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HOW THE PARAMETER ϵ INFLUENCE THE GROWTH RATES OF THE PARTIAL QUOTIENTS IN GCF$_\epsilon$ EXPANSIONS

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Abstract. For generalized continued fraction (GCF) with parameter $\epsilon(k)$, we consider the size of the set whose partial quotients increase rapidly, namely the set

$$E_\epsilon(\alpha) := \Big\{x \in (0,1] : k_{n+1}(x) \ge k_n(x)^\alpha \text{ for all } n \ge 1\Big\},$$

where $\alpha > 1$. We in [6] have obtained the Hausdorff dimension of $E_\epsilon(\alpha)$ when $\epsilon(k)$ is constant or $\epsilon(k) \sim k^\beta$ for any $\beta \ge 1$. As its supplement, now we show that:

$$\dim_H E_\epsilon(\alpha) = \begin{cases} \frac{1}{\alpha}, & \text{when } -k^\delta \le \epsilon(k) \le k \text{ with } 0 \le \delta < 1; \\ \frac{1}{\alpha+1}, & \text{when } -k-\rho < \epsilon(k) \le -k \text{ with } 0 < \rho < 1; \\ \frac{1}{\alpha+2}, & \text{when } \epsilon(k) = -k-1+\frac{1}{k}. \end{cases}$$

So the bigger the parameter function $\epsilon(k_n)$ is, the larger the size of $E_\epsilon(\alpha)$ becomes.

1. Introduction

In 2003, F. Schweiger [2] introduced a new class of continued fractions with parameters, called generalized continued fractions (GCF$_\epsilon$), which are induced by the map $T_\epsilon : (0,1] \to (0,1]$

$$T_\epsilon(x) := \frac{-1+(k+1)x}{1+\epsilon-k\epsilon x} \quad \text{for } x \in \Big(\frac{1}{k+1}, \frac{1}{k}\Big], \tag{1.1}$$

where the parameter $\epsilon : \mathbb{N} \to \mathbb{R}$ satisfies

$$\epsilon(k) + k + 1 > 0 \quad \text{for all } k \ge 1. \tag{1.2}$$

For any $x \in (0,1]$, its partial quotients $\{k_n\}_{n\ge1}$ in the GCF$_\epsilon$ expansion are defined as

$$k_1 = k_1(x) := \Big\lfloor \frac{1}{x} \Big\rfloor, \quad \text{and} \quad k_n = k_n(x) := k_1\big(T_\epsilon^{n-1}(x)\big).$$

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By the algorithm (1.1), it follows [2] that

$$x = \frac{A_n + B_n T_\epsilon^n(x)}{C_n + D_n T_\epsilon^n(x)} \text{ for all } n \geq 1,$$

where the numbers A_n, B_n, C_n, D_n are given by the recursive relations

$$\begin{pmatrix} C_0 & D_0 \\ A_0 & B_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} C_n & D_n \\ A_n & B_n \end{pmatrix} = \begin{pmatrix} C_{n-1} & D_{n-1} \\ A_{n-1} & B_{n-1} \end{pmatrix} \begin{pmatrix} k_n + 1 & k_n\epsilon(k_n) \\ 1 & 1 + \epsilon(k_n) \end{pmatrix}, \quad n \geq 1, \tag{1.3}$$

It is interesting to see how the parameter functions ϵ influence the growth rates of the partial quotients in GCF_ϵ. Under the condition (1.2), it is easy to see that for any $x \in [0, 1)$, $k_{n+1}(x) \geq k_n(x)$. In [5], it was shown that when $-1 < \epsilon(k) \leq 1$ for all $k \geq 1$, for almost all $x \in [0, 1)$

$$\lim_{n\to\infty} \frac{\log k_n(x)}{n} = 1.$$

As far as a general parameter ϵ is concerned, there is no general result concerning the growth rate of $k_n(x)$. However, it is believed that the bigger the parameter ϵ is, the faster the growth rate of $k_n(x)$ should be. In this paper, we consider this question from the view point of Hausdorff dimension. Namely, we consider the size of the following set:

$$E_\epsilon(\alpha) := \Big\{x \in [0, 1) : k_{n+1}(x) \geq k_n(x)^\alpha \text{ for all } n \geq 1\Big\}.$$

In [6], Zhong and Tang showed that:

Theorem 1.1.

$$\dim_H E_\epsilon(\alpha) = \begin{cases} \frac{1}{\alpha}, & \textit{when } \epsilon(k) \equiv \epsilon_0 \ (\textit{constant}); \\ \frac{1}{\alpha-\beta+1}, & \textit{when } \epsilon(k) \sim k^\beta \textit{ and } \alpha \geq \beta \geq 1; \\ 1, & \textit{when } \epsilon(k) \sim k^\beta \textit{ and } \alpha \leq \beta, \end{cases}$$

where $\dim_H$ *denotes the Hausdorff dimension.*

In this paper, we will prove that:

Theorem 1.2.

$$\dim_H E_\epsilon(\alpha) = \begin{cases} \frac{1}{\alpha}, & \textit{when } -k^\delta \leq \epsilon(k) \leq k \textit{ with } 0 \leq \delta < 1; \\ \frac{1}{\alpha+1}, & \textit{when } -k-\rho < \epsilon(k) \leq -k \textit{ with } 0 < \rho < 1; \\ \frac{1}{\alpha+2}, & \textit{when } \epsilon(k) = -k - 1 + \frac{1}{k}. \end{cases}$$

The above two theorems imply that

(1) The bigger ϵ is, the larger the set in question. This gives some evidence that the bigger ϵ is, the faster the partial quotients k_n grows.

(2) If and only if $-k^\delta \leq \epsilon(k) \leq ck$ (where $0 \leq \delta < 1$ and c is constant), the set $E_\epsilon(\alpha)$ is of Hausdorff dimension $\frac{1}{\alpha}$. This is the same with the Engel series expansion (see [4]).

(3) If $\epsilon = \epsilon(k,t) = -k^t$, then $t = 1$ is a jump discontinuity of $\dim_H E_\epsilon(\alpha)$. In fact, it follows from Theorem 1.2 that

$$\dim_H E_\epsilon(\alpha) = \begin{cases} \frac{1}{\alpha}, & \text{when } 0 \le t < 1; \\ \frac{1}{\alpha+1}, & \text{when } t = 1. \end{cases}$$

2. Preliminary

In this section, we present some simple facts about GCF$_\epsilon$ expansion for later use. The first lemma concerns the relationships between A_n, B_n, C_n, D_n which are recursively defined by (1.3).

Lemma 2.1 ([2, 3, 5])**.** *For all $n \ge 1$ we have*

(i) $C_n = (k_n + 1)C_{n-1} + D_{n-1} > 0$, $C_0 = 1$.
(ii) $D_n = k_n\epsilon(k_n)C_{n-1} + (1 + \epsilon(k_n))D_{n-1}$, $D_0 = 0$, *and* $D_n \ge 0$ *when* $\epsilon \ge 0; D_n < 0$ *when* $\epsilon < 0$.
(iii) $k_nC_n + D_n = (k_nC_{n-1} + D_{n-1})(k_n + 1 + \epsilon(k_n)) > 0$.
(iv) $B_nC_n - A_nD_n = (B_NC_N - A_ND_N)\prod_{i=N+1}^{n}(k_i + 1 + \epsilon(k_i)) > 0, \forall\ 0 \le N < n$.

Now we define the cylinder set as follows. For any non-decreasing integer vector $(k_1, \ldots, k_n)$, define the n-th order cylinders as follows

$$B(k_1, \ldots, k_n) = \{x \in (0, 1] : k_j(x) = k_j, \forall 1 \le j \le n\}.$$

Then it is just the interval with the endpoints $L_n = \frac{A_n}{C_n}$ and $R_n = \frac{k_nA_n+B_n}{k_nC_n+D_n}$. As a consequence, the length of $B(k_1, \ldots, k_n)$ is

$$|B(k_1, k_2, \ldots, k_n)| = \frac{B_nC_n - A_nD_n}{C_n(k_nC_n + D_n)}. \tag{2.1}$$

A further calculation shows ($k_{n+1} = k$)

$$|B(k_1, k_2, \ldots, k_n, k)| = \frac{B_nC_n - A_nD_n}{(kC_n + D_n)((k + 1)C_n + D_n)}. \tag{2.2}$$

From now on until the end of this paper, we fix a point $x \in E_\epsilon(\alpha)$, and let $k_n = k_n(x)$ be the nth partial quotient of x. The numbers A_n, B_n, C_n, D_n be recursively defined by (1.3) for x.

The following simple inequalities will be used frequently. For easy reference, we list them as a lemma.

Lemma 2.2. *When n is large enough (say $n \ge N_1$) we have,*

(a) $k_n - k_{n-1} \ge \frac{k_n}{2}$.
(b) $k_n - k_n^\delta \ge \frac{k_n}{2}$ *for* $\delta < 1$.
(c) $k_n^\alpha - k_n \ge \frac{k_n^\alpha}{2}$.

Proof. Since $x \in E_\epsilon(\alpha)$ with $\alpha > 0$, and k_n is integer, it's obvious that

$$k_n \geq k_n^\alpha \Rightarrow k_n \geq k_{n-1} + 1 \Rightarrow k_n \geq n.$$

So when $n \geq \max\{2^{\frac{1}{\alpha-1}}, 2^{\frac{1}{1-\delta}}\}$, all of the inequalities (a), (b) and (c) hold. □

The next result concerns the growth of $C_n = C_n(x)$.

Lemma 2.3. *For all $n \geq N_1$ we have*

$$C_n \geq \frac{k_n}{2}\frac{k_{n-1}}{2}\cdots\frac{k_{N_1+1}}{2}\frac{k_{N_1}}{2}C_{N_1-1}.$$

Proof. Since $k_i \geq k_{i-1}$ for all $i > 1$, so by using Lemma 2.1(iii) n times we get

$$k_nC_n + D_n \geq (k_1C_0 + D_0)\prod_{i=1}^{n}(k_i + 1 + \epsilon(k_i)) > 0.$$

This gives

$$D_n \geq -k_nC_n \quad \text{for all } n \geq 1. \tag{2.3}$$

Then by Lemma 2.1(i) and Lemma 2.2(a), we get

$$\begin{aligned} C_n &\geq (k_n + 1)C_{n-1} - k_{n-1}C_{n-1} \\ &\geq (k_n + 1 - k_{n-1})C_{n-1} \\ &\geq \frac{k_n}{2}C_{n-1} \qquad \text{when } n \geq N_1. \end{aligned} \tag{2.4}$$

Iterating this process enables us to conclude the result. □

The second one concerns the growth of $k_n^\alpha C_n + D_n$.

Lemma 2.4. *For any $n \geq N_1$ we have*

$$k_n^\alpha C_n + D_n \geq \frac{k_n^\alpha}{2}\frac{k_{n-1}^\alpha}{2}\frac{k_{n-2}^\alpha}{2}\cdots\frac{k_{N_1}^\alpha}{2}C_{N_1}.$$

Proof. By (2.3) and Lemma 2.2(c), we get

$$k_n^\alpha C_n + D_n \geq (k_n^\alpha - k_n)C_n \geq \frac{k_n^\alpha}{2}C_n, \quad \text{when } n \geq N_1. \tag{2.5}$$

Using $k_n \geq k_{n-1}^\alpha$, the result (2.4) also gives that

$$\begin{aligned} C_n &\geq \frac{k_n}{2}C_{n-1} \geq \frac{k_{n-1}^\alpha}{2}C_{n-1} \\ &\geq \cdots \\ &\geq \frac{k_{n-1}^\alpha}{2}\frac{k_{n-2}^\alpha}{2}\cdots\frac{k_{N_1}^\alpha}{2}C_{N_1}. \end{aligned} \tag{2.6}$$

Substituting (2.6) into (2.5) to get the result. □

The following corollary will be used for getting the upper bound of $\dim_H E_\epsilon(\alpha)$.

Corollary 2.5. *Let*

$$L_1 = \frac{3^{N_1-1}k_1k_2\cdots k_{N_1-1}}{C_{N_1}C_{N_1-1}}, L_2 = \frac{1}{C_{N_1}C_{N_1-1}} \text{ and } L_3 = \frac{2/(k_1k_2\cdots k_{N_1-1})}{C_{N_1}C_{N_1-1}},$$

where N_1 is given by Lemma 2.2. Then for any $n \geq N_1$ we have

(1) *If $\epsilon(k_n) \leq k_n$ for all $n \geq 1$, then*

$$\frac{B_nC_n - A_nD_n}{C_n(k_n^\alpha C_n + D_n)} \leq L_1 \cdot \frac{12}{k_n^\alpha}\frac{12}{k_{n-1}^\alpha}\cdots\frac{12}{k_{N_1}^\alpha};$$

(2) *If $\epsilon(k_n) \leq -k_n$ for all $n \geq 1$, then*

$$\frac{B_nC_n - A_nD_n}{C_n(k_n^\alpha C_n + D_n)} \leq L_2 \cdot \frac{4}{k_n^{1+\alpha}}\frac{4}{k_{n-1}^{1+\alpha}}\cdots\frac{4}{k_{N_1}^{1+\alpha}};$$

(3) *If $\epsilon(k_n) = -k_n - 1 + \frac{1}{k}$ for all $n \geq 1$, then*

$$\frac{B_nC_n - A_nD_n}{C_n(k_n^\alpha C_n + D_n)} \leq L_3 \cdot \frac{4}{k_n^{2+\alpha}}\frac{4}{k_{n-1}^{2+\alpha}}\cdots\frac{4}{k_{N_1}^{2+\alpha}}.$$

Proof. From Lemma 2.1(iv) we can get immediate

$$B_nC_n - A_nD_n \leq 3^n k_1k_2\cdots k_n \qquad \text{when } \epsilon(k_n) \leq k_n \text{ for all } n \geq 1;$$
$$B_nC_n - A_nD_n \leq 1 \qquad \text{when } \epsilon(k_n) \leq -k_n \text{ for all } n \geq 1;$$
$$B_nC_n - A_nD_n = \frac{1}{(k_1k_2\cdots k_n)} \qquad \text{when } \epsilon(k_n) = -k_n - 1 + \frac{1}{k} \text{ for all } n \geq 1.$$

Combining these with Lemma 2.3 and the Lemma 2.4, we get the three results. □

The following results will be used for getting the lower bound of $\dim_H E_\epsilon(\alpha)$

Lemma 2.6. *For any $n \geq N_1$ we have,*

(1) $\frac{(k_n^\alpha - k_n)C_n}{k_n^\alpha C_n + D_n} \geq \frac{1}{4}$ *if $\epsilon(k_n) \leq k_n$,*

(2) $C_n(k_nC_n + D_n) \leq 2^{2n-2N_1+1}k_n(k_nk_{n-1}\cdots k_{N_1+1}C_{N_1})^2$ *if $\epsilon(k_n) \leq k_n$,*

(3) $C_n(k_nC_n + D_n) \leq 2^{2n-2N_1}(k_nk_{n-1}\cdots k_{N_1+1}C_{N_1})^2$ *if $\epsilon(k_n) \leq -k_n$.*

Proof. We first show $D_n \leq k_nC_n$ for $\epsilon(k_n) \leq k_n$. This is true for $n = 1$.

Furthermore, suppose $D_{n-1} \leq k_{n-1}C_{n-1}$. Then by Lemma 2.1 and $\epsilon(k_{n-1}) \leq k_{n-1}$,

$$\begin{aligned} D_n &= k_n\epsilon(k_n)C_{n-1} + \epsilon(k_n)D_{n-1} + D_{n-1} \\ &\leq k_n^2C_{n-1} + k_nD_{n-1} + k_{n-1}C_{n-1} \\ &\leq k_n^2C_{n-1} + k_nD_{n-1} + k_nC_{n-1} = k_nC_n. \end{aligned}$$

Thus when $\epsilon(k_n) \leq k_n$ we have

$$k_n^\alpha C_n + D_n \leq (k_n^\alpha + k_n)C_n \leq 2k_n^\alpha C_n, \tag{2.7}$$

and by Lemma 2.1(i),

$$C_n = (k_n + 1)C_{n-1} + D_{n-1}$$

$$\leq (k_n + 1 + k_{n-1})C_{n-1}$$
$$\leq 2k_n C_{n-1}.$$

By induction one has that, for any $1 \leq N_1 < n$,

$$C_n \leq 2^{n-N_1}(k_n k_{n-1} \cdots k_{N_1+1})C_{N_1}. \tag{2.8}$$

From Lemma 2.2(c) we have $k_n^\alpha - k_n \geq \frac{k_n^\alpha}{2}$ when $n \geq N_1$. Combine this with (2.7) to get that

$$\frac{(k_n^\alpha - k_n)C_n}{k_n^\alpha C_n + D_n} \geq \frac{\frac{1}{2}k_n^\alpha C_n}{2k_n^\alpha C_n} = \frac{1}{4} \quad \text{when } n \geq N_1.$$

Combine (2.7) and (2.8), we get

$$C_n(k_n C_n + D_n) \leq 2^{2n-2N_1+1}k_n(k_n k_{n-1} \cdots k_{N_1+1}C_{N_1})^2 \quad \text{when } \epsilon(k_n) \leq k_n.$$

In case of $\epsilon(k_n) \leq -k_n$. By using Lemma 2.1(i),(ii), we get

$$\begin{aligned} D_n &\leq -k_n^2 C_{n-1} + (1 - k_n)D_{n-1} \\ &= -k_n C_n + k_n C_{n-1} + D_{n-1} \\ &= -k_n C_n + C_n - C_{n-1}. \end{aligned}$$

It gives that

$$D_n + k_n C_n \leq C_n \quad \text{when } \epsilon(k_n) \leq -k_n.$$

Thus by (2.8), we obtain

$$C_n(k_n C_n + D_n) \leq C_n^2 \leq 2^{2n-2N_1}(k_n k_{n-1} \cdots k_{N_1+1}C_{N_1})^2 \quad \text{when } \epsilon(k_n) \leq -k_n.$$

□

Corollary 2.7. *Let* $L_4 = \frac{B_{N_1}C_{N_1} - A_{N_1}D_{N_1}}{2^{3-3N_1}(C_{N_1})^2}$, $L_5 = \frac{(1-\rho)^{N_1}}{2^{2-2N_1}(C_{N_1})^2}$ *and* $L_6 = \frac{1/(k_1 k_2 \cdots k_{N_1-1})}{2^{2-2N_1}(C_{N_1})^2}$ *be three constants, where* N_1 *is given by Lemma 2.2. Then for any* $n \geq N_1$ *we have*

(1) *If* $-k_n^\delta \leq \epsilon(k_n) \leq k_n$ *with* $0 \leq \delta < 1$ *for all* $n \geq 1$*, then*

$$\frac{B_n C_n - A_n D_n}{C_n(k_n C_n + D_n)} \cdot \frac{(k_n^\alpha - k_n)C_n}{k_n^\alpha C_n + D_n} \geq L_4 \cdot \frac{1}{2^{3n}k_n(k_n k_{n-1} \cdots k_{N_1+1})};$$

(2) *If* $-k_n - \rho < \epsilon(k_n) \leq -k_n$ *for all* $n \geq 1$*, then*

$$\frac{B_n C_n - A_n D_n}{C_n(k_n C_n + D_n)} \cdot \frac{(k_n^\alpha - k_n)C_n}{k_n^\alpha C_n + D_n} \geq L_5 \cdot \frac{(1-\rho)^{n-N_1}}{2^{2n}(k_n k_{n-1} \cdots k_{N_1+1})^2};$$

(3) *If* $\epsilon(k_n) = -k_n - 1 + \frac{1}{k_n}$ *for all* $n \geq 1$*, then*

$$\frac{B_n C_n - A_n D_n}{C_n(k_n C_n + D_n)} \cdot \frac{(k_n^\alpha - k_n)C_n}{k_n^\alpha C_n + D_n} \geq L_6 \cdot \frac{1}{2^{2n}(k_n k_{n-1} \cdots k_{N_1+1})^3}.$$

Proof. By Lemma 2.1(iv) and Lemma 2.2(b), when $n \geq N_1$ we have,

(1) If $\epsilon(k_n) \geq -k_n^\delta$ with $0 \leq \delta < 1$, then

$$B_nC_n - A_nD_n \geq \frac{k_n}{2}\frac{k_{n-1}}{2}\cdots\frac{k_{N_1+1}}{2}(B_{N_1}C_{N_1} - A_{N_1}D_{N_1});$$

(2) If $-k-\rho < \epsilon(k_n) \leq -k_n$ with $0 < \rho < 1$, then

$$B_nC_n - A_nD_n \geq (1-\rho)^n;$$

(3) If $\epsilon(k_n) = -k_n - 1 + \frac{1}{k_n}$, then

$$B_nC_n - A_nD_n = \frac{1}{(k_1k_2\cdots k_n)}.$$

Combine these with Lemma 2.6 to get the above three results. □

3. The Hausdorff dimension of $E_\epsilon(\alpha)$

The proof of Theorem 1.2 is divided into two parts: one for upper bound, the other for lower bound.

3.1. Upper bound

For any non-decreasing integer vector $(k_1, k_2, \ldots, k_n)$, define

$$I(k_1, k_2, \ldots, k_n) = \bigcup_{k=k_n^\alpha}^{\infty} cl\{x \in (0,1) : k_i(x) = k_i,\ \forall 1 \leq i \leq n,\ k_{n+1}(x) = k\}.$$

Then it is clear that

$$E_\epsilon(\alpha) \subset \bigcap_{n=1}^{\infty} \bigcup_{\substack{k_{i+1} \geq k_i^\alpha \\ 1 \leq i \leq n-1}} I(k_1, k_2, \ldots, k_n). \tag{3.1}$$

So by (2.2), one has

$$\begin{aligned} |I(k_1, k_2, \ldots, k_n)| &= \sum_{k=k_n^\alpha}^{\infty} \frac{B_nC_n - A_nD_n}{((k+1)C_n + D_n)(kC_n + D_n)} \\ &= \frac{B_nC_n - A_nD_n}{C_n} \sum_{k=k_n^\alpha}^{\infty} \Big(\frac{1}{kC_n + D_n} - \frac{1}{(k+1)C_n + D_n}\Big) \\ &= \frac{B_nC_n - A_nD_n}{C_n(k_n^\alpha C_n + D_n)}. \end{aligned} \tag{3.2}$$

(i) Let's first consider the case of $-k^\delta \leq \epsilon(k) \leq k$ with $0 \leq \delta < 1$.

Since the series $\sum_{k=1}^{\infty}\left(\frac{1}{k^\alpha}\right)^{s_1}$ is convergent for any $s_1 > 1/\alpha$, there exists an integer n_1 large enough such that for any $n \geq n_1$,

$$\sum_{k=n}^{\infty}\Big(\frac{12}{k^\alpha}\Big)^{s_1} \leq 1. \tag{3.3}$$

By (3.1), (3.2), (3.3) and Corollary 2.5(1), we find that for any $n \geq N = \max\{N_1, n_1\}$, the s_1-dimensional Hausdorff measure of $E_\epsilon(\alpha)$ can be estimated as

$$\begin{aligned}
&\mathcal{H}^{s_1}(E_\epsilon(\alpha)) \\
\leq\ &\liminf_{n\to\infty} \sum_{\substack{k_{i+1} \geq k_i^\alpha \\ 1 \leq i \leq n-1}} |I(k_1, k_2, \ldots, k_n)|^{s_1} \\
\leq\ &\liminf_{n\to\infty} \sum_{\substack{k_{i+1} \geq k_i^\alpha \\ 1 \leq i \leq N-1}} L_1(\alpha)^{s_1} \sum_{k_{N+1} \geq k_N^\alpha} \left(\frac{12}{k_{N+1}^\alpha}\right)^{s_1} \cdots \sum_{k_n \geq k_{n-1}^\alpha} \left(\frac{12}{k_{n-1}^\alpha}\right)^{s_1} \\
\leq\ &\liminf_{n\to\infty} \sum_{\substack{k_{i+1} \geq k_i^\alpha \\ 1 \leq i \leq N-1}} L_1(\alpha)^{s_1} < \infty.
\end{aligned}$$

which gives that $\dim_H E_\epsilon(\alpha) \leq s_1$. Since $s_1 > \frac{1}{\alpha}$ is arbitrary, we get $\dim_H E_\epsilon(\alpha) \leq \frac{1}{\alpha}$.

(ii) In case of $-k - \rho < \epsilon(k) \leq -k$ with constant $0 < \rho < 1$. By using Corollary 2.5(2), we can prove, in the same way as we prove (i) that $\dim_H E_\epsilon(\alpha) \leq \frac{1}{\alpha+1}$.

(iii) In case of $\epsilon(k) = -k - 1 + \frac{1}{k}$. By using Corollary 2.5(3), we can also prove, in the same way as we prove (i) that $\dim_H E_\epsilon(\alpha) \leq \frac{1}{\alpha+2}$.

3.2. Lower bound

In order to estimate the lower bound, we recall the classical dimensional result concerning a specially defined Cantor set.

Lemma 3.1 (Falconer [1]). *Let $I = E_0 \supset E_1 \supset E_2 \supset \cdots$ be a decreasing sequence of sets, with each E_n, a union of a finite number of disjoint closed intervals. If each interval of E_{n-1} contains at least m_n intervals of $E_n (n = 1, 2, \ldots)$ which are separated by gaps of at least η_n, where $0 < \eta_{n+1} < \eta_n$ for each n. Then the lower bound of the Hausdorff dimension of E can be given by the following inequality*:

$$\dim_H \Big(\cap_{n\geq 1} E_n \Big) \geq \liminf_{n\to\infty} \frac{\log(m_1 m_2 \cdots m_{n-1})}{-\log(m_n \eta_n)}.$$

Let $f_n(\alpha) = 2^{\alpha+\alpha^2+\cdots+\alpha^n} = 2^{\frac{\alpha^{n+1}-\alpha}{\alpha-1}}$. Define

$$E(f) = \{x \in (0,1) : \ f_n(\alpha) \leq k_n(x) < 2f_n(\alpha) \ \forall n \geq 1\}. \tag{3.4}$$

It is easy to see that, when $x \in E(f)$, we have

$$k_n(x)^\alpha < (2f_n(\alpha))^\alpha = f_{n+1}(\alpha) \leq k_{n+1}(x).$$

This implies that

$$E(f) \subset E_\epsilon(\alpha).$$

For each $n \geq 1$, let $E_n(f)$ be the collection of cylinders

$$\bigcup_{j=k_n^\alpha}^{\infty} \big\{B(k_1, \ldots, k_n): \ f_i(\alpha) \leq k_i(x) < 2f_i(\alpha) \ \forall \ 1 \leq i \leq n, \ k_{n+1}(x) = j\big\}.$$

Then

$$E(f) = \bigcap_{n=1}^{\infty} E_n(f)$$

and $E(f)$ fulfills the construction of the Cantor set in Lemma 3.1. Now we specify the integers $\{m_n, n \geq 1\}$ and the real numbers $\{\eta_n, n \geq 1\}$.

Due to the definition of E_n, each interval of E_{n-1} contains $m_n = f_n(\alpha) = 2^{\frac{\alpha^{n+1}-\alpha}{\alpha-1}}$ intervals of E_n, and

$$m_1 m_2 \cdots m_{n-1} \geq 2^{\sum_{i=1}^{n-1} \frac{\alpha^{i+1}-\alpha}{\alpha-1}} = 2^{\frac{\alpha^{n+1}-n\alpha^2+(n-1)\alpha}{(\alpha-1)^2}}. \tag{3.5}$$

In addition, any two of intervals in E_n are separated by at least an interval $J_n(f)$ defined by

$$\bigcup_{j=k_n}^{k_n^\alpha - 1} \big\{B(k_1, \ldots, k_n): \ f_i(\alpha) \leq k_i(x) < 2f_i(\alpha), \ \forall \ 1 \leq i \leq n, \ k_{n+1}(x) = j\big\}.$$

From (2.2) we get

$$\begin{aligned} |J_n(f)| &= \sum_{j=k_n}^{k_n^\alpha - 1} \frac{B_n C_n - A_n D_n}{((j+1)C_n + D_n)(jC_n + D_n)} \\ &= \frac{B_n C_n - A_n D_n}{C_n} \sum_{j=k_n}^{k_n^\alpha - 1} \Big(\frac{1}{jC_n + D_n} - \frac{1}{(j+1)C_n + D_n}\Big) \\ &= \frac{B_n C_n - A_n D_n}{C_n} \Big(\frac{1}{k_n C_n + D_n} - \frac{1}{k_n^\alpha C_n + D_n}\Big) \\ &= \frac{B_n C_n - A_n D_n}{C_n(k_n C_n + D_n)} \cdot \frac{(k_n^\alpha - k_n) C_n}{(k_n^\alpha C_n + D_n)}. \end{aligned} \tag{3.6}$$

Thus and by Corollary 2.7(1), we get when $-k^\delta \leq \epsilon(k) \leq k$ with $0 < \delta < 1$,

$$|J_n(f)| \geq L_4 \cdot \frac{1}{2^{3n} k_n (k_n k_{n-1} \cdots k_{N+1})}; \tag{3.7}$$

And when $-k - \rho < \epsilon(k) \leq -k$ with $0 < \rho < 1$, by Corollary 2.7(2), we get

$$|J_n(f)| \geq L_5 \cdot \frac{(1-\rho)^{n-N_1}}{2^{2n} (k_n k_{n-1} \cdots k_{N+1})^2}. \tag{3.8}$$

When $\epsilon(k) = -k - 1 + \frac{1}{k}$, by Corollary 2.7(3), we get

$$|J_n(f)| \geq L_6 \cdot \frac{1}{2^{2n} (k_n k_{n-1} \cdots k_{N+1})^3}. \tag{3.9}$$

In view of (3.4), the partial quotients k_n satisfying that $2^{\frac{\alpha^{n+1}-\alpha}{\alpha-1}} \le k_n \le 2^{\frac{\alpha^{n+1}-1}{\alpha-1}}$ for all $n \ge 1$. Therefore,

(i) when $-k^\delta \le \epsilon(k) \le k$ with $0 < \delta < 1$,

$$\begin{aligned}|J_n(f)| &\ge L_4 \left(2^{3n} \cdot 2^{\frac{\alpha^{n+1}-1}{\alpha-1}} \cdot \left(2^{\sum_{i=N+1}^{n} \frac{\alpha^{i+1}-1}{\alpha-1}}\right)\right)^{-1} \\ &\ge L_4 \left(2^{3n} \cdot 2^{\frac{\alpha^{n+1}-1}{\alpha-1}} \cdot \left(2^{\frac{\alpha^{n+2}-\alpha^{N+2}-(n-N)(\alpha-1)}{(\alpha-1)^2}}\right)\right)^{-1} =: \eta_n.\end{aligned}$$

(ii) when $-k-\rho < \epsilon(k) \le -k$ with $0 < \rho < 1$,

$$\begin{aligned}|J_n(f)| &\ge L_5 \cdot (1-\rho)^{n-N_1} \left(2^{2n} \cdot \left(2^{\sum_{i=N+1}^{n} \frac{\alpha^{i+1}-1}{\alpha-1}}\right)^2\right)^{-1} \\ &\ge L_5 \cdot (1-\rho)^{n-N_1} \left(2^{2n} \cdot \left(2^{\frac{\alpha^{n+2}-\alpha^{N+2}-(n-N)(\alpha-1)}{(\alpha-1)^2}}\right)^2\right)^{-1} =: \eta_n'.\end{aligned}$$

(iii) when $\epsilon(k) = -k-1+\frac{1}{k}$,

$$\begin{aligned}|J_n(f)| &\ge L_6 \left(2^{2n} \cdot \left(2^{\sum_{i=N+1}^{n} \frac{\alpha^{i+1}-1}{\alpha-1}}\right)^3\right)^{-1} \\ &\ge L_6 \left(2^{2n} \cdot \left(2^{\frac{\alpha^{n+2}-\alpha^{N+2}-(n-N)(\alpha-1)}{(\alpha-1)^2}}\right)^3\right)^{-1} =: \eta_n''.\end{aligned}$$

As a result of (3.5), in the case (i), we get

$$\begin{aligned}\liminf_{n\to\infty} \frac{\log_2(m_1\cdots m_{n-1})}{\alpha^{n+1}} &\ge \frac{1}{(\alpha-1)^2}, \\ \lim_{n\to\infty} \frac{-\log_2 m_n\eta_n}{\alpha^{n+1}} &= \frac{\alpha}{(\alpha-1)^2}.\end{aligned}$$

Combining this with Lemma 3.1, we get when $-k^\delta \le \epsilon(k) \le k$ with $0 < \delta < 1$,

$$\dim_H E_\epsilon(\alpha) \ge \dim_H E(f) \ge \frac{1}{\alpha}.$$

Similarly, when $-k-\rho < \epsilon(k) \le -k$ with $0 < \rho < 1$ (case ii),

$$\dim_H E_\epsilon(\alpha) \ge \frac{1}{\alpha+1}.$$

And when $\epsilon(k) = -k-1-\frac{1}{k}$ (case iii),

$$\dim_H E_\epsilon(\alpha) \ge \frac{1}{\alpha+2}.$$

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