-\rho)[-\alpha g(X,Z) \\ &+ \eta(X)D(Z) + (B(\xi) + \eta(\sigma))\eta(X)\eta(Z)]. \end{aligned} \tag{4.3}$$

Replacing X by ϕX and Z by ϕZ in (4.3), we have

$$[\alpha + B(\xi)]S(\phi X, \phi Z) = \alpha(n-1)(\alpha^2-\rho)g(\phi X, \phi Z). \tag{4.4}$$

By virtue of (2.8) and (2.13), equation (4.4) becomes

$$S(X,Z) = kg(X,Z) + l\eta(X)\eta(Z), \tag{4.5}$$

where

$$k = \frac{\alpha(n-1)(\alpha^2-\rho)}{\alpha+B(\xi)} \quad \textit{and} \quad l = \frac{(n-1)(\alpha^2-\rho)B(\xi)}{\alpha+B(\xi)},$$

provided $\alpha + B(\xi) \neq 0$.
Hence we can state,

Theorem 4.4. *A weakly ϕ-Ricci symmetric $(LCS)_n$-manifolds is an η-Einstein manifold.*

From Theorem 4.4., it can be easily seen that

Corollary 4.1. *A ϕ-Ricci symmetric $(LCS)_n$-manifold is an Einstein manifold.*

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