

Meromorphic Functions Sharing a Nonzero Value with their Derivatives

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ABSTRACT. Let f be a transcendental meromorphic function of finite order in the plane such that $f^{(m)}$ has finitely many zeros for some positive integer $m \geq 2$. Suppose that $f^{(k)}$ and f share a CM, where $k \geq 1$ is a positive integer, $a \neq 0$ is a finite complex value. Then f is an entire function such that $f^{(k)} - a = c(f - a)$, where $c \neq 0$ is a nonzero constant. The results in this paper are concerning a conjecture of Brück [5]. An example is provided to show that the results in this paper, in a sense, are the best possible.

1. Introduction and Main Results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations in Nevanlinna theory of meromorphic functions as explained in [12, 16, 33, 34]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function h , we denote by $T(r, h)$ the Nevanlinna characteristic of h and by $S(r, h)$ any

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Received July 30, 2014; accepted November 19, 2014

2010 Mathematics Subject Classification: 30D35, 30D30.

Key words and phrases: Meromorphic functions, Order of growth, Shared values, Uniqueness theorems.

This work is supported by the NSFC (Grant No. 11171184), the NSFC (Grant No. 11461042), the NSF of Shandong Province, China (Grant No. ZR2014AM011) and the NSF of Shandong Province, China (Grant No. ZR2013AM026).

quantity satisfying $S(r, h) = o\{T(r, h)\}$, as $r \rightarrow \infty$ and $r \notin E$.

Let f and g be two nonconstant meromorphic functions, and let a be a finite value. We say that f and g share the value a CM, provided that $f - a$ and $g - a$ have the same zeros and each common zero of $f - a$ and $g - a$ has the same multiplicity related to f and g . Similarly, we say that f and g share a IM, provided that $f - a$ and $g - a$ have the same zeros, and each common zero of $f - a$ and $g - a$ is counted only once. In this paper, we also need the following definition:

Definition 1.1.([16],[34]) For a nonconstant entire function f , the order $\rho(f)$ and the hyper-order $\rho_2(f)$ are defined as

$$\rho(f) = \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r} = \limsup_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log r}$$

and

$$\rho_2(f) = \limsup_{r \rightarrow \infty} \frac{\log \log T(r, f)}{\log r} = \limsup_{r \rightarrow \infty} \frac{\log \log \log M(r, f)}{\log r}$$

respectively, where and in what follows, $M(r, f) = \max_{|z|=r} |f(z)|$.

In 1977, Rubel-Yang [28] proved that if an entire function f shares two distinct finite complex numbers CM with its derivative f' , then $f = f'$. What is the relation between f and f' , if an entire function f shares one finite complex number a CM with its derivative f' ? In 1996, Brück [5] made a conjecture that if f is a nonconstant entire function satisfying $\rho_2(f) < \infty$, where $\rho_2(f)$ is not a positive integer, and if f and f' share one finite complex number a CM, then $f - a = c(f' - a)$ for some constant $c \neq 0$. For the case that $a = 0$, the above conjecture had been proved by Brück [5]. Brück [5] also proved the above conjecture is true, provided that $a \neq 0$ and $N(r, 1/f') = S(r, f)$, where f is an entire function. Later on, Gundersen-Yang [11], Chen-Shon [8] proved that the above conjecture is true, provided that $\rho(f) < \infty$ and $\rho_2(f) < 1/2$ respectively, where f is an entire function. In 2005, Al-Khaladi [1] showed that the conjecture remains true for a nonconstant meromorphic function f such that $N(r, 1/f') = S(r, f)$. In this direction, some other research works have been obtained, see, e.g., Al-Khaladi [2, 3], Banerjee-Bhattacharjee [4], Chang-Zhu[6], Chang-Fang[7], Heittokangas-Korhonen-Laine-Rieppo-Zhang[13], Lahiri-Sarkar [15], Li-Gao [19, 20], Li-Yi [21-26], Wang [29], Wang-Laine [30], Wang-Li [31], Xiao-Li [32], Zhang [35], Zhang-Yang [36-37]. But the conjecture remains open by now.

We first recall the following result due to Gundersen and Yang:

Theorem A.([11,Theorem 1]) Let f be a nonconstant entire function of finite order, and let $a \neq 0$ be a finite complex number. If f and f' share a CM, then $f' - a = c(f - a)$ for some nonzero constant c .

Wang [29] obtained the following result to improve Theorem A:

Theorem B.([29,Theorem 1]) Let f be a nonconstant entire function of finite order, let P be a polynomial with degree $p \geq 1$, and let k be a positive integer. If $f - P$ and $f^{(k)} - P$ share 0 CM, then $f^{(k)} - P = c(f - P)$ for some nonzero constant c .

One may ask, what can be said about the relationship between a meromorphic function f and $f^{(k)}$, if f and $f^{(k)}$ share a CM, where f is a nonconstant meromorphic function of finite order, $k \geq 1$ is a positive integer and $a \neq 0$ is constant? In this direction, we will prove the following result:

Theorem 1.1. *Let f be a transcendental meromorphic function of finite order such that $f^{(m)}$ has finitely many zeros for some $m \geq 2$, and let $a \neq 0$ be a finite complex value. If f and $f^{(k)}$ share a CM, then f is a transcendental entire function such that $f^{(k)} - a = c(f - a)$ for some nonzero constant c .*

If we remove the assumption “ f is of finite order” in Theorem 1.1, then we have the following result by Lemma 2.2 in Section 2 of this paper:

Theorem 1.2. *Let f be a transcendental meromorphic function such that $f^{(m)}$ and $f^{(n)}$ have finitely many zeros for some two distinct nonnegative integers m and n satisfying $0 \leq m \leq n - 2$, and let $a \neq 0$ be a finite complex value. If f and $f^{(k)}$ share a CM, where $k \geq 1$ is a positive integer, then f is a transcendental entire function such that $f^{(k)} - a = c(f - a)$ for some complex constant $c \neq 0$.*

The following example shows that the assumption “ $f^{(m)}$ has finitely many zeros for some $m \geq 2$ ” in Theorem 1.1 is necessary:

Example 1.1.([11]) Let

$$f(z) = \frac{2e^z + z + 1}{e^z + 1}.$$

Then $\rho(f) = 1$ and

$$f(z) - 1 = \frac{e^z + z}{e^z + 1}, \quad f'(z) - 1 = -\frac{e^z(e^z + z)}{(e^z + 1)^2},$$

and

$$f''(z) = \frac{[(z - 3)e^z - (z + 1)]e^z}{(e^z + 1)^3}.$$

Therefore, f and f' share 1 CM such that

$$\frac{f'(z) - 1}{f(z) - 1} = -\frac{e^z}{e^z + 1},$$

$$\overline{N}\left(r, \frac{1}{f''}\right) = 2\overline{N}\left(r, \frac{1}{e^z + 1}\right) + O(\log r) = 2T(r, e^z) + O(\log r),$$

which implies that f'' has infinitely many zeros in the complex plane, and that the conclusion of Theorem 1.1 is invalid.

In 1995, Yi-Yang[34] posed the following question.

Question 1.1.([34, p.398]) Let f be a nonconstant meromorphic function, and let a be a finite nonzero complex constant. If f , $f^{(n)}$ and $f^{(m)}$ share the value a CM, where n and m ($n < m$) are distinct positive integers not all even or odd, then can we get the result $f = f^{(n)}$?

Gundersen-Yang [11] proved the following result to deal with Question 1.1:

Theorem C.([11, Theorem 2]) Let f be a nonconstant entire function of finite order, let $a \neq 0$ be a complex number, and let k be a positive integer. If a is shared by f , $f^{(k)}$ and $f^{(k+1)}$ IM, and shared by $f^{(k)}$ and $f^{(k+1)}$ CM, then $f = f'$.

From Theorem 1.1 and Theorem C we can get the following result:

Theorem 1.3. Let f be a transcendental meromorphic function of finite order such that $f^{(m)}$ has finitely many zeros for some $m \geq 2$, and let $a \neq 0$ be a finite complex value. Suppose that 0 is shared by $f - a$, $f^{(k)} - a$ and $f^{(k+1)} - a$ IM, and shared by $f^{(k)} - a$ and $f^{(k+1)} - a$ CM, where $k \geq 1$ is a positive integer. Then, f is a transcendental entire function such that $f = f'$.

2. Preliminaries

In order to prove our theorems in the present paper, we need the following preliminary results:

Lemma 2.1.([17, Theorem 1.2]) Suppose that f is meromorphic of finite order in the plane, and that $f^{(m)}$ has finitely many zeros, for some $m \geq 2$. Then f has finitely many poles.

Lemma 2.2.([18]) Suppose that $m \geq 0$ and $k \geq 2$, and that f is meromorphic in the plane such that $f^{(m)}$ and $f^{(m+k)}$ each have finitely many zeros. Then $f^{(m+1)}/f^{(m)}$ is a rational function. In particular, f has finite order and finitely many poles.

Lemma 2.3.([16, Corollary 2.3.4]) Let f be a transcendental meromorphic function and $k \geq 1$ be an integer. Then $m(r, f^{(k)}/f) = O(\log(rT(r, f)))$, outside of a possible exceptional set E of finite linear measure, and if f is of finite order of growth, then $m(r, f^{(k)}/f) = O(\log r)$.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function. Next we define by $\mu(r) = \max\{|a_n| r^n : n = 0, 1, 2, \dots\}$ the maximum term of f , and define by $\nu(r, f) = \max\{m : \mu(r) = |a_m| r^m\}$ the central index of f (c.f.[14, p. 33-35]).

Lemma 2.4.(Wiman-Valiron,[14, p.187-199]) Let g be a transcendental entire function, and let $0 < \delta < 1/4$. Then there exists a set $E \subset \mathbb{R}^+$ of finite logarithmic measure, i.e., $\int_E dt/t < +\infty$, such that for all z with $|z| = r \notin E$ and

$$|g(z)| > M(r, g)\nu(r, g)^{-\frac{1}{4}+\delta},$$

one has

$$g^{(m)}(z) = \left(\frac{\nu(r, g)}{z}\right)^m \{1 + o(1)\}g(z),$$

where $m \geq 0$ is an integer.

Lemma 2.5.([27, Corollary 1]) *Let $f(z)$ be an entire function of finite order and let $\{w_n\}$ be an unbounded sequence. Assume that $\bigcup_{n=1}^{\infty} \{z : f(z) = w_n\}$ has only $k < \infty$ distinct limiting directions, then $f(z)$ is a polynomial of degree at most k .*

Lemma 2.6.([9, Lemma 2] or [10, Lemma 4]) *If f is a transcendental entire function of hyper order $\rho_2(f)$, then*

$$\rho_2(f) = \limsup_{r \rightarrow \infty} \frac{\log \log \nu(r, f)}{\log r}.$$

3. Proof of Theorems

Proof of Theorem 1.1. First of all, by the assumption $\rho(f) < \infty$ and the assumption that $f^{(m)}$ has finitely many zeros for some $m \geq 2$ we have from Lemma 2.1 that f has finite many poles in the complex plane. We consider the following two cases:

Case 1. Suppose that f is not an entire function. Then, from the assumption that $f - a$ and $f^{(k)} - a$ share 0 CM we have

$$(3.1) \quad \frac{f^{(k)} - a}{f - a} = h,$$

where

$$(3.2) \quad h = \frac{e^{\alpha_1}}{(z - \omega_1)^k (z - \omega_2)^k \cdots (z - \omega_{n-1})^k (z - \omega_n)^k}.$$

Here α_1 is an entire function. From (3.1) and (3.2) we have

$$(3.3) \quad e^{\alpha_1} = (z - \omega_1)^k (z - \omega_2)^k \cdots (z - \omega_{n-1})^k (z - \omega_n)^k \cdot \frac{f^{(k)} - a}{f - a}.$$

From (3.3), Lemma 2.3 and the assumption $\rho(f) < \infty$ we deduce

$$(3.4) \quad \rho(e^{\alpha_1}) \leq 2T(r, f) + O(\log r),$$

as $r \rightarrow \infty$. From (3.4) and Definition 1.1 we have $\rho(e^{\alpha_1}) \leq \rho(f) < \infty$, which implies that α_1 is a polynomial. From (3.2) we have

$$(3.5) \quad \rho(e^{\alpha_1}) = \rho(h).$$

Now we let

$$(3.6) \quad F = P_0 f,$$

where P_0 is a nonconstant polynomial such that P_0 and $1/f$ share 0 CM. Then, F is a transcendental entire function. By calculation we get from (3.6) that

$$(3.7) \quad \frac{f^{(k)} - a}{f - a} = \frac{\sum_{j=0}^k \binom{k}{j} \frac{R_0^{(j)}}{R_0} \cdot \frac{F^{(k-j)}}{F} - \frac{aP_0}{F}}{1 - \frac{aP_0}{F}},$$

where and in what follows,

$$(3.8) \quad R_0 = \frac{1}{P_0}.$$

By calculating we get from (3.8) and the definition of P_0 that

$$(3.9) \quad \frac{R_0'}{R_0} = \frac{m_1}{z - \omega_1} + \frac{m_2}{z - \omega_2} + \dots + \frac{m_n}{z - \omega_n},$$

where m_1, m_2, \dots, m_n are negative integers. By induction we get from (3.9) that

$$(3.10) \quad \frac{R_0^{(j)}}{R_0} = \frac{\{(-1)^{j-1}(j-1)! \sum_{l=1}^n m_l\}(1 + o(1))}{z^j},$$

as $|z| \rightarrow \infty$, where j is a positive integer satisfying $1 \leq j \leq k$. Noting that F is a transcendental entire function, we know from the proposition of the central index (c.f.[14, p.33-35]) we can see that

$$(3.11) \quad \nu(r, F) \rightarrow +\infty.$$

Let

$$(3.12) \quad M(r, F) = |F(z_r)|,$$

where $z_r = re^{i\theta(r)}$, $\theta(r) \in [0, 2\pi)$ is some nonnegative real number. From (3.12) and Lemma 2.4 we know that there exists a subset $E_j \subset (1, \infty)$ of finite logarithmic measure, i.e., $\int_{E_j} \frac{dt}{t} < \infty$, such that for the points $z_r = re^{i\theta(r)}$, $\theta(r) \in [0, 2\pi)$, as $|z_r| = r \notin E_j$ and $M(r, F) = |F(z_r)|$, we have

$$(3.13) \quad \frac{F^{(j)}(z_r)}{F(z_r)} = \left(\frac{\nu(r, F)}{z_r}\right)^j \{1 + o(1)\}.$$

From (3.7), (3.10)-(3.13) we get

$$\begin{aligned} \frac{f^{(k)}(z_r) - a}{f(z_r) - a} &= \frac{\sum_{j=0}^k \binom{k}{j} \frac{R_0^{(j)}}{R_0} \cdot \frac{F^{(k-j)}}{F} - \frac{aP_0}{F}}{1 - \frac{aP_0}{F}} \Bigg|_{z=z_r} \\ &= \frac{\{\nu(r, F)\}^k \{1 + o(1)\} + N_n \sum_{j=1}^{k-1} \binom{k}{j} \sum_{j=1}^{k-1} (-1)^{j-1} (j-1)! \{\nu(r, F)\}^{k-j} \{1 + o(1)\}}{z_r^k} \end{aligned}$$

$$(3.14) \quad = \left(\frac{\nu(r, F)}{z_r} \right)^k \{1 + o(1)\},$$

as $r \notin \cup_{j=1}^k E_j$ and $r \rightarrow \infty$, where $N_n = \sum_{l=1}^n m_l$.

Next we prove that α_1 , and so e^{α_1} is a nonzero constant. Indeed, suppose that α_1 is a nonconstant polynomial. Then

$$(3.15) \quad \alpha_1(z) = p_l z^l + p_{l-1} z^{l-1} + \dots + p_1 z + p_0,$$

where $p_l, p_{l-1}, \dots, p_1, p_0$ are complex numbers and $p_l = \gamma_l e^{i\theta_l} \neq 0, \gamma_l > 0$. Given a positive number ε , we set

$$(3.16) \quad T_\varepsilon = \bigcup_{j=0}^{l-1} \{z : |\arg z - \theta_j| < \varepsilon\},$$

where

$$(3.17) \quad \theta_j = \left(\frac{2j}{l} + \frac{1}{2l} \right) \pi - \frac{\theta_l}{l}, \quad 0 \leq j \leq l-1.$$

Next we let $w_j = f(z_{r_j}), j = 1, 2, \dots$. Then $\{w_j\}$ is an unbounded sequence. We discuss the following two subcases:

Subcase 1.1. Suppose that T are the only l distinct limiting directions of $\bigcup_{j=1}^\infty \{z : F(z) = w_j\}$. First of all, from (3.6) we have $\rho(F) = \rho(f) < \infty$. This together with Lemma 2.5 implies that F is a nonconstant polynomial. Combining this with (3.6), we can see that f is a rational function, which contradicts the assumption of Theorem 1.1.

Subcase 1.2. Suppose that there exists some sufficiently small positive number ε_0 and there exist some infinite subsequence of the points z_{r_j} , say itself such that

$$(3.18) \quad \{z_{r_j}\} \subset C \setminus T_{\varepsilon_0}.$$

Noting that $\cos(\theta_l + l\theta_j) = 0$ for $0 \leq j \leq l-1$, we can deduce from (3.12)-(3.18) that there exists a positive number $\delta_1(l, \varepsilon_0) \in (0, \gamma_l)$ that depends only upon l and ε_0 such that

$$(3.19) \quad |\operatorname{Re}\alpha_1(z_{r_j})| \geq \delta_1(l, \varepsilon_0)r_j^l \quad \text{or} \quad |\operatorname{Re}\alpha_1(z_{r_k})| \leq -\delta_1(l, \varepsilon_0)r_j^l,$$

as $z_{r_j} \in C \setminus T_{\varepsilon_0}, r_j \notin E$ and $r_j \rightarrow \infty$. Noting that $|z_{r_j}| = r_j$, we have from (3.15), (3.17) and (3.19) that

$$\begin{aligned}
 & \delta_1(l, \varepsilon_0)r_j^l \\
 \leq & \left| \log |e^{\alpha_1(z_{r_j})}| \right| \\
 \leq & \log \left| \frac{(z_{r_j} - \omega_1)^k(z_{r_j} - \omega_2)^k \cdots (z_{r_j} - \omega_{n-1})^k(z_{r_j} - \omega_n)^k}{z_{r_j}^k} \{\nu(r_j, F)\}^k \right| + o(1) \\
 \leq & k \log \nu(r_j, F) + kn \log r,
 \end{aligned}$$

as $z_{r_j} \in C \setminus T_{\varepsilon_0}$, $r_j \notin E$ and $r_j \rightarrow \infty$. From (3.15), (3.20) and Lemma 2.6 we have

$$(3.21) \quad \rho(e^{\alpha_1}) = l \leq \limsup_{r \rightarrow \infty} \frac{\log \log \nu(r, f)}{\log r} = \rho_2(f).$$

Again from the assumption $\rho(f) < \infty$ and Definition 1.1 we have $\rho_2(f) = 0$. Combining this with (3.15) and (3.21), we have $l = \deg(\alpha_1) = 0$, and so α_1 is a constant. Therefore, from (3.1) and (3.2) we have

$$(3.22) \quad \frac{f^{(k)} - a}{f - a} = \frac{c}{(z - \omega_1)^k(z - \omega_2)^k \cdots (z - \omega_{n-1})^k(z - \omega_n)^k},$$

where $c = e^{\alpha_1}$. From (3.17) and (3.22) we have

$$(3.23) \quad \left(\frac{\nu(r, F)}{z_r} \right)^k \{1 + o(1)\} = \frac{c}{(z_r - \omega_1)^k(z_r - \omega_2)^k \cdots (z_r - \omega_{n-1})^k(z_r - \omega_n)^k},$$

as $|z_r| = r \notin \cup_{j=1}^k E_j$ and $r \rightarrow \infty$. From (3.23) we have

$$(3.24) \quad \{\nu(r, F)\}^k \leq \left| \frac{2z_r^k}{(z_r - \omega_1)^k(z_r - \omega_2)^k \cdots (z_r - \omega_{n-1})^k(z_r - \omega_n)^k} \right|,$$

as $|z_r| = r \notin \cup_{j=1}^k E_j$ and $r \rightarrow \infty$. By letting $|z_r| = r \rightarrow \infty$ and $|z_r| = r \notin \cup_{j=1}^k E_j$ on two sides of (3.24), we have

$$\{\nu(r, F)\}^k < 3,$$

as $|z_r| = r \notin \cup_{j=1}^k E_j$ and $r \rightarrow \infty$. This contradicts (3.11).

Case 2. Suppose that f is an entire function. Then,

$$(3.25) \quad \frac{f^{(k)} - a}{f - a} = e^{\alpha_2},$$

where α_2 is an entire function. From (3.25) we can see that f is a transcendental entire function. From (3.25) and Lemma 2.3 we have

$$(3.26) \quad T(r, e^{\alpha_2}) \leq 2T(r, f) + O(\log r),$$

as $r \rightarrow \infty$. From (3.26) and Definition 1.1 we deduce $\rho(e^{\alpha_2}) \leq \rho(f) < \infty$, and so α_2 is a polynomial. If α_2 is a constant, then the conclusion of Theorem 1.1 is valid. Next we suppose that α_2 is nonconstant polynomial. Then, in the same manner as in the proof of (3.21) we have $\rho(e^{\alpha_2}) \leq \rho_2(f) = 0$, which implies that α_2 is a constant, this is impossible. This completes the proof of Theorem 1.1.

Acknowledgements. The authors wish to express their thanks to the referee for his/her valuable suggestions and comments.

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