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Some Additive Maps on Sigma Prime Rings

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Abstract. The purpose of this paper is to prove some results which are of independent interest and related to additive maps on σ-prime rings. Further, examples are given to demonstrate that the restrictions imposed on the hypotheses of these results are not superfluous.

1. Introduction

During the past few decades, there has been an ongoing interest concerning the relationship between the commutativity of a ring and the behavior of a special mapping on the ring. An example due to Oukhtite [9], shows that every prime ring can be injected in σ-prime ring and from this point of view σ-prime rings constitute a more general class of prime rings. Recently, a major breakthrough has been achieved by Oukhtite et al. [10], where the important results by Posner[12], Herstein[5] and Bell[2] have been proved for σ-prime rings. They are spree of developing and extending more and more results which hold for a prime ring (see, e.g.,[7, 11], where further references can be found). A continuous approach in this direction is still on. In this context, we establish some results concerning additive map with additional conditions in non commutative σ-prime ring. Further, we show that some additive maps on a σ-prime ring with additional conditions that acts as a homomorphism or anti-homomorphism on a non zero ideal in the ring, is zero map or the identity map. In particular, our research can be viewed as a new more elementary approach. At the end, an example is given to demonstrate that the restrictions imposed on the hypotheses of the results are not superfluous.

Throughout this note, R will represent an associative ring with multiplicative center $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ stand for the commutator $xy - yx$.

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Recall that a ring R is prime if for any $x, y \in R$, $xRy = \{0\}$ implies $x = 0$ *or* $y = 0$. A ring R equipped with an involution σ is to be σ-prime if $xRy = xR\sigma(y) = \{0\}$ $\Rightarrow x = 0$ *or* $y = 0$. An ideal U is a σ-ideal if U is invariant under σ, i.e.,$\sigma(U) = U$. Assume that $F : R \to R$ is a map associated with another map $\delta : R \to R$ so that $F(xy) = F(x)y + x\delta(y)$ holds for all $x, y \in R$. If F is additive and δ is a derivation of R, then F is said to be a generalized derivation of R that was introduced by Brešar[4]. In [6], Hvala gave an algebraic study of generalized derivations of prime rings. We note that if R has the property that $Rx = (0)$ implies $x = 0$ and $g : R \to R$ is any function, and $d : R \to R$ is any additive map such that $d(xy) = d(x)y + xg(y)$ for all $x, y \in R$, then d is uniquely determined by g and moreover g must be a derivation by [4, Remark 1]. Let S be any nonempty subset of R and f be any map of R. If $f(xy) = f(x)f(y)$ or $f(xy) = f(y)f(x)$ for all $x, y \in S$ then f is called a map that acts as a homomorphism or anti-homomorphism on S, respectively.

We shall make extensive use of the following basic identities without any specific mention; For all $x, y, z \in R$,

(i) $[xy, z] = x[y, z] + [x, z]y$,

(ii) $[x, yz] = y[x, z] + [x, y]z$.

2. Main Results

We facilitate our discussion with the simple lemma which is required for developing the proofs of our main theorems.

Lemma A.([8, Lemma 2.2]) *Let R be a a σ-prime ring and $I \neq 0$ a σ-ideal of R. If $a, b \in R$ are such that $aIb = 0 = aI\sigma(b)$, then $a = 0$ or $b = 0$.*

We begin with the following one.

Theorem 2.1. *Suppose R is a noncommutative σ-prime ring and $I \neq 0$ a σ-ideal of R. Let F, D are two additive mappings (not necessary a derivation) on R such that $F(xy) = F(x)y + xD(y)$, for all $x, y \in R$ with additional condition $\sigma D = D\sigma$. If $a \in R$ and $[F(x), a] = 0$, for all $x \in I$, then $F([x, a]) = 0$ or $a \in Z(R)$.*

Proof. By the hypothesis, we have

$$[F(x), a] = 0, \ \forall\ x \in I. \tag{2.1}$$

Taking xr instead of x in (2.1) and use it to arrive at

$$F(x)[r, a] + x[D(r), a] + [x, a]D(r) = 0, \ \forall\ x \in I, r \in R. \tag{2.2}$$

In (2.2) replace x by xy, where $y \in I$ then, from (2.2), we obtain

$$(F(x)y + xD(y) - xF(y))[r, a] + [x, a]yD(r) = 0. \tag{2.3}$$

Take a instead of r in (2.3), we have

$$[x, a]yD(a) = 0, \ \forall\ x \in I,$$

and so,

$$[x, a]ID(a) = 0, \ \forall\ x \in I,$$

Since D is commute with σ. Thus, by Lemma A, it follows that

$$[x, a] = 0 \ \text{ or } \ D(a) = 0, \ \forall\ x \in I.$$

Assume that $a \notin Z(R)$, then $D(a) = 0$. If we substitute sx, $s \in R$ for x in (2.3), we obtain

(2.4)$(F(s)x + sD(x) - sF(x))y[r, a] + [s, a]xyD(r) = 0, \ \forall\ x, y \in I, r \in R.$

Replacing s by a in (2.4), we arrive at

$$(F(a)x + aD(x) - aF(x))y[r, a] = 0, \ \forall\ x, y \in I, r \in R.$$

Or

$$(F(a)x + aD(x) - aF(x))I[r, a] = 0, \ \forall\ x \in I, r \in R.$$

Using the fact $a \notin Z(R)$ together with Lemma A, we obtain

$$F(a)x + aD(x) - aF(x) = 0, \ \forall\ x \in I.$$

This implies that

$$F(ax) = aF(x), \ \forall\ x \in I.$$

On the other hand, if $D(a) = 0$, we see that the relation

$$F(xa) = F(x)a, \ \forall\ x \in I.$$

Combining the last two equality we arrive at

$$F[x, a] = 0, \ \forall\ x \in I.$$

This completes the proof. □

Theorem 2.2. *Suppose R is a noncommutative σ-prime ring and $I \neq 0$ a σ-ideal of R. Let F, D are two additive mappings (not necessary a derivation) on R such that $F(xy) = F(x)y + xD(y)$, for all $x, y \in R$ with additional condition $\sigma D = D\sigma$. If $a \in R$ and $F([x, a]) = 0$, for all $x \in I$, then $[F(x), a] = 0$ or $a \in Z(R)$.*

Proof. We begin with the situation

$$F([x, a]) = 0, \ \text{for all } x \in I.$$

We replace x by xa in the above defining equation to obtain

$$[x, a]D(a) = 0, \ \text{for all } x \in I. \tag{2.5}$$

Taking xy instead of x in (2.5) and using (2.5) we obtain

$$[x,a]yD(a)=0,\ \text{for all } x,y\in I,$$

and so,

$$[x,a]ID(a)=0,\ \text{for all } x\in I. \tag{2.6}$$

Since D is commute with σ. Thus, by Lemma A, it follows that

$$[x,a]=0 \quad \text{or} \quad D(a)=0,\ \text{for all } x\in I.$$

If $a\notin Z(R)$, then $D(a)=0$. Now, we replace x by xy in the hypothesis and using hypothesis to obtain

$$F(x)[y,a]+[x,a]D(y)+xD([y,a])=0,\ \text{for all } x,y\in I. \tag{2.7}$$

From associative law and calculating $F(xya)$ in two different ways, we obtain

$$xD(ya)=xD(y)a+xyD(a),\ \text{for all } x,y\in I.$$

Similarly,

$$xD(ay)=xD(a)y+xaD(y),\ \text{for all } x,y\in I.$$

Comparing last two equation to obtain

$$xD[y,a]=x[D(y),a]+x[y,D(a)],\ \text{for all } x,y,z\in I. \tag{2.8}$$

From (2.7) & (2.8), and using $D(a)=0$ we arrive at

$$F(x)[y,a]+[x,a]D(y)+x[D(y),a]=0,\ \text{for all } x,y\in I. \tag{2.9}$$

Substitute yz instead of y in (2.9) and use it to obtain

$$(F(x)y+xD(y))[z,a]+[x,a]yD(z)-xF(y)[z,a]=0,\ \text{for all } x,y,z\in I$$

and so,

$$(F(x)y+xD(y)-xF(y))[z,a]+[x,a]yD(z)=0,\ \text{for all } x,y,z\in I. \tag{2.10}$$

Replace x by ax in (2.10) and use it, it yields

$$(F(a)x+aD(x)-aF(x))y[z,a]=0,\ \text{for all } x,y,z\in I.$$

Hence, we get

$$(F(ax)-aF(x))I[z,a]=0,\ \text{for all } x,z\in I.$$

Since $a \notin Z(R)$ and from Lemma A we obtain

$$F(ax) = aF(x), \text{ for all } x \in I. \tag{2.11}$$

On other hand, as $D(a) = 0$,

$$F(xa) = F(x)a, \text{ for all } x \in I. \tag{2.12}$$

Combining (2.11) & (2.12), we have

$$[F(x), a] = 0, \text{ for all } x \in I.$$

Thus, the proof is complete. □

Following Corollaries are the immediate consequences of the above theorems.

Corollary 2.3. *If R is a σ-prime ring and $I \neq 0$ a σ-ideal of R. Suppose F, D are two additive mappings (not necessary a derivation) on R such that $F(xy) = F(x)y + xD(y)$, for all $x, y \in R$ with additional condition $\sigma D = D\sigma$. If $[F(x), a] = 0$, for all $x \in I, a \in R$, then $F([x, a]) = 0$ or R is commutative.*

Corollary 2.4. *If R is a σ-prime ring and $I \neq 0$ a σ-ideal of R. Suppose F, D are two additive mappings (not necessary a derivation) on R such that $F(xy) = F(x)y + xD(y)$, for all $x, y \in R$ with additional condition $\sigma D = D\sigma$. If $F([x, a]) = 0$, for all $x \in I, a \in R$, then $[F(x), a] = 0$ or R is commutative.*

In [3], Bell and Kappe proved that *if δ is a derivation of prime ring $\Re$ which acts as a homomorphism or an anti-homomorphism on a nonzero ideal I of $\Re$, then $\delta = 0$ on $\Re$.* Further, Albas and Argac [1] extended this result to generalized derivation. Recently, Oukhtite[9] proved that the result is also true for σ-prime ring, as following:

Theorem A.([9, Theorem 1.1]) Suppose R is a 2-torsion free σ-prime ring, $U \neq 0$ a σ-ideal and (f, d) is a generalized derivation with additional condition that $\sigma D = D\sigma$.

(i) If $F(xy) = F(x)F(y)$, for all $x, y \in I$, then $d = 0$. Moreover, if F commute with σ then F is an identity map.

(ii) If $F(xy) = F(y)F(x)$, for all $x, y \in I$ and $d \neq 0$, then R is commutative.

It would be interesting to know that whether above theorem can be proved without the assumption that R is 2-torsion free. In this context, we prove the following:

Theorem 2.5. *Suppose R is a σ-prime ring and $I \neq 0$ a σ-ideal of R. Let F, D are two additive mappings (not necessary a derivation) on R such that $F(xy) = F(x)y + xD(y)$, for all $x, y \in R$ with additional condition $\sigma D = D\sigma$. If F acts as a homomorphism or anti-homomorphism, then $D = 0$. Moreover, if F commutes with σ then either $F = 0$ or F is an identity map on R.*

Proof. We break the proof of theorem in two parts.

(i) If $F|I$ is a homomorphism, we have

$$F(xy) = F(x)F(y), \ \forall \ x, y \in I.$$

From associative law and calculating $F(xyz)$ in two different ways, we obtain easily

$$(F(x) - x)yD(z) = 0, \ \forall \ x, y, z \in I.$$

This implies that

$$(F(x) - x)ID(z) = 0, \ \forall \ x, z \in I.$$

Since D commutes with σ thus, by Lemma A, it follow that either $(F(x) - x) = 0$ or $D(z) = 0$ for all $x, z \in I$. If $D|I \neq 0$ then $F(x) = x$, for all $x \in I$. From this our hypothesis becomes $xD(y) = 0$ for all $x, y \in I$. Since D commutes with σ and I is a σ-ideal then from Lemma A we obtain $D = 0$, a contradiction. Hence $D|I = 0$. In this situation, our hypothesis $F(xy) = F(x)F(y)$ for all $x, y \in I$ force to

$$F(x)y = F(x)F(y), \ \forall \ x, y \in I.$$

Replacing x by yz we get

$$F(x)z(y - F(y)) = 0, \ \forall \ x, y, z \in I.$$

As F commutes with σ and I is a σ-ideal then, by Lemma A, we have either $F(y) = y$ or $F(x) = 0$ for all $x, y \in I$.

If $F(x) = 0$ for all $x \in I$, then $0 = F(rx) = F(r)x + rD(x) = F(r)x$, $\forall \ x \in I \ \& \ r \in R$, hence, F is a zero map on R. On the other hand, if $F(y) = y$ for all $y \in I$ then $rx = F(rx) = F(r)x + rD(x) = F(r)x$, $\forall \ x \in I \ \& \ r \in R$. Therefore, by Lemma A and using the fact F commutes with σ, we obtain

$$F(r) = r, \ \forall \ r \in R.$$

It remain to prove that $D = 0$ on R. Let $F = 0$ (although it is sufficient to assume $F|I = 0$), we get

$$0 = F(rx) = F(r)x + rD(x) = rD(x), \ \forall \ x \in I \ \ \& \ \ r \in R.$$

Since D commutes with σ and $I \neq 0$ is a σ-ideal then, by Lemma A we have $D(r) = 0$ for all $r \in R$. In second case, if F be an identity map (although it is sufficient to assume $F|I$ is the identity), we get

$$rx = F(rx) = F(r)x + rD(x) = rx + rD(x), \ \forall \ x \in I \ \ \& \ \ r \in R.$$

This implies that $D(r) = 0$ for all $r \in R$.

(ii) if $F|I$ is an anti-homomorphism. As in [9] we get

$$[F(z), y]xD(z) = 0, \ \forall \ x, y, z \in I.$$

That is,

$$[F(z), y]ID(z) = 0, \ \forall \ y, z \in I.$$

Since D commutes with σ thus, by Lemma A, it follow that

$$[F(z), y] = 0 \ \text{ or } \ D(z) = 0, \ \forall \ y, z \in I.$$

If $D(z) \neq 0$ for some $z \in I$, then $[F(z), y] = 0$ for all $y, z \in I$. Now, we have

$$\begin{aligned} F(xy)z + xyD(z) &= F(xyz) = F(z)F(y)F(x) = F(y)F(z)F(x) \\ &= F(y)F(xz) = F(y)(F(x)z + xD(z)) \\ &= F(xy)z + F(y)xD(z), \ \forall \ x, y \in I. \end{aligned}$$

This implies that

$$xyD(z) = F(y)xD(z), \ \forall \ x, y \in I. \tag{2.13}$$

Taking rx in place of x in (2.13) we get

$$rxyD(z) = F(y)rxD(z), \ \forall \ x, y, z \in I \ r \in R. \tag{2.14}$$

Multiplying (2.13) by r, we have

$$rxyD(z) = rF(y)xD(z), \ \forall \ x, y, z \in I \ r \in R. \tag{2.15}$$

From (2.14) and (2.15), we obtain

$$(F(y)r - rF(y))xD(z) = 0, \ \forall \ x, y, z \in I \ r \in R.$$

Since D commutes with σ such that $D(z) \neq 0$ and $I \neq 0$ is a σ-ideal then, by Lemma A we have

$$F(y)r = rF(y), \ \forall \ x, y, z \in I \ r \in R.$$

Therefore $F|I$ is an homomorphism. Using (i) we get $D = 0$. This is a contradiction, so that $D|I = 0$. In this situation, we have

$$\begin{aligned} F(x)zy = F(xz)y &= F(z)F(x)y \\ &= F(z)F(xy) = F(xyz) = F(x)yz, \ \forall \ x, y, z \in I. \end{aligned}$$

i.e.

$$F(x)w(zy - yz) = 0, \ \forall \ x, y, z, w \in I.$$

or

$$F(x)I(zy - yz) = 0, \ \forall\ x, y, z, w \in I.$$

Since F commutes with σ and I be a σ-ideal then, by Lemma A we have $F(x) = 0$ for all $x \in I$ or $zy = yz$, for all $y, z \in I$. The second case implies that R is commutative and $F|I$ is homomorphism. Using (i), we get $F(x) = x$ for all $x \in I$ or $F = 0$ on R. Finally, we get, as in (i) that $D = 0$ on R. This completes the proof. □

We can close this paper with some examples which shows that the restrictions in the hypothesis of several results are not superfluous.

Example 2.6. Let $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & c & 0 \end{pmatrix} : a, b, c \in \mathbb{S} \right\}$ and $I = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c & 0 \end{pmatrix} : b \in \mathbb{S} \right\}$, where $\mathbb{S}$ is a non commutative ring. We define the following maps:

$\sigma \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & c & 0 \end{pmatrix} = \begin{pmatrix} 0 & b & a \\ 0 & 0 & 0 \\ 0 & c & 0 \end{pmatrix}$, $F \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & c & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $D \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & c & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c & 0 \end{pmatrix}$. Then it can be seen easily that R is a non commutative ring with involution σ and I is a σ-ideal of R. It is straightforward to check that F and D satisfy all the requirements of Theorem 2.1, but neither $F[X, A] = 0$ nor $A \notin Z(R)$.

Example 2.7. Let $\mathbb{G} = \{a_1 + a_2 i + a_3 j + a_4 k : a_1, a_2, a_3, a_4 \in \mathbb{R}\}$, where i, j, k comes from the Quaternion group Q_8, and multiply accordingly, form a ring under natural addition and multiplication. Consider $H = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & c & 0 \end{pmatrix} : a, b, c \in \mathbb{G} \right\}$ and $I = \left\{ \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : b \in \mathbb{G} \right\}$. We define the following maps:

$\sigma \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & c & 0 \end{pmatrix} = \begin{pmatrix} 0 & -a & b \\ 0 & 0 & 0 \\ 0 & -c & 0 \end{pmatrix}$, $F \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & c & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $D \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & c & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -c & 0 \end{pmatrix}$. Then it can be seen easily that R is a non-commutative ring with involution σ and I is a σ-ideal of R. It is straightforward to check that F and D satisfy all the requirements of Theorem 2.2, but neither $[F(X), A] = 0$ nor $A \notin Z(R)$.

Example 2.8. Let $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$ and $I = \left\{ \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : b \in \mathbb{Z} \right\}$. We define the following maps:

$\sigma \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -a & b \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix}$, $F \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $D \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix}$. Then it can be seen easily that R is a ring with involution σ and I is a σ-ideal of R. It is straightforward to check that F and D satisfy all the requirements of Theorem 2.5, but neither $D = F = 0$ nor F is an identity map.

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