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LINEAR PRESERVERS OF BOOLEAN RANK BETWEEN DIFFERENT MATRIX SPACES

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Abstract. The Boolean rank of a nonzero $m \times n$ Boolean matrix A is the least integer k such that there are an $m \times k$ Boolean matrix B and a $k \times n$ Boolean matrix C with $A = BC$. We investigate the structure of linear transformations $T : \mathbb{M}_{m,n} \to \mathbb{M}_{p,q}$ which preserve Boolean rank. We also show that if a linear transformation preserves the set of Boolean rank 1 matrices and the set of Boolean rank k matrices for any k, $2 \leq k \leq \min\{m, n\}$ (or if T strongly preserves the set of Boolean rank 1 matrices), then T preserves all Boolean ranks.

1. Introduction

In 1897, Frobenius initiated the study of linear preservers when he investigated linear operators on the space of square matrices that preserve the determinant [5]. He found that if T leaves the determinant function invariant, then for some fixed nonsingular matrices U and V such that $\det UV = 1$, $T(X) = UXV$ for any matrix X. Since that time, a lot of effort has gone into investigations of linear operators that leave various functions, sets or relations invariant. In 1959, Marcus and Moyls [8] studied rank preservers and rank 1 preservers. They showed that if the matrices are over an algebraically closed field of characteristic 0, then rank preservers or preservers of the set of rank 1 matrices are of the form $T(X) = UXV$ for some fixed nonsingular matrices U and V, or $T(X) = UX^tV$ for some fixed nonsingular matrices U and V.

The study of preserver problems between different matrix spaces (or tensor spaces) over fields began in 1967 with Westwick [9] investigating the preservers of decomposable tensors (rank one matrices). This was continued in 1975 by Lim [7] and more recently by Li, Rodman and Šemrl [6] investigating rank k preservers and preservers of matrices of rank at most k. The characterizations

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that they give for such preservers over fields are very similar to those in our Theorem 5.1.

Beasley and his colleagues ([1]-[3]) have investigated the linear operators that preserve Boolean rank of a Boolean matrix. Recently, Beasley and Song obtained characterizations of the linear transformations that preserve term rank between different matrix spaces over semirings containing the binary Boolean semiring in [4]. These characterizations show that the forms of linear preservers are more complicated than forms of linear preservers between the same matrix spaces.

In this paper we investigate Boolean linear transformations $T : \mathbb{M}_{m,n} \to \mathbb{M}_{p,q}$ that preserve Boolean rank between different matrix spaces. We show that a Boolean linear transformation $T : \mathbb{M}_{m,n} \to \mathbb{M}_{p,q}$ preserves Boolean rank if and only if it preserves Boolean ranks 1 and k for any k with $2 \leq k \leq \min\{m, n\}$. We also obtain forms of Boolean linear transformations that preserve Boolean rank between different matrix spaces.

2. Preliminaries

The *binary Boolean semiring* consists of the set $\mathbb{B} = \{0, 1\}$ equipped with two binary operations, addition and multiplication. The operations are defined as usual except that $1 + 1 = 1$.

Let $\mathbb{M}_{m,n}$ denote the set of all $m \times n$ Boolean matrices, that is, the set of $m \times n$ matrices with entries in $\mathbb{B}$. The usual definitions for adding and multiplying matrices apply to Boolean matrices as well. From now on, we assume that $1 \leq m \leq n$.

The *Boolean rank*, $b(A)$, of nonzero $A \in \mathbb{M}_{m,n}$ is the least integer k such that there are Boolean matrices $B \in \mathbb{M}_{m,k}$ and $C \in \mathbb{M}_{k,n}$ with $A = BC$. It follows that $1 \leq b(A) \leq m$ for all nonzero $A \in \mathbb{M}_{m,n}$. The Boolean rank of the zero Boolean matrix O is 0. A Boolean rank 1 matrix is of the form $\boldsymbol{x}\boldsymbol{y}^t$ for nonzero vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. It is easily seen that if A has Boolean rank k, then A is the sum of k matrices of Boolean rank 1.

A mapping $T : \mathbb{M}_{m,n} \to \mathbb{M}_{p,q}$ is called a *Boolean linear transformation* if $T(\alpha A + \beta B) = \alpha T(A) + \beta T(B)$ for all $A, B \in \mathbb{M}_{m,n}$ and for all $\alpha, \beta \in \mathbb{B}$.

Let $1 \leq k \leq m$. For a Boolean linear transformation $T : \mathbb{M}_{m,n} \to \mathbb{M}_{p,q}$, we say that

(1) T *preserves Boolean rank* k if $b(T(X)) = k$ whenever $b(X) = k$ for all X;
(2) T *strongly preserves Boolean rank* k provided that $b(T(X)) = k$ if and only if $b(X) = k$ for all X;
(3) T *preserves Boolean rank* if it preserves Boolean rank k for all $k \in \{1, \ldots, m\}$.

Further, T is *nonsingular* if and only if $T(X) = O$ implies that $X = O$. Note that over semirings, a linear transformation may be nonsingular but not be

invertible. For example, if $T(X) = A$ for some fixed nonzero A and for all $X \in \mathbb{M}_{m,n}$, then T is nonsingular, but not invertible.

The matrix $J_{m,n}$ is the $m \times n$ Boolean matrix all of whose entries are 1. When $n = 1$ the matrix $J_{m,1}$ is thought of as a vector and is written $\boldsymbol{j}_m$. A matrix in $\mathbb{M}_{m,n}$ is called a *cell* if it has exactly one 1 entry. We denote the cell whose one 1 entry is in the $(i,j)^{th}$ position by $E_{i,j}$. Further we let $\mathbb{E}_{m,n}$ be the set of all cells in $\mathbb{M}_{m,n}$. That is, $\mathbb{E}_{m,n} = \{E_{i,j} \mid 1 \le i \le m,\ 1 \le j \le n\}$. For sets G and H, $G \setminus H = \{x \in G : x \notin H\}$ so that $H \subseteq G$ is not required.

If A and B are matrices in $\mathbb{M}_{m,n}$, we say that A *dominates* B (written $B \sqsubseteq A$ or $A \sqsupseteq B$) if $a_{i,j} = 0$ implies $b_{i,j} = 0$ for all i and j. This provides a reflexive, antisymmetric and transitive relation on $\mathbb{M}_{m,n}$. For matrices A and B in $\mathbb{M}_{m,n}$ with $B \sqsubseteq A$, we define $A \setminus B$ to be the matrix C such that $c_{i,j} = 1$ if and only if $a_{i,j} = 1$ and $b_{i,j} = 0$ for all i and j. Further, $R_i = \sum_{l=1}^{n} E_{i,l}$ is the ith (*full*) *row matrix* and $C_j = \sum_{s=1}^{m} E_{s,j}$ is the jth (*full*) *column matrix*. In particular, $J_{m,n} \setminus R_i$ is the matrix with zeros in the ith row and 1's elsewhere.

We shall use the notation $[\![1,\ell]\!]$ to denote the set $\{1, 2, \ldots, \ell\}$. In the following we use "rank" and "linear transformation" instead of "Boolean rank" and "Boolean linear transformation", respectively.

3. Preservers of rank 1

Recall that a mapping $\mathbf{f}$ is nonsingular if and only if $\mathbf{f}(X) = O$ implies that $X = O$. For $m = 1$ we observe the following:

Theorem 3.1. *Let $L : \mathbb{M}_{1,n} \to \mathbb{M}_{p,q}$ be a mapping. Then L preserves rank 1 if and only if there exist nonsingular mappings $\mathbf{f} : \mathbb{M}_{1,n} \to \mathbb{M}_{p,1}$ and $\mathbf{g} : \mathbb{M}_{1,n} \to \mathbb{M}_{1,q}$ such that $L(X) = \mathbf{f}(X)\mathbf{g}(X)$ for all $X \in \mathbb{M}_{1,n}$.*

Proof. If L preserves rank 1, then each element in the image is rank 1 and hence can be factored as an outer product of vectors, i.e., $L(X) = \boldsymbol{u}(X)\boldsymbol{v}(X)^t$. Letting $\mathbf{f}(X) = \boldsymbol{u}(X)$ and $\mathbf{g}(X) = \boldsymbol{v}(X)^t$ defines the needed mappings. The converse is obvious. □

One may observe by the above theorem that if $T : \mathbb{M}_{1,n} \to \mathbb{M}_{p,q}$ is any linear transformation that preserves rank 1, then the image of T must be a rank 1 space. In [1, Theorem 5.1] there is a classification of all rank 1 subspaces of $\mathbb{M}_{p,q}$. However, here, the dimension of the subspace can be at most n. Thus, a rank 1 preserver $T : \mathbb{M}_{1,n} \to \mathbb{M}_{p,q}$ can be classified as any linear transformation whose image is one of those subspaces of dimension at most n. Below in Theorem 3.4 we give a constructive characterization. We first need some definitions and results from [1].

Let $A = \boldsymbol{u}\boldsymbol{v}^t$ be a rank 1 matrix. The *perimeter* of A, $p(A)$, is k where k is the number of nonzero entries in $\boldsymbol{u}$ plus the number of nonzero entries in $\boldsymbol{v}$. Two rank 1 matrices, A and B, have a common left factor if there are vectors $\boldsymbol{u}, \boldsymbol{x}$ and $\boldsymbol{y}$ such that $A = \boldsymbol{u}\boldsymbol{x}^t$ and $B = \boldsymbol{u}\boldsymbol{y}^t$. They have a common right factor if there are vectors $\boldsymbol{v}, \boldsymbol{x}$ and $\boldsymbol{y}$ such that $A = \boldsymbol{x}\boldsymbol{v}^t$ and $B = \boldsymbol{y}\boldsymbol{v}^t$. Further we

say that they have a *common factor* if they have either a common right factor or a common left factor.

Lemma 3.2 ([1, Lemma 2.6.1])**.** *If $A \sqsubseteq B$ and $b(A) = b(B) = 1$, then $p(A) < p(B)$ unless $A = B$.*

Lemma 3.3 ([1, Lemma 2.6.2])**.** *If A, B and $A + B$ are rank 1 matrices and neither $A \sqsubseteq B$ nor $B \sqsubseteq A$, then A, B and $A + B$ have a common factor.*

Note that any nonsingular linear transformation $T : \mathbb{M}_{m,n} \to \mathbb{M}_{p,q}$ preserves rank 1 if $\min\{p, q\} = 1$. The case of $m = 1$ is characterized in:

Theorem 3.4. *Let $T : \mathbb{M}_{1,n} \to \mathbb{M}_{p,q}$ be a linear transformation. Then T preserves rank 1 if and only if for some $r \leq n$, there exists an ordered partition $(F_1, \ldots, F_r)$ of $[\![1, n]\!]$, together with two ordered, pairwise disjoint sets of indices, $(I_1, \ldots, I_r)$ of $[\![1, p]\!]$ and $(J_1, \ldots, J_r)$ of $[\![1, q]\!]$ such that for all $k \in [\![1, r]\!]$, either*

- *there is a cover $\{A_{k,i} : i \in F_k\}$ of subsets of I_k such that for all $i \in F_k$,*
$$T(E_{1,i}) = \sum_{s \in I_1 \cup \cdots \cup I_{k-1} \cup A_{k,i}} \ \sum_{t \in J_1 \cup \cdots \cup J_k} E_{s,t}$$

or

- *there is a cover $\{B_{k,i} : i \in F_k\}$ of subsets of J_k such that for all $i \in F_k$,*
$$T(E_{1,i}) = \sum_{s \in I_1 \cup \cdots \cup I_k} \ \sum_{t \in J_1 \cup \cdots \cup J_{k-1} \cup B_{k,i}} E_{s,t}.$$

Proof. A routine examination shows that the image of T defined in either case contains only O and rank 1 matrices. Thus T preserves rank 1.

To prove the direct implication, we proceed by induction on n. If $n = 1$ or $n = 2$, the theorem is easily verified. Assume the conclusion holds for any $k < n$. For each $i = 1, \ldots, n$, let $T(E_{1,i}) = M_i$. Choose i_0 such that M_{i_0} has minimum perimeter, and let $M_{i_0} = \boldsymbol{u}\boldsymbol{v}^t$ be a rank 1 factorization of M_{i_0}. By permuting we may assume without loss of generality that $u_1 = v_1 = 1$. For each $i \neq i_0$, either $M_i \sqsupseteq M_{i_0}$ or M_i and M_{i_0} have a common factor. Suppose that there exist i and j such that $M_i \not\sqsubseteq M_{i_0}$ and M_i and M_{i_0} have a common left factor, and $M_j \not\sqsubseteq M_{i_0}$ and M_j and M_{i_0} have a common right factor. Then for some fixed $\boldsymbol{x}$ and $\boldsymbol{y}$, such that $M_i = \boldsymbol{u}\boldsymbol{x}^t$ and $M_j = \boldsymbol{y}\boldsymbol{v}^t$, there is some k such that $u_k = 0$ and $y_k = 1$ and some ℓ such that $v_\ell = 0$ and $x_\ell = 1$. Thus, the submatrix of $M_{i_0} + M_i + M_j$ on rows 1 and k and columns 1 and ℓ is $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Thus, the rank of $M_{i_0} + M_i + M_j$ is at least 2, a contradiction. Hence there are two possibilities: for all $i \neq i_0$, if $M_i \not\sqsupseteq M_{i_0}$, then M_i and M_{i_0} have a common left factor; or, for all $i \neq i_0$, if $M_i \not\sqsupseteq M_{i_0}$, then M_i and M_{i_0} have a common right factor. We will assume only the first as the second case has a parallel argument.

Let $F_1 = \{i \in [\![1, n]\!] : M_i = \boldsymbol{u}\boldsymbol{x}^t \text{ for some vector } \boldsymbol{x}\}$. For each $i \in F_1$, let $M_i = \boldsymbol{u}\boldsymbol{x}^{(i)^t}$ where $\boldsymbol{x}^{(i)^t} = (x_1^{(i)}, \ldots, x_q^{(i)})$, and let $A_{1,i} = \{j : x_j^{(i)} = 1\}$. Let $J_1 = \cup_{i \in F_1} A_{1,i}$ and $I_1 = \{i : u_i = 1\}$.

Let $G = [\![1,n]\!] \setminus F_1$. By permuting we may assume that $G = \{1,\dots,k\}$ where F_1 has $n-k$ indices in $[\![1,n]\!]$. Let $\widehat{\mathbb{M}} = \mathbb{M}_{1,k}$, so that $\widehat{A}$ represents a matrix in $\widehat{\mathbb{M}}$. Define $\widehat{T} : \widehat{\mathbb{M}} \to \mathbb{M}_{p,q}$ by $\widehat{T}(\widehat{X}) = T([\widehat{X}, O_{1,n-k}])$. Then $\widehat{T} : \mathbb{M}_{1,k} \to \mathbb{M}_{p,q}$ is linear and preserves rank 1. So, by induction, for some $2 \le r \le n$, there exists an ordered partition $(F_2,\dots,F_r)$ of G, together with two ordered, pairwise disjoint sets of indices, $(\widehat{I}_2,\dots,\widehat{I}_r)$ of $[\![1,p]\!]$ and $(\widehat{J}_2,\dots,\widehat{J}_r)$ of $[\![1,q]\!]$ such that for all $k \in [\![2,r]\!]$, either

- there is a cover $\{\widehat{A}_{k,i} : i \in F_k\}$ of subsets of $\widehat{I}_k$ such that for all $i \in F_k$,

$$\widehat{T}(\widehat{E}_{1,i}) = \sum_{s \in \hat{I}_2 \cup \cdots \cup \hat{I}_{k-1} \cup \hat{A}_{k,i}} \ \sum_{t \in \hat{J}_2 \cup \cdots \cup \hat{J}_k} E_{s,t}$$

or

- there is a cover $\{\widehat{B}_{k,i} : i \in F_k\}$ of subsets of $\widehat{J}_k$ such that for all $i \in F_k$,

$$\widehat{T}(\widehat{E}_{1,i}) = \sum_{s \in \hat{I}_2 \cup \cdots \cup \hat{I}_k} \ \sum_{t \in \hat{J}_2 \cup \cdots \cup \hat{J}_{k-1} \cup \hat{B}_{k,i}} E_{s,t}.$$

Let $M_0 = \sum_{i \in F_1} M_i$. Since for each $j \in F_k$ with $k \ge 2$, $M_j \sqsupset M_{i_0}$, thus, the left factor of M_j strictly dominates $\boldsymbol{u}$ by the definition of F_1. Further, either $M_j \sqsupseteq M_0$ or M_j and M_0 have a common factor (it must be the right factor). Thus, $M_j \sqsupseteq M_0$ for all $j \in [\![1,k]\!]$.

For $\ell = 2,\dots,r$, let $I_\ell = \widehat{I}_\ell \setminus I_1$, $A_{\ell,i} = \widehat{A}_{\ell,i} \setminus I_1$, $J_\ell = \widehat{J}_\ell \setminus J_1$ and $B_{\ell,i} = \widehat{B}_{\ell,i} \setminus J_1$. The theorem then follows. □

We end this section with a lemma about strong preservers of rank 1.

Lemma 3.5. *Let $2 \le m \le n$. If $T : \mathbb{M}_{m,n} \to \mathbb{M}_{p,q}$ is a linear transformation that strongly preserves rank 1, then T preserves rank 2.*

Proof. Let A be an arbitrary matrix in $\mathbb{M}_{m,n}$ with $b(A) = 2$. Then we have that $A = B + C$ with rank 1 matrices B and C in $\mathbb{M}_{m,n}$. Thus $T(A) = T(B) + T(C)$ has rank 1 or 2. But T strongly preserves rank 1 and hence $T(A)$ cannot be rank 1. Thus $b(T(A)) = 2$. Since A is an arbitrary matrix of rank 2, the result follows. □

4. Preservers of ranks 1 and k

In this section, we consider linear transformations $T : \mathbb{M}_{m,n} \to \mathbb{M}_{p,q}$ that preserve ranks 1 and k and we obtain lemmas needed to prove the main theorem in the next section.

If A is a Boolean matrix and α is a set of rows and β is a set of columns, then $A[\alpha|\beta]$ is the Boolean matrix obtained from A by deleting all rows not in α and all columns not in β. Further the transpose of A is denoted by A^t.

Hereafter, we assume that $2 \le k \le m \le n$. We begin by proving a crucial lemma.

Lemma 4.1. *Suppose that $T : \mathbb{M}_{m,n} \to \mathbb{M}_{p,q}$ is a linear transformation, T preserves ranks 1 and k, and $T(J_{m,n}) = J_{p,q}$. Then there are permutation matrices P and Q of orders p and q, respectively, such that*

$$(PT(X)Q)[1,\dots,m|1,\dots,n] = X$$

for all $X \in \mathbb{M}_{m,n}$, or

$$(PT(X)Q)[1,\dots,n|1,\dots,m] = X^t$$

for all $X \in \mathbb{M}_{m,n}$.

Proof. We will divide the proof into four preliminary steps before the final argument.

Step 1. For any line matrix (a row matrix or a column matrix) L, we have that $T(L)$ has no zero row or no zero column but not both.

Proof of Step 1. First, suppose that for some line matrix L, say $L = R_1$ or $L = C_1$, $T(L)$ has neither a zero row nor a zero column. Then we must have that $T(L) = J_{p,q}$ since T preserves rank 1. But then $T(L+E_{2,2}+\cdots+E_{k,k}) = J_{p,q}$, a contradiction since T preserves rank k and $L + E_{2,2} + \cdots + E_{k,k}$ has rank k. Thus the image of every line matrix has either a zero row or a zero column.

Next, suppose that the image of some line matrix L_1, say $L_1 = R_1$ or $L_1 = C_1$, has both a zero row and a zero column. Then, by permuting, we may assume that $T(L_1) = \left[\begin{smallmatrix} J_{r,s} & O \\ O & O \end{smallmatrix}\right]$ for some $r < p$ and $s < q$. Since $T(J_{m,n}) = J_{p,q}$ there is some cell $E_{i,j}$ whose image dominates $E_{p,q}$. By permuting we may assume that $i = 2$ or $j = 2$ according as $L_1 = R_1$ or $L_1 = C_1$. Let $L_2 = R_2$ or $L_2 = C_2$ according as $L_1 = R_1$ or $L_1 = C_1$. But then we must have that $T(L_2)$ dominates

$$\begin{bmatrix} J_{r,s} & O & \boldsymbol{j}_r \\ O & O & O \\ \boldsymbol{j}_s{}^t & O & 1 \end{bmatrix}.$$

Otherwise, $T(L_1 + L_2)$ would have rank 2. But then $T(E_{1,1} + L_2) = T(L_2)$ is a rank one matrix. Thus we have a contradiction for the case of $k = 2$, $m = 2$ or $n = 2$. Hence we assume that $3 \le k \le m \le n$. Now, $T(E_{1,1} + L_2 + E_{3,3} + \cdots + E_{k,k}) = T(E_{1,1} + L_2) + T(E_{3,3}) + \cdots + T(E_{k,k})$ is the sum of $k-1$ rank one matrices. Thus $T(E_{1,1} + L_2 + E_{3,3} + \cdots + E_{k,k})$ has rank at most $k-1$, a contradiction since $E_{1,1} + L_2 + E_{3,3} + \cdots + E_{k,k}$ has rank k.

It follows that the image of any line matrix has no zero row or no zero column, but not both. □

Henceforth, we assume without loss of generality that $T(R_1)$ has a zero row but no zero column.

Step 2. The image of any row matrix has a zero row but no zero column and the image of any column matrix has a zero column and no zero row.

Proof of Step 2. If for some $i \neq 1$, $T(R_i)$ has a zero column, then $T(R_i)$ has no zero row by Step 1. Since $T(R_1)$ has a zero row but no zero column, $T(R_1+R_i)$ must have rank 2, a contradiction. Hence the image of any row matrix has a zero row but no zero column.

Suppose that the image of some column matrix has no zero column. Without loss of generality we may assume that $T(C_1)$ has no zero column. By Step 1, since $T(R_1)$ and $T(C_1)$ are rank 1, every nonzero row of $T(R_1)$ and every nonzero row of $T(C_1)$ consists entirely of ones. But then, $T(R_1 + C_1)$ is also rank 1. Now, T preserves rank 1 so that $T(R_1 + C_1 + E_{3,3} + \cdots + E_{k,k}) = T(R_1 + C_1) + T(E_{3,3}) + \cdots + T(E_{k,k})$ has rank at most $k-1$, a contradiction since $R_1 + C_1 + E_{3,3} + \cdots + E_{k,k}$ has rank k. Hence the image of any column matrix has a zero column and no zero row. $\square$

In the following we shall use the notation $\widehat{C}_j = \sum_{k=j}^{n} E_{k,j}$. Thus, $\widehat{C}_1 = C_1$, $\widehat{C}_2 = C_2 \setminus E_{1,2}$, $\widehat{C}_3 = C_3 \setminus (E_{1,3} + E_{2,3})$, etc.

Step 3. For $i = 1, \ldots, m$, $T(J_{m,n} \setminus R_i)$ has a zero row and no zero column, and for $j = 1, \ldots, n$, $T(J_{m,n} \setminus C_j)$ has a zero column and no zero row.

Proof of Step 3. Since the column case is parallel to the row case, we only consider the row case. Since $b(J_{m,n} \setminus R_i) = 1$ and T preserves rank 1, we have that $b(T(J_{m,n} \setminus R_i)) = 1$. If we choose an index i' in $\{1, \ldots, m\} \setminus \{i\}$, then $T(J_{m,n} \setminus R_i)$ dominates $T(R_{i'})$. Since, by Step 2, $T(R_{i'})$ has no zero column, $T(J_{m,n} \setminus R_i)$ has no zero column. Thus, by permuting, we may assume that $T(J_{m,n} \setminus R_i) = \left[\begin{smallmatrix} J_{r,q} \\ O \end{smallmatrix}\right]$ for some $r \leq p$. Without loss of generality, we assume that $i = 1$ and hence $T(J_{m,n} \setminus R_1) = \left[\begin{smallmatrix} J_{r,q} \\ O \end{smallmatrix}\right]$. Now we will show that $r < p$.

Suppose that $T(C_1 \setminus E_{1,1})$ has no zero row. Then, by Step 2, $T((C_1 \setminus E_{1,1}) + C_2)$ is a sum of full column matrices and hence $T((C_1 \setminus E_{1,1}) + C_2)$ has rank 1. But then $T((C_1 \setminus E_{1,1}) + C_2 + \widehat{C}_3 + \cdots + \widehat{C}_k) = T((C_1 \setminus E_{1,1}) + C_2) + T(\widehat{C}_3) + \cdots + T(\widehat{C}_k)$ must have rank at most $k-1$ while $(C_1 \setminus E_{1,1}) + C_2 + \widehat{C}_3 + \cdots + \widehat{C}_k$ has rank k, a contradiction to the fact that T preserves rank k. Thus $T(C_1 \setminus E_{1,1})$ must have a zero row.

Now, suppose that $T(C_1 \setminus E_{1,1}) \sqsubseteq T(C_j \setminus E_{1,j})$ for some $j \geq 2$. Without loss of generality, we assume that $j = 2$. Then $T(C_1 \setminus E_{1,1}) \sqsubseteq T(C_2)$ and hence $T((C_1 \setminus E_{1,1}) + C_2 + \widehat{C}_3 + \cdots + \widehat{C}_k) = T(C_2 + \widehat{C}_3 + \cdots + \widehat{C}_k) = T(C_2) + T(\widehat{C}_3) + \cdots + T(\widehat{C}_k)$. But the rank of $T(C_2 + \widehat{C}_3 + \cdots + \widehat{C}_k) = T(C_2) + T(\widehat{C}_3) + \cdots + T(\widehat{C}_k)$ is at most $k-1$ since T preserves rank 1, while the rank of $(C_1 \setminus E_{1,1}) + C_2 + \widehat{C}_3 + \cdots + \widehat{C}_k$, and hence its image, is k, a contradiction. Thus, $T(C_1 \setminus E_{1,1})$ is not dominated by $T(C_j \setminus E_{1,j})$ for all $j \geq 2$. That is, there is some nonzero column of $T(C_1 \setminus E_{1,1})$ that is not dominated by $T(C_j \setminus E_{1,j})$. Thus, since $(C_1 \setminus E_{1,1}) + (C_j \setminus E_{1,j})$ is rank 1, every zero row of $T(C_1 \setminus E_{1,1})$ is a zero row of $T(C_j \setminus E_{1,j})$. Since $T(J_{m,n} \setminus R_1) = \sum_{k=1}^{n} T(C_k \setminus E_{1,k})$, it follows that $T(J_{m,n} \setminus R_1)$ has a row of zeros. Therefore we conclude that $r < p$. Thus, for $i = 1, \ldots, m$, $T(J_{m,n} \setminus R_i)$ has a zero row and no zero column. $\square$

Step 4. $m \leq p$, $n \leq q$ and there exist permutation matrices P of order p and Q of order q such that for each $\ell = 1, \ldots, m$, $PT(J_{m,n} \setminus R_\ell)$ has a zero ℓ^{th} row, and for each $j = 1, \ldots, n$, $T(J_{m,n} \setminus R_j)Q$ has a zero j^{th} column.

Proof of Step 4. By Step 3, $T(J_{m,n} \setminus R_\ell)$ has a zero row for each $\ell = 1, \ldots, m$. Now, suppose that $T(J_{m,n} \setminus R_j)$ and $T(J_{m,n} \setminus R_\ell)$ have a common zero row for some $j \neq \ell$. Then $T(J_{m,n}) = T(J_{m,n} \setminus R_j) + T(J_{m,n} \setminus R_\ell)$ must have a zero row, a contradiction since $T(J_{m,n} \setminus R_j + J_{m,n} \setminus R_\ell) = T(J_{m,n}) = J_{p,q}$. Thus $T(J_{m,n} \setminus R_j)$ and $T(J_{m,n} \setminus R_\ell)$ cannot have a common zero row for all $j \neq \ell$. This shows that $T(J_{m,n} \setminus R_i)$ defines a distinct zero row for each $i = 1, \ldots, m$. Thus we must have that $p \geq m$. Using a parallel argument, $q \geq n$.

We now define a mapping $f_o : \{1, \ldots, m\} \to \{1, \ldots, p\}$ by $f_o(\ell) = j$ if and only if the first zero row in $T(J_{m,n} \setminus R_\ell)$ is the jth. From the above paragraph, $f_o(\ell)$ is distinct from $f_o(j)$ unless $j = \ell$ so that f_o is injective. Extend f_o to $f : \{1, \ldots, p\} \to \{1, \ldots, p\}$ in any way so that f is a bijection. Let $P = [p_{i,j}]$ be the permutation matrix defined by $p_{i,j} = \delta_{f(i),j}$ where δ is the Kronecker delta. Then the ℓth row of $PT(J_{m,n} \setminus R_\ell)$ has all zero entries. The column case is parallel. □

Note that $m \leq p$ and $n \leq q$ by Step 4 and that the ith row of $PT(R_i)Q$ is all ones and the jth column of $PT(C_j)Q$ is all ones. Further, that the only rows with nonzero entries in $PT(R_i)Q$ are the ith and rows numbered bigger than m and the only columns with nonzero entries in $PT(C_j)Q$ are the jth and columns numbered bigger than n. Since $E_{i,j}$ lies in the intersection of the ith row and j^{th} column, the nonzero entries of $PT(E_{i,j})Q$ must be dominated by both $PT(R_i)Q$ and $PT(C_j)Q$. Thus, the nonzero entries of $PT(E_{i,j})Q$ lie only in the intersection of the ith row and rows numbered bigger than m and the jth column and columns numbered bigger than n. That is, we have that $(PT(E_{i,j})Q)[1, \ldots, m|1, \ldots, n] = E_{i,j}$ since $J_{p,q} = T(J_{m,n}) = \sum_{i,j} T(E_{i,j})$. Hence, $(PT(X)Q)[1, \ldots, m|1, \ldots, n] = X$ for all $X \in \mathbb{M}_{m,n}$.

Had we assumed before Step 2 that $T(R_1)$ has a zero column but no zero row, we would conclude that $(PT(X)Q)[1, \ldots, n|1, \ldots, m] = X^t$. □

To illustrate the above lemma, we give the following example:

Example 4.2. Define a transformation $T : \mathbb{M}_{3,4} \to \mathbb{M}_{5,6}$ by

$$T(X) = \begin{bmatrix} I_3 \\ V \end{bmatrix} X \begin{bmatrix} I_4 & U \end{bmatrix} = \begin{bmatrix} X & XU \\ VX & VXU \end{bmatrix}$$

for all $X \in \mathbb{M}_{3,4}$, where $V = \left[\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{smallmatrix}\right] \in \mathbb{M}_{2,3}$ and $U = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] \in \mathbb{M}_{4,2}$. Then for any $X = \left[\begin{smallmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} \end{smallmatrix}\right] \in \mathbb{M}_{3,4}$, we have that $T(X)$ becomes:

$$\begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,3} & x_{1,2} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & x_{2,3} & x_{2,2} \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & x_{3,3} & x_{3,2} \\ x_{1,1}+x_{2,1} & x_{1,2}+x_{2,2} & x_{1,3}+x_{2,3} & x_{1,4}+x_{2,4} & x_{1,3}+x_{2,3} & x_{1,2}+x_{2,2} \\ x_{2,1}+x_{3,1} & x_{2,2}+x_{3,2} & x_{2,3}+x_{3,3} & x_{2,4}+x_{3,4} & x_{2,3}+x_{3,3} & x_{2,2}+x_{3,2} \end{bmatrix}.$$

Notice that $T(J_{3,4}) = J_{5,6}$. Further T preserves the rank of any matrix in $\mathbb{M}_{3,4}$ (see Theorem 5.1 in the next section).

For a matrix $X \in \mathbb{M}_{m,n}$, define $X^{(i)}$ to be the ith row of X, and $X_{(j)}$ to be the jth column of X. Let $\boldsymbol{e}_{r,i}$ denote the column vector of size r, whose i^{th} entry equals one, and the others are zero. Then $\boldsymbol{e}_{r,i}^t$ is a row vector of size r with one 1 in the ith entry.

Lemma 4.3. *Suppose that $T : \mathbb{M}_{m,n} \to \mathbb{M}_{p,q}$ is a linear transformation that preserves rank 1. If $T(X) = \begin{bmatrix} X & B(X) \\ C(X) & D(X) \end{bmatrix}$, where $B(X) \in \mathbb{M}_{m,q-n}$, $C(X) \in \mathbb{M}_{p-m,n}$ and $D(X) \in \mathbb{M}_{p-m,q-n}$ are matrices dependent on X, then there are matrices $U \in \mathbb{M}_{n,q-n}$ and $V \in \mathbb{M}_{p-m,m}$ such that*

$$B(X) = XU, \quad C(X) = VX \quad \text{and} \quad D(X) = VXU.$$

Proof. Define a matrix $U \in \mathbb{M}_{n,q-n}$ by $U^{(i)} = T(E_{1,i})[1|n+1,\dots,q]$ for $i = 1,\dots,n$, and a matrix $V \in \mathbb{M}_{p-m,m}$ by $V_{(j)} = T(E_{j,1})[m+1,\dots,p|1]$ for $j = 1,\dots,m$.

Now, the ith row of $T(E_{i,j})$ is $\begin{bmatrix} \boldsymbol{e}_{n,j}^t & \boldsymbol{x}^t \end{bmatrix}$ for some column vector $\boldsymbol{x}$ of size $q-n$, and the first row of $T(E_{1,j})$ is $\begin{bmatrix} \boldsymbol{e}_{n,j}^t & U^{(j)} \end{bmatrix}$. Since $T(E_{1,j}+E_{i,j})$ must be rank 1, we must have that $\boldsymbol{x}^t = U^{(j)}$. It now follows that $T(E_{i,j})[1,\dots,m|1,\dots,q] = \begin{bmatrix} E_{i,j} & E_{i,j}U \end{bmatrix}$. Similarly, the jth column of $T(E_{i,j})$ is $\begin{bmatrix} \boldsymbol{e}_{m,i} \\ \boldsymbol{y} \end{bmatrix}$ for some column vector $\boldsymbol{y}$ of size $p-m$, and the first column of $T(E_{1,j})$ is $\begin{bmatrix} \boldsymbol{e}_{m,i} \\ V_{(i)} \end{bmatrix}$. Since $T(E_{i,1}+E_{i,j})$ must be rank 1, we must have that $\boldsymbol{y} = V_{(i)}$. It now follows that $T(E_{i,j})[1,\dots,q|1,\dots,n] = \begin{bmatrix} E_{i,j} \\ VE_{i,j} \end{bmatrix}$.

Now, since $T(E_{i,j})$ has rank 1, if a row of $D(E_{i,j})$ is nonzero, it must be identical to the nonzero row in $E_{i,j}U$, and if a column of $D(E_{i,j})$ is nonzero, it must be identical to the nonzero column of $VE_{i,j}$. But then, $D(E_{i,j}) = VE_{i,j}U$. That is $T(E_{i,j}) = \begin{bmatrix} E_{i,j} & E_{i,j}U \\ VE_{i,j} & VE_{i,j}U \end{bmatrix}$ for each $i = 1,\dots,m$ and $j = 1,\dots,n$.

Let $X \in \mathbb{M}_{m,n}$. Then $X = \sum_{i=1}^m \sum_{j=1}^n x_{i,j}E_{i,j}$, so that by the linearity of T, we have

$$T(X) = T\left(\sum_{i=1}^m \sum_{j=1}^n x_{i,j}E_{i,j}\right) = \begin{bmatrix} X & XU \\ VX & VXU \end{bmatrix}.$$

Hence the result follows. □

The number of nonzero entries of a matrix X is denoted by $\sharp(X)$. Notice that if $r \leq p$ and P is a matrix in $\mathbb{M}_{p,r}$ with $\sharp(P) = b(P) = r$, then there are permutation matrices P_1 and P_2 of orders p and r, respectively, such that $P = P_1 \left[\begin{smallmatrix} P_2 \\ O \end{smallmatrix}\right]$.

Theorem 4.4. *Let $2 \leq k \leq m \leq n$ and $T : \mathbb{M}_{m,n} \to \mathbb{M}_{p,q}$ be a linear transformation. Then if T preserves ranks 1 and k, then either*

(1) *$p \geq m$ and $q \geq n$, and for some r, s with $m \leq r \leq p$ and $n \leq s \leq q$, there are matrices $P \in \mathbb{M}_{p,r}$ and $Q \in \mathbb{M}_{s,q}$ with $\sharp(P) = b(P) = r$ and $\sharp(Q) = b(Q) = s$, such that for some fixed matrices $V \in \mathbb{M}_{r-m,m}$ and $U \in \mathbb{M}_{n,s-n}$,*
$$T(X) = P \begin{bmatrix} X & XU \\ VX & VXU \end{bmatrix} Q$$
for all $X \in \mathbb{M}_{m,n}$; or

(2) *$q \geq m$ and $p \geq n$, and for some r, s with $n \leq r \leq p$ and $m \leq s \leq q$, there are matrices $P \in \mathbb{M}_{p,r}$ and $Q \in \mathbb{M}_{s,q}$ with $\sharp(P) = b(P) = r$ and $\sharp(Q) = b(Q) = s$, such that for some fixed matrices $V \in \mathbb{M}_{r-n,n}$ and $U \in \mathbb{M}_{m,s-m}$,*
$$T(X) = P \begin{bmatrix} X^t & X^tU \\ VX^t & VX^tU \end{bmatrix} Q$$
for all $X \in \mathbb{M}_{m,n}$.

Proof. Since T preserves rank 1, there are permutation matrices P_1 and Q_1 of orders p and q, respectively, such that $T(J_{m,n}) = P_1 \left[\begin{smallmatrix} J_{r,s} & O \\ O & O \end{smallmatrix}\right] Q_1$ for some r and s. Let $T_\alpha : \mathbb{M}_{m,n} \to \mathbb{M}_{r,s}$ be defined by
$$T_\alpha(X) = (P_1^{-1}T(X)Q_1^{-1})[1, \ldots, r|1, \ldots, s],$$
so that T_α preserves rank 1 and rank k and $T_\alpha(J_{m,n}) = J_{r,s}$. By Lemma 4.1, $T_\alpha(J_{m,n} \setminus R_i)$ has either a zero row or a zero column for all $i = 1, \ldots, m$.

Case 1. Suppose that $T_\alpha(J_{m,n} \setminus R_i)$ has a zero row for some i. By Lemma 4.1, $m \leq r \leq p$, $n \leq s \leq q$, and there are permutation matrices P_2 and Q_2 of orders r and s, respectively, such that $(P_2^{-1}T_\alpha(X)Q_2^{-1})[1, \ldots, m|1, \ldots, n] = X$, so that $T_\alpha(X) = P_2 \left[\begin{smallmatrix} X & B(X) \\ C(X) & D(X) \end{smallmatrix}\right] Q_2$ for some matrices $B(X) \in \mathbb{M}_{m,s-n}$, $C(X) \in \mathbb{M}_{r-m,n}$ and $D(X) \in \mathbb{M}_{r-m,s-n}$ which depend upon X. By Lemma 4.3, there are matrices $U \in \mathbb{M}_{n,s-n}$ and $V \in \mathbb{M}_{r-m,m}$ such that $\left[\begin{smallmatrix} X & B(X) \\ C(X) & D(X) \end{smallmatrix}\right] = \left[\begin{smallmatrix} X & XU \\ VX & VXU \end{smallmatrix}\right]$. Thus, we have that
$$\begin{aligned} T(X) &= P_1 \begin{bmatrix} T_\alpha(X) & O \\ O & O \end{bmatrix} Q_1 \\ &= P_1 \begin{bmatrix} I_r \\ O \end{bmatrix} T_\alpha(X) \begin{bmatrix} I_s & O \end{bmatrix} Q_1 \\ &= P_1 \begin{bmatrix} I_r \\ O \end{bmatrix} P_2 \begin{bmatrix} X & XU \\ VX & VXU \end{bmatrix} Q_2 \begin{bmatrix} I_s & O \end{bmatrix} Q_1 \end{aligned}$$

$$= P_1 \begin{bmatrix} P_2 \\ O \end{bmatrix} \begin{bmatrix} X & XU \\ VX & VXU \end{bmatrix} \begin{bmatrix} Q_2 & O \end{bmatrix} Q_1.$$

Letting $P = P_1 \left[\begin{smallmatrix} P_2 \\ O \end{smallmatrix}\right]$ and $Q = \begin{bmatrix} Q_2 & O \end{bmatrix} Q_1$, we have arrived at conclusion (1).

Case 2. Suppose that $T_\alpha(J_{m,n} \setminus R_i)$ has a zero column for some i. Following a parallel argument to that of Case 1 above, we arrive at conclusion (2). □

Corollary 4.5. *Let $2 \le k \le m \le n$ and $T : \mathbb{M}_{m,n} \to \mathbb{M}_{p,q}$ be a linear transformation. If T preserves ranks 1 and k, then T preserves rank.*

Proof. The structure of T from the above lemma gives that T preserves all ranks. □

5. The main theorem

Theorem 5.1. *Let $2 \le k \le m \le n$ and $T : \mathbb{M}_{m,n} \to \mathbb{M}_{p,q}$ be a linear transformation. The following conditions are equivalent:*

(i) *T preserves rank;*
(ii) *T strongly preserves rank 1;*
(iii) *T preserves ranks 1 and k;*
(iv) *T satisfies*
  (1) *$p \ge m$ and $q \ge n$, and for some r, s with $m \le r \le p$ and $n \le s \le q$, there are matrices $P \in \mathbb{M}_{p,r}$ and $Q \in \mathbb{M}_{s,q}$ with $\sharp(P) = b(P) = r$ and $\sharp(Q) = b(Q) = s$, such that for some fixed matrices $V \in \mathbb{M}_{r-m,m}$ and $U \in \mathbb{M}_{n,s-n}$,*
$$T(X) = P \begin{bmatrix} X & XU \\ VX & VXU \end{bmatrix} Q$$
  for all $X \in \mathbb{M}_{m,n}$; or
  (2) *$q \ge m$ and $p \ge n$, and for some r, s with $n \le r \le p$ and $m \le s \le q$, there are matrices $P \in \mathbb{M}_{p,r}$ and $Q \in \mathbb{M}_{s,q}$ with $\sharp(P) = b(P) = r$ and $\sharp(Q) = b(Q) = s$, such that for some fixed matrices $V \in \mathbb{M}_{r-n,n}$ and $U \in \mathbb{M}_{m,s-m}$,*
$$T(X) = P \begin{bmatrix} X^t & X^tU \\ VX^t & VX^tU \end{bmatrix} Q$$
  for all $X \in \mathbb{M}_{m,n}$.

Proof. It is obvious that (i) implies (ii), and (iv) implies (i). Further, (iii) implies (iv) by Lemma 4.4. Suppose that T strongly preserves rank 1. Then T preserves rank 2 by Lemma 3.5. By Corollary 4.5, for $k = 2$ we have that T preserves all ranks. Thus (ii) implies (iii). □

Thus, we have characterized Boolean linear transformations that preserve Boolean rank between different matrix spaces. These characterizations extend those of the linear operators that preserve Boolean rank between the same Boolean matrix space.

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