

A Note on Skew-commuting Automorphisms in Prime Rings

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ABSTRACT. Let R be a prime ring with center Z , I a nonzero ideal of R , and σ a non-trivial automorphism of R such that $\{(x \circ y)^\sigma - (x \circ y)\}^n \in Z$ for all $x, y \in I$. If either $\text{char}(R) > n$ or $\text{char}(R) = 0$, then R satisfies s_4 , the standard identity in 4 variables.

1. Introduction

Throughout this paper, unless specifically stated, R will be an associative ring, Z the center of R , Q its two sided Martindale quotient ring and C its extended centroid. We refer the reader to [3] for the definitions and related properties of these objects. For $x, y \in R$, we denote $[x, y] = xy - yx$, the commutator of x and y and $x \circ y = xy + yx$, the anti-commutator (skew-commutator) of x and y . Recall that a ring R is prime if for any $a, b \in R$, $aRb = 0$ implies that $a = 0$ or $b = 0$. Let $n \geq 1$ be a fixed integer and s_4 be the standard identity in 4 variables. For a subset S of R , a mapping $f : S \rightarrow R$ is called commuting (centralizing) if $[f(x), x] = 0$ (resp. $[f(x), x] \in Z$) for all $x \in S$. A mapping $f : S \rightarrow R$ is called skew-commuting (skew-centralizing) if $f(x) \circ x = 0$ (resp. $f(x) \circ x \in Z$) holds for all $x \in S$. The study of commuting and centralizing mappings goes back to 1955 when Divinsky [10] proved that a simple artinian ring is commutative if it has a commuting automorphism different from the identity mapping. Two year later Posner [17] showed that a prime ring must be commutative if it possesses a nonzero centralizing derivation. In 1970 Luh [14] generalized Divinsky's result to prime rings. Later, Mayne [16] obtained the analogous result of Posner for non identity centralizing automorphisms. Similar results extended to the case of left ideal by Bell and Martindale [4] and Lanski [13].

In [5] Brešar obtained a characterization of commuting additive mappings on

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prime rings. Based on this result, Brešar initiated the study of functional identities. We refer the reader to Brešar [7] for an introductory account on functional identities and their applications. In [6], Brešar proved that there are no nonzero skew-commuting additive mappings on a 2-torsion free semiprime rings. In other words, if R is a 2-torsion free semiprime ring and $f : R \rightarrow R$, an additive mapping such that $f(x) \circ x = 0$, for all $x \in R$, then $f = 0$.

In [2] Ashraf and Rehman proved that, If R is a prime ring, I a nonzero ideal of R and d is a derivation of R such that $d(x \circ y) - (x \circ y) = 0$ for all $x, y \in I$, then R is commutative. Further in [1] Argac and Inceboz generalized this result, and obtained the following result, if R is a prime ring, I a nonzero ideal of R and d is a derivation of R , n is a fixed positive integer and $(d(x \circ y))^n - (x \circ y) = 0$ or is central for all $x, y \in I$, then the ring is commutative. The present paper is motivated by the previous results and we here continue this line of investigation by examining what happens in case the derivation is replaced by automorphism. Explicitly we shall prove the following theorem:

Theorem A. Let R be a prime ring with center Z , I a nonzero ideal of R , and σ a nontrivial automorphism of R such that $\{(x \circ y)^\sigma - (x \circ y)\}^n \in Z$ for all $x, y \in I$. If either $\text{char}(R) > n$ or $\text{char}(R) = 0$, then R satisfies s_4 , the standard identity in 4 variables.

2. Main Result

We begin our discussion with the following well-known results.

Lemma 2.1. ([18, Lemma 2.1]) *Let R be a prime ring with the extended centroid C . Then the following conditions are equivalent:*

- (i) $\dim_C RC \leq 4$;
- (ii) R satisfies s_4 , the standard identity in 4 variables;
- (iii) R is commutative or R embeds in $M_2(F)$ for F a field;
- (iv) R is algebraic of bounded degree 2 over C ;
- (v) R satisfies $[[x^2, y], [x, y]]$.

An automorphism σ of R is called Q -inner if there exists an invertible element $b \in Q$ such that $x^\sigma = bxb^{-1}$ for all $x \in R$. Otherwise, σ is called outer. We denote by A the group of all automorphisms of R and by G_i the group consisting of all Q -inner automorphisms of R . Recall that a subset S of A is said to be independent (modulo G_i) if for any $g_1, g_2 \in S$, $g_1g_2^{-1} \in G_i$ implies $g_1 = g_2$. For instance, if g is an outer automorphism of R , then 1 and g are independent (modulo G_i). The following useful result is due to Chuang [9]. We only state a special case for our proofs in the sequel and refer its proof to that of Chuang [9].

Lemma 2.2. ([9, Theorem 3]) *Suppose that R is a prime ring and S an independent subset of A modulo Gi . Let $\phi = \psi(x_i^{g_j}) = 0$ be a generalized identity with automorphisms of R reduced with respect to S . If, for all $x_i \in X$, $g_j \in S$, the $x_i^{g_j}$ - word degree of $\phi = \psi(x_i^{g_j})$ is strictly less than $\text{char}(R)$ when $\text{char}(R) \neq 0$, then $\psi(z_{ij}) = 0$ is also a generalized polynomial identity of R .*

We begin our discussion with the following result, which is essential for developing the proof of our main theorem.

Theorem 2.1. *Let R be a prime ring, I a nonzero ideal of R , and σ a nontrivial automorphism of R such that $\{(x \circ y)^\sigma - (x \circ y)\}^n = 0$ for all $x, y \in I$, n be fixed positive integer. If either $\text{char}(R) > n$ or $\text{char}(R) = 0$, then R satisfies s_4 .*

Proof. We assume that $\dim_C RC > 4$. By assumption, we get that

$$\{(x \circ y)^\sigma - (x \circ y)\}^n = 0 \quad \text{for all } x, y \in I.$$

Since σ is an automorphism of R such that $\sigma \neq 1$, we have

$$(2.1) \quad \{(x^\sigma \circ y^\sigma) - (x \circ y)\}^n = 0 \quad \text{for all } x, y \in I.$$

If σ is an Q -outer automorphism of R , by Chuang ([9], Main Theorem), R also satisfies identity (2.1). Since either $\text{char}(R) > n$ or $\text{char}(R) = 0$, it follows from Lemma 2.2 that

$$(2.2) \quad \{(z \circ w) - (x \circ y)\}^n = 0 \quad \text{for all } x, y, z, w \in R.$$

It is well known that there exists a field F such that R and F_m satisfy the same polynomial identities (see Jacobson, [12], p. 57, p. 89). Let e_{ij} be a matrix unit with 1 in (i, j) -entry and zero elsewhere. Suppose $m \geq 2$, if we choose $z = e_{11}, w = e_{22}, x = e_{12}, y = e_{21}$, then we get a contradiction:

$$\begin{aligned} 0 &= \{(e_{11} \circ e_{22}) - (e_{12} \circ e_{21})\}^n \\ &= (-1)^n e_{11} + (-1)^n e_{22} \neq 0. \end{aligned}$$

This forces $m = 1$, that is R is commutative, a contradiction.

We now assume that σ is Q -inner. So there exists an invertible element $b \in Q$ such that $x^\sigma = bxb^{-1}$ for all $x \in R$. We note that $b \notin C$ since $\sigma \neq 1$. By a theorem of Chuang [8], Q and I satisfy the same generalized polynomial identities (GPI's). From (2.1) we see that

$$(2.3) \quad \{b(x \circ y)b^{-1} - (x \circ y)\}^n = 0 \quad \text{for all } x, y \in Q.$$

Since Q remains prime by the primeness of R , replacing R by Q we may assume that $b \in R$ and C is just the center of R . Note that R is a centrally closed prime C -algebra in the present situation (Erickson et al., [11]), i.e., $RC = R$.

Since $b \notin C$, it is obvious that (2.3) is a nontrivial GPI on R . By Martindale's theorem [15], R is a strongly primitive ring. Let V_R be a faithful irreducible right R -module with commuting division ring D . By the density theorem, R acts densely on ${}_D V$.

For any given $v \in V$, we claim that v and vb are D -dependent. If $vb = 0$, then v and vb are D -dependent and we are done in this case. Suppose that $vb \neq 0$, v and vb are D -independent. We consider the following two cases.

Case 1. Assume that v, vb, vb^{-1} are D -independent, by the density of R , there exist $x, y \in R$ such that

$$\begin{aligned} vx = 0, \quad vbx = vb^{-1}, \quad vb^{-1}x = 0, \\ vy = 0, \quad vby = 0, \quad vb^{-1}y = vb. \end{aligned}$$

From (2.3), we get a contradiction:

$$0 = v\{b(x \circ y)b^{-1} - (x \circ y)\}^n = v \neq 0.$$

Case 2. Otherwise, v, vb, vb^{-1} are D -dependent. Since v and vb are D -independent, we have $vb^{-1} = d_1v + d_2vb$, for some $d_1, d_2 \in D$. Moreover, we claim that $d_2 \neq 0$. Indeed, if $d_2 = 0$, then $vb^{-1} = d_1v$ and $v = d_1vb$, a contradiction. By the density of R there exist $x, y \in R$ such that

$$\begin{aligned} vx = v, \quad vbx = 0, \\ vy = -v, \quad vby = vb \end{aligned}$$

and so from (2.3), we get a contradiction:

$$0 = v\{b(x \circ y)b^{-1} - (x \circ y)\}^n = v2^n \neq 0.$$

From the above, we have proven that v and vb are D -dependent, i.e., $vb = \alpha(v)v$ for all $v \in V$, where $\alpha(v) \in D$ depends on $v \in V$. In fact, it is easy to check that $\alpha(v)$ is independent of the choice of $v \in V$. i.e., there exists $\lambda \in D$ such that $vb = \lambda v$ for all $v \in V$. We claim $\lambda \in Z(D)$, the center of D . Indeed, for any $\beta \in D$,

$$(\beta v)b = \lambda(\beta v) = (\lambda\beta)v$$

and on the other hand

$$(\beta v)b = \beta(vb) = \beta(\lambda v) = (\beta\lambda)v.$$

From the above two expression we can get $\lambda\beta = \beta\lambda$, which implies $\lambda \in Z(D)$. So $b \in C$, a contradiction. Thus $\dim_C RC \leq 4$. By Lemma 2.1, R satisfies s_4 . This completes the proof. \square

Now, we are well equipped to prove our main result.

Theorem 2.2. Let R be a prime ring, I a nonzero ideal of R , and σ a nontrivial automorphism of R such that $\{(x \circ y)^\sigma - (x \circ y)\}^n \in Z$ for all $x, y \in I$, n be fixed positive integer. If either $\text{char}(R) > n$ or $\text{char}(R) = 0$, then R satisfies s_4 .

Proof. We assume that $\dim_C RC > 4$. By assumption we get that,

$$[\{(x \circ y)^\sigma - (x \circ y)\}^n, r] = 0 \quad \text{for all } x, y, r \in I.$$

Since σ is an automorphism of R with $\sigma \neq 1$, an identity map, we have

$$(2.4) \quad [\{ (x^\sigma \circ y^\sigma) - (x \circ y) \}^n, r] = 0 \quad \text{for all } x, y, r \in I.$$

If σ is an Q -outer automorphism of R , by Chuang ([9], Main Theorem), R also satisfies identity (2.4). Since either $\text{char}(R) > n$ or $\text{char}(R) = 0$, it follows from Lemma 2.2 that

$$(2.5) \quad [\{ (z \circ w) - (x \circ y) \}^n, r] = 0 \quad \text{for all } x, y, r, z, w \in R.$$

It is well known that there exists a field F such that R and F_m satisfy the same polynomial identities (see Jacobson, [12], p. 57, p. 89). Let e_{ij} be a matrix unit with 1 in (i, j) -entry and zero elsewhere. Since $\dim_C RC > 4$, we see that $m > 2$. If we choose $z = e_{11}$, $w = e_{22}$, $x = e_{12}$, $y = e_{21}$, $r = e_{13}$, then by (2.5) we get a contradiction as follows

$$0 = [\{ (e_{11} \circ e_{22}) - (e_{12} \circ e_{21}) \}^n, e_{13}] = (-1)^n e_{13} \neq 0.$$

We now assume that σ is Q -inner. So there exists an invertible element $b \in Q$ such that $x^\sigma = bxb^{-1}$ for all $x \in R$. Since $\sigma \neq 1$, we see that $b \notin C$. By a theorem of Chuang [8], Q and I satisfy the same generalized polynomial identities. It follows from (2.4) that

$$(2.6) \quad [\{ b(x \circ y)b^{-1} - (x \circ y) \}^n, r] = 0 \quad \text{for all } x, y, r \in Q.$$

Since $b \notin C$, it is obvious that (2.6) is a nontrivial GPI on Q . By Martindale's theorem [15], Q is a strongly primitive ring. Let V_Q be a faithful irreducible right Q -module with commuting ring $D = \text{End}(V_Q)$, a finite dimensional division algebra over C . By the density theorem, Q acts densely on ${}_D V$.

If $\dim_D V = \infty$, then $\{b(x \circ y)b^{-1} - (x \circ y)\}^n = 0$ holds on H , the socle of Q , and hence it also holds on Q . Thus, by above Theorem 2.1, we prove the theorem in this case. So we must have $\dim_D V < \infty$. Thus Q is isomorphic to D_m , the $m \times m$ matrix ring over D for some m .

If C is finite, then D is a finite division ring and thus is a field by Wedderburn's theorem. In this case, $Q = C_m$. On the other hand, if C is infinite and F is a

maximal subfield of D , then by a Vandermonde determinant argument, we know that the condition

$$\{b(x \circ y)b^{-1} - (x \circ y)\}^n \in C \quad \text{for all } x, y, \in Q.$$

carries over to

$$\{b(x \circ y)b^{-1} - (x \circ y)\}^n \in F \quad \text{for all } x, y, \in Q \otimes_C F.$$

But $Q \otimes_C F = D_m \otimes F = (D \otimes_C F)_m = F_k$ for some k . In either case, we may suppose that $Q = F_k$ for some $k > 1$. Since $\dim_C Q > 4$, we see that $k > 2$. Let V be a k -dimensional vector space over F and Q can be realized as a ring consisting of all F -linear transformations of V . For any given $v \in V$, we claim that v and vb are F -dependent. Suppose, on the contrary, that v and vb are F -independent. We reduce the proof to the following two cases.

Case 1. Assume that v, vb, vb^{-1} are F -independent, we extend v, vb, vb^{-1} to be an F -base $v, vb, vb^{-1}, v_i, \dots, v_k$ of ${}_F V$, where $i = 4, 5, \dots, k$, by the density of Q , there exist $x, y \in Q$ such that

$$\begin{aligned} vx = 0, \quad vbx = vb^{-1}, \quad vb^{-1}x = 0, \quad v_i x = 0, \\ vy = 0, \quad vby = 0, \quad vb^{-1}y = vb, \quad v_i y = 0, \end{aligned}$$

for $i = 4, 5, \dots, k$. And so

$$v(x \circ y) = 0, \quad vb(x \circ y) = vb, \quad vb^{-1}(x \circ y) = vb^{-1}, \quad v_i(x \circ y) = 0,$$

for $i = 4, 5, \dots, k$. Hence $\{(b(x \circ y)b^{-1}) - (x \circ y)\}^n$ is of rank at most 2. Being in F , we get that $\{(b(x \circ y)b^{-1}) - (x \circ y)\}^n = 0$. Thus we have a contradiction:

$$0 = v\{(b(x \circ y)b^{-1}) - (x \circ y)\}^n = v \neq 0.$$

Case 2. Otherwise, v, vb, vb^{-1} are F -dependent. Since v and vb are F -independent, we extend v, vb to be an F -base v, vb, v_i, \dots, v_k of ${}_F V$, where $i = 3, 4, 5, \dots, k$. We have $vb^{-1} = d_1v + d_2vb$, for some $d_1, d_2 \in F$. Moreover, we claim that $d_2 \neq 0$. Indeed, if $d_2 = 0$, then $vb^{-1} = d_1v$ and $v = d_1vb$, a contradiction. By the density of Q there exist $x, y \in Q$ such that

$$\begin{aligned} vx = v, \quad vbx = 0, \quad v_i x = 0, \\ vy = -v, \quad vby = vb, \quad v_i y = 0, \end{aligned}$$

for $i = 3, 4, 5, \dots, k$. And so

$$v(x \circ y) = -2v, \quad vb(x \circ y) = 0, \quad v_i(x \circ y) = 0,$$

for $i = 3, 4, 5, \dots, k$. Hence $\{(b(x \circ y)b^{-1}) - (x \circ y)\}^n$ is of rank at most 2. Being in F , we get that $\{(b(x \circ y)b^{-1}) - (x \circ y)\}^n = 0$. Thus we have a contradiction:

$$0 = v\{(b(x \circ y)b^{-1}) - (x \circ y)\}^n = v2^n \neq 0.$$

From the above, we have proven that v and vb are F -dependent for every $v \in V$. Following the same proof as that of above theorem, we obtain that $b \in F$, a contradiction. Thus $\dim_C RC \leq 4$. In view of Lemma 2.1, R satisfies s_4 . \square

By taking a special case for $n = 1$ of Theorem 2.2, we have the following corollary.

Corollary 2.1. *Let R be a prime ring, I a nonzero ideal of R , and σ a nontrivial automorphism of R such that $\{(x \circ y)^\sigma - (x \circ y)\} \in Z$ for all $x, y \in I$, then R satisfies s_4 .*

The following example demonstrates that R to be prime is essential in Theorem 2.2.

Example 2.1. Let \mathbb{Z} be the ring of all integers. Let

$$R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}, \quad I = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{Z} \right\} \text{ and define}$$

$$\sigma : R \rightarrow R \text{ by } \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}^\sigma = \begin{pmatrix} 0 & -a & b \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix}. \text{ Then } I \text{ is a nonzero ideal of } R,$$

and it is easy to see that σ is an automorphism of R such that $\{(x \circ y)^\sigma - (x \circ y)\}^n \in Z$ for all $x, y \in I$, but R is not commutative. Hence, in theorem the hypothesis of primeness cannot be omitted.

References

- [1] N. Argac, H. G. Inceboz, *Derivation of prime ring and semiprime rings*, J. Korean Math. Soc., **46(5)**(2009), 997-1005.
- [2] M. Ashraf, N. Rehman, *On commutativity of rings with derivations*, Results Math., **42(1-2)**(2002), 3-8.
- [3] K. I. Beidar, W. S. Martindale III, A. V. Mikhalev, *Rings with Generalized Identities*, New York-Basel-Hong Kong: Marcel Dekker, Inc., (1996), 51-95.
- [4] H. E. Bell, W. S. Martindale, *Centralizing mappings of semiprime rings*, Can. Math. Bull., **30**(1987), 92-101.
- [5] M. Brešar, *centralizing mappings and derivations in prime rings*, J. Algebra, **156**(1993), 385-394.
- [6] M. Brešar, *On skew commuting mappings of rings*, Bull. Aust. Math. Soc., **47**(1993), 291-296.
- [7] M. Brešar, *Functional identities: a survey*, Contemporary Math., **259**(2000), 93-109.
- [8] C. L. Chuang, *GPIs having coefficients in Utumi quotient rings*, Proc. Amer. Math. Soc., **103(3)**(1988), 723-728,.

- [9] C. L. Chuang, *Differential identities with automorphisms and antiautomorphisms II*, J. Algebra, **160**(1993), 130-171.
- [10] N. Divinsky, *On commuting automorphisms of rings*, Trans. R. Soc. Can. Sect. III, **49**(1955), 19-22.
- [11] J. S. Erickson, W. S. Martindale, J. M. Osborn, *Prime nonassociative algebras*, Pacific J. Math., **60**(1975), 49-63.
- [12] N. Jacobson, *PI-algebras: An Introduction*, Lecture Notes in Mathematics, Vol. **441**. Berlin HeidelbergNew York: Springer Verlag, (1975).
- [13] C. Lanski, *Left ideals and derivations in semiprime ring*, J. Algebra, **277**(2004), 658-667.
- [14] J. Luh, *A note on commuting automorphisms of rings*, Amer. Math. Mont., **77**(1970), 61-62.
- [15] W. S. Martindale, *Prime rings satisfying a generalized polynomial identity*, J. Algebra, **12**(1969), 576-584.
- [16] J. H. Mayne, *Centralizing automorphisms of prime ring*, Canad. Math. Bull., **19**(1976), 113-115.
- [17] E. C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc., **8**(1957), 1093-1100.
- [18] Y. Wang, *Power-centralizing automorphisms of Lie ideals in prime ring*, Comm. Algebra, **34**(2006), 609-615.