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TRANSLATION AND HOMOTHETICAL SURFACES IN EUCLIDEAN SPACE WITH CONSTANT CURVATURE

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Abstract. We study surfaces in Euclidean space which are obtained as the sum of two curves or that are graphs of the product of two functions. We consider the problem of finding all these surfaces with constant Gauss curvature. We extend the results to non-degenerate surfaces in Lorentz-Minkowski space.

1. Introduction

In this paper we study two types of surfaces in Euclidean space $\mathbb{R}^3$. The first kind of surfaces are translation surfaces which were initially introduced by S. Lie. A *translation surface* S is a surface that can be expressed as the sum of two curves $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$, $\beta : J \subset \mathbb{R} \to \mathbb{R}^3$. In a parametric form, the surface S writes as $X(s,t) = \alpha(s) + \beta(t)$, $s \in I$, $t \in J$. See [2, p. 138]. A translation surface S has the property that the translations of a parametric curve $s =$ *constant* by $\beta(t)$ remain in S (similarly for the parametric curves $t =$ *constant*). It is an open problem to classify all translation surfaces with constant mean curvature (CMC) or constant Gauss curvature (CGC). A first example of a CMC translation surface is the Scherk surface

$$z(x,y) = \frac{1}{a}\log\left(\left|\frac{\cos(ay)}{\cos(ax)}\right|\right),\ a > 0.$$

This surface is minimal ($H = 0$) and belongs to a more general family of Scherk surfaces ([7, pp. 67–73]). In this case, the curves α and β lie in two orthogonal planes and after a change of coordinates, the surface is locally described as the graph of $z = f(x)+g(y)$. Other examples of CMC or CGC translation surfaces given as a graph $z = f(x) + g(y)$ are: planes ($H = K = 0$), circular cylinders

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($H = constant \neq 0$, $K = 0$) and cylindrical surfaces ($K = 0$). The progress on this problem has been as follows:

(1) If α and β lie in orthogonal planes, the only minimal translation surfaces are the plane and the Scherk surface [8].
(2) If α and β lie in orthogonal planes, the only CMC translation surfaces are the plane, the Scherk surface and the circular cylinder [5].
(3) If α and β lie in orthogonal planes, the only CGC translation surfaces have $K = 0$ and are cylindrical surfaces [5].
(4) If both curves α and β are planar, the only minimal translation surfaces are the plane or a surface which belongs to the family of Scherk surfaces [3].
(5) If one of the curves α or β is planar and the other one is not, there are no minimal translation surfaces [3].

Our first result concerns the case when the Gauss curvature K is constant. We prove that, without any assumption on the curves α and β, the only flat ($K = 0$) translation surfaces are cylindrical surfaces. By a cylindrical surface we mean a ruled surface whose directrix is contained in a plane and the rulings are parallel to a fixed direction in $\mathbb{R}^3$.

Theorem 1.1. (1) *The only translation surfaces with zero Gauss curvature are cylindrical surfaces.*
(2) *There are no translation surfaces with constant Gauss curvature $K \neq 0$ if one of the generating curves is planar.*

When $K = 0$, we give a complete classification CGC of translation surfaces and for $K \neq 0$, we extend the result given in [3] for CMC translation surfaces.

A second kind of surfaces of our interest are the homothetical surfaces, where we replace $f(x) + g(x)$ by $f(x)g(x)$ in the definition of a translation surface.

Definition. A homothetical surface S in Euclidean space $\mathbb{R}^3$ is a surface that is a graph of a function $z = f(x)g(y)$, where $f : I \subset \mathbb{R} \to \mathbb{R}$ and $g : J \subset \mathbb{R} \to \mathbb{R}$ are two smooth functions.

As far as the authors know, the first approach to this kind of surfaces appeared in [9], when studying the problem of finding minimal homothetical nondegenerate surfaces in Lorentz-Minkowski space $\mathbb{L}^3$ (see also [10]). Some authors have considered minimal homothetical hypersurfaces in Euclidean space and in semi-Euclidean spaces ([4, 10]). Our first result concerns minimal surfaces. Van de Woestyne proved in [9] that the only minimal homothetical non-degenerate surfaces in $\mathbb{L}^3$ are planes and helicoids. At the end of [9] the author asserted that, up to small changes in the proof, a similar result can be obtained in Euclidean space $\mathbb{R}^3$. In the present paper we do a different proof of the Euclidean version and in Section 3 we prove:

Theorem 1.2. *Planes and helicoids are the only minimal homothetical surfaces in Euclidean space.*

The parametrization of the helicoid is not the usual one as for a ruled surface which has a helix as base, but

$$z(x,y) = (x+b)\tan(cy+d), \tag{1.1}$$

where $b, c, d \in \mathbb{R}$, $c \neq 0$ ([7, p. 20]).

The third result considers homothetical surfaces in Euclidean space with constant Gauss curvature, obtaining a complete classification.

Theorem 1.3. *Let S be a homothetical surface in Euclidean space $\mathbb{R}^3$ with constant Gauss curvature K. Then $K = 0$. Furthermore, the surface is a plane, a cylindrical surface or a surface whose parametrization is*:

(1) $z(x,y) = ae^{bx+cy}$, *$a, b, c > 0$, or*
(2)
$$z(x,y) = \left(\frac{bx}{m} + d\right)^m \left(\frac{cy}{m-1} + e\right)^{1-m},$$
with $b, c, d, e, m \in \mathbb{R}$, $b, c \neq 0$, $m \neq 0, 1$.

This theorem is proved in Section 4. Finally, in Section 5 we extend Theorems 1.1 and 1.3 from Euclidean to Lorentz-Minkowski space, obtaining similar results.

2. Proof of Theorem 1.1

Throughout this article, we consider the rectangular coordinates (x, y, z) of Euclidean space $\mathbb{R}^3$. Assume S is the sum of the curves $\alpha(s)$ and $\beta(t)$. Locally, α and β are graphs on the axis coordinates of $\mathbb{R}^3$, so we may assume that $\alpha(s) = (s, f_1(s), f_2(s))$ and $\beta(t) = (g_1(t), t, g_2(t))$, $s \in I$, $t \in J$, for some functions f_1, f_2, g_1, g_2. Let us observe that if we replace the functions f_i or g_i by an additive constant, the surface changes by a translation of Euclidean space and thus, in what follows, we will take these functions up to additive constants. The Gauss curvature in local coordinates $X = X(s,t)$ writes as

$$K = \frac{ln - m^2}{EG - F^2},$$

where $\{E, F, G\}$ and $\{l, m, n\}$ are the coefficients of the first and second fundamental form with respect to X, respectively. In our case, the parametrization of S is $X(s,t) = \alpha(s) + \beta(t)$ and as $\partial^2_{st}X = 0$, then $m = 0$. The computation of K leads to

$$K = \frac{(f_2'' - f_1''g_2' + g_1'(f_1''f_2' - f_1'f_2''))\,(g_2'' - f_2'g_1'' + f_1'(g_1''g_2' - g_1'g_2''))}{\left((1 + f_1'^2 + f_2'^2)(1 + g_1'^2 + g_2'^2) - (f_1' + g_1' + f_2'g_2')^2\right)^2}. \tag{2.1}$$

2.1. Case $K = 0$

Then $l = 0$ or $n = 0$. Assume $l = 0$ (the following argument will be similar if $n = 0$). Thus

$$f_2'' - f_1''g_2' + g_1'(f_1''f_2' - f_1'f_2'') = 0. \tag{2.2}$$

We distinguish several cases.

(1) Assume $f_1'' = 0$. Then $f_1(s) = as$, $a \in \mathbb{R}$, and (2.2) gives $f_2''(1 - ag_1') = 0$. If $f_2'' = 0$, then f_2 is linear, proving that the curve α is a straight-line and the surface is a cylindrical surface whose base curve is β. If $f_2'' \neq 0$, then $a \neq 0$. Solving for g_1, we obtain $g_1(t) = t/a$. Then $X(s,t) = (s + t/a, as + t, f_2(s) + g_2(t))$ and the surface is the plane of equation $ax - y = 0$.

(2) Assume $f_1'' \neq 0$ and $g_1'' = 0$. Then $g_1(t) = at$, $a \in \mathbb{R}$, and (2.2) implies

$$\frac{f_2'' + a(f_1''f_2' - f_1'f_2'')}{f_1''} = g_2'. \tag{2.3}$$

As the left hand-side of this equation depends only on s, while the right hand-side only on t, we conclude that both functions in (2.3) must be equal to the same constant $b \in \mathbb{R}$. In particular, $g_2(t) = bt$. Now the curve β is a straight-line and the surface is a cylindrical surface with the curve α as base. Let us notice that under these conditions, equation (2.3) does not yield further information on the curve α.

(3) Assume $f_1''g_1'' \neq 0$. Differentiating (2.2) with respect to t, we have $-f_1''g_2'' + g_1''(f_1''f_2' - f_1'f_2'') = 0$. With a similar argument as above, one proves that there exists $a \in \mathbb{R}$ such that

$$\frac{f_1''f_2' - f_1'f_2''}{f_1''} = a = \frac{g_2''}{g_1''}.$$

The identity $g_2'' = ag_1''$ implies that $\det(\beta', \beta'', \beta''') = 0$ and this means that the torsion of β is 0 identically. This proves that β is a planar curve. Now we come back to the beginning of the proof assuming that β is included in the yz-plane (or equivalently, $g_1 = 0$). We compute K again obtaining

$$g_2''(f_2'' - f_1''g_2') = 0.$$

If $g_2'' = 0$, then g_2 is linear and β is a straight-line, proving that S is a cylindrical surface with the curve α as base. If $g_2'' \neq 0$, then $f_2'' - f_1''g_2' = 0$ and it follows that there exists $a \in \mathbb{R}$ such that

$$\frac{f_2''}{f_1''} = a = g_2',$$

and so, $g_2'' = 0$, a contradiction.

2.2. Case $K \neq 0$

We will follow the same ideas as in [3] by distinguishing two cases: first, we suppose that both curves are planar, and second, we assume that only one is planar.

(1) Case when α and β are planar curves. By the result of Liu in [5], we only consider the case when the curves α and β cannot lie in planes mutually orthogonal. Let us notice that if the curves lie in parallel planes, the translation surface is (part) of a plane. Without loss of generality we can assume that α lies in the xz-plane and β in the plane of equation $x\cos\theta - y\sin\theta = 0$, with $\cos\theta,\ \sin\theta \neq 0$. Then α and β are written as

$$\alpha(s) = (s, 0, f(s)),\ \beta(t) = (t\sin\theta, t\cos\theta, g(t))$$

with f and g smooth functions on s and t, respectively. The computation of K leads to

$$K = \frac{\cos^2\theta f''g''}{(f'^2 + g'^2 + \cos^2\theta - 2\sin\theta f'g')^2}.$$

Notice that $K \neq 0$ implies $f''g'' \neq 0$. Differentiating with respect to s and with respect to t, we obtain respectively

$$\cos^2\theta f'''g'' = 4K(f'^2 + g'^2 + \cos^2\theta - 2\sin\theta f'g')(f'f'' - \sin\theta f''g'),$$

$$\cos^2\theta f''g''' = 4K(f'^2 + g'^2 + \cos^2\theta - 2\sin\theta f'g')(g'g'' - \sin\theta f'g'').$$

Using $f''g'' \neq 0$, we have

$$\frac{f'''}{f''^2}(g' - \sin\theta f') = \frac{g'''}{g''^2}(f' - \sin\theta g'). \tag{2.4}$$

Differentiating now with respect to s and next with respect to t, we get

$$\left(\frac{f'''}{f''^2}\right)' g'' = f''\left(\frac{g'''}{g''^2}\right)'.$$

Dividing by $f''g''$, we have an identity of two functions, one depending only on s and the other one depending only on t. Then both functions are equal to a same constant and there exist $a, b, c \in \mathbb{R}$ such that

$$\frac{f'''}{f''^2} = af' + b,\quad \frac{g'''}{g''^2} = ag' + c.$$

Substituting this into (2.4), we get

$$a\sin\theta f'^2 + b\sin\theta f' + cf' = a\sin\theta g'^2 + c\sin\theta g' + bg'.$$

Again we have two functions, one depending only on s and other one depending only on t. Therefore both functions are constant and hence, f' and g' are constant, which is in contradiction with $f''g'' \neq 0$.

(2) Assume that α is a planar curve and β does not lie in a plane. After a change of coordinates, we may suppose

$$\alpha(s) = (s, 0, f(s)),\ \beta(t) = (g_1(t), t, g_2(t))$$

for smooth functions f, g_1 and g_2. This will lead to a contradiction, which forces that β is a planar curve. For this reason, let us first observe that β is planar if and only if its torsion vanishes for all s, that is, $\det(\beta'(t), \beta''(t), \beta'''(t)) = 0$ for all t, or equivalently,

$$g_1'''g_2'' - g_1''g_2''' = 0. \tag{2.5}$$

We compute K obtaining

$$K = \frac{f''(g_2'' - f'g_1'')}{(1 + g_2'^2 + f'^2 + f'^2 g_1'^2 - 2f'g_1'g_2')^2}. \tag{2.6}$$

As $K \neq 0$, we have $f'' \neq 0$. We move f'' to the left hand-side of equation (2.6) and we obtain a function depending only on the variable s. Then the derivative of the right hand-side with respect to t is 0. This means

$$4(f'g_1' - g_2')(f'g_1'' - g_2'')^2 - (f'g_1''' - g_2''')(1 + f'^2(1 + g_1'^2) - 2f'g_1'g_2' + g_2'^2) = 0.$$

For each fixed t, we can view this expression as a polynomial equation on $f'(s)$ and thus, all coefficients vanish. The above equation writes precisely as $\sum_{n=0}^{3} A_n(t) f'(s)^n = 0$. The computations of A_n give:

$$\begin{aligned}
A_0 &= (1 + g_2'^2)g_2''' - 4g_2'g_2''^2,\\
A_1 &= 8g_1''g_2'g_2'' + 4g_1'g_2''^2 - (1 + g_2'^2)g_1''' - 2g_1'g_2'g_2''',\\
A_2 &= -8g_1'g_1''g_2'' - 4g_1''^2 g_2' + 2g_1'g_2'g_1''' + (1 + g_1'^2)g_2''',\\
A_3 &= -(1 + g_1'^2)g_1''' + 4g_1'g_1''^2.
\end{aligned}$$

From $A_0 = 0$ and $A_3 = 0$ we get for $i = 1, 2$,

$$(1 + g_i'^2)g_i''' - 4g_i'g_i''^2 = 0, \tag{2.7}$$

that is,

$$\frac{g_i'''}{g_i''} = \frac{4g_i'g_i''}{1 + g_i'^2}.$$

Integration with respect to t leads to

$$g_i'' = \lambda_i(1 + g_i'^2)^2,\ \lambda_i > 0,\ i = 1, 2. \tag{2.8}$$

In particular, from (2.7),

$$g_i''' = 4\lambda_i^2 g_i'(1 + g_i'^2)^3.$$

This together with (2.5), we shall prove that β is a planar curve. In terms of g_1' and g_2', and using (2.8), the equation (2.5) is equivalent to

$$\lambda_1 g_1'(1 + g_1'^2) - \lambda_2 g_2'(1 + g_2'^2) = 0. \tag{2.9}$$

From the data obtained for g_i'' and g_i''', we now substitute into the coefficients A_1 and A_2. After some manipulations, the identity $A_1 g_2'(1+ g_1'^2) + A_2 g_1'(1 + g_2'^2) = 0$ simplifies into

$$\left[(\lambda_1 g_2'(1 + g_1'^2) + \lambda_2 g_1'(1 + g_2'^2)\right] \left[\lambda_1 g_1'(1 + g_1'^2) - \lambda_2 g_2'(1 + g_2'^2)\right] = 0.$$

If the second factor is zero, then β is planar by (2.9), obtaining a contradiction. If the first factor vanishes, then

$$1 + g_2'^2 = -\frac{\lambda_1}{\lambda_2}\frac{g_2'}{g_1'}(1 + g_1'^2).$$

We place this information together with (2.8) into the coefficient A_1, and we obtain that $A_1 = 0$ is equivalent to the identity

$$g_1'^4 + g_2'^4 + g_1'^2 + g_2'^2 + 2g_1'^2 g_2'^2 = 0.$$

Then $g_1' = g_2' = 0$, that is, the curve β is planar, obtaining a contradiction again. This finishes the proof of Theorem 1.1 for the case $K \neq 0$.

3. Proof of Theorem 1.2

Assume that S is a homothetical surface which is the graph of $z = f(x)g(y)$ and let $X(x, y) = (x, y, f(x)g(y))$ be a parametrization of S. The computation of $H = 0$ leads to

$$f''g(1 + f^2 g'^2) - 2ff'^2 gg'^2 + fg''(1 + f'^2 g^2) = 0. \tag{3.1}$$

Since the roles of f and g in (3.1) are symmetric, we only discuss the cases according to the function f. We distinguish several cases.

(1) Case $f' = 0$. Then $f(x) = \lambda$, $\lambda \in \mathbb{R}$ and (3.1) gives $fg'' = 0$. If $f = 0$, S is the horizontal plane of equation $z = 0$. If $g'' = 0$, then $g(y) = ay + b$, $a, b \in \mathbb{R}$ and $X(x, y)$ parametrizes the plane of equation $\lambda ay - z = \lambda b$.

(2) Case $f'' = 0$, $f' \neq 0$, and by symmetry, $g' \neq 0$. Then $f(x) = ax + b$, for $a, b \in \mathbb{R}$, $a \neq 0$. Now (3.1) reduces into

$$-2a^2 gg'^2 + g''(1 + a^2 g^2) = 0.$$

Then

$$\frac{g''}{g'} = 2a^2 \frac{gg'}{1 + a^2 g^2}.$$

By integrating, we obtain that there exists a constant $k > 0$ such that

$$g' = k(1 + a^2 g^2).$$

Solving this ODE, we get

$$g(y) = \frac{1}{a}\tan(aky + d),\ d \in \mathbb{R}.$$

It only remains to see that what we have is a helicoid. To this end, the parametrization of S is

$$X(x,y) = (x, y, f(x)g(y)) = (0, y, bg(y)) + x(1, 0, ag(y)),$$

which indicates that the surface is ruled. But it is well known that the only ruled minimal surfaces in $\mathbb{R}^3$ are planes and helicoids ([1]) and since g is not a constant function, S must be a helicoid.

(3) Case $f'' \neq 0$. We will prove that this case is not possible. By symmetry in the discussion of the case, we also suppose $g'' \neq 0$. If we divide (3.1) by $ff'^2gg'^2$, we have

$$\frac{f''}{ff'^2g'^2} + \frac{f''f}{f'^2} - 2 + \frac{g''}{f'^2gg'^2} + \frac{gg''}{g'^2} = 0.$$

Let us differentiate with respect to x and then with respect to y, to see

$$\left(\frac{f''}{ff'^2}\right)'\left(\frac{1}{g'^2}\right)' + \left(\frac{1}{f'^2}\right)'\left(\frac{g''}{gg'^2}\right)' = 0. \tag{3.2}$$

Since $f''g'' \neq 0$, we divide (3.2) by $(1/g'^2)'(1/f'^2)'$ and we conclude that there exists a constant $a \in \mathbb{R}$ such that

$$\left(\frac{f''}{ff'^2}\right)' \frac{1}{\left(\frac{1}{f'^2}\right)'} = a = -\left(\frac{g''}{gg'^2}\right)' \frac{1}{\left(\frac{1}{g'^2}\right)'}.$$

Hence there are constants $b, c \in \mathbb{R}$ such that

$$\frac{f''}{ff'^2} = a\frac{1}{f'^2} + b, \quad -\frac{g''}{gg'^2} = a\frac{1}{g'^2} + c,$$

or equivalently,

$$f'' = f(a + bf'^2), \quad g'' = -g(a + cg'^2). \tag{3.3}$$

Taking into account (3.3), we replace f'' and g'' in (3.1), obtaining

$$(a + bf'^2)(1 + f^2g'^2) - 2f'^2g'^2 - (a + cg'^2)(1 + f'^2g^2) = 0.$$

If we divide by $f'^2g'^2$, we get

$$\frac{c - af^2}{f'^2} + 2 - bf^2 = \frac{b - ag^2}{g'^2} - cg^2.$$

We use again the fact that each side of this equation depends only on x and only on y respectively, hence there exists $\lambda \in \mathbb{R}$ such that

$$f'^2 = \frac{c - af^2}{\lambda + bf^2 - 2}, \quad g'^2 = \frac{b - ag^2}{\lambda + cg^2}. \tag{3.4}$$

Differentiating with respect to x and y, respectively, we have

$$f'' = -\frac{f(bc + a(\lambda - 2))}{(\lambda + bf^2 - 2)^2}, \quad g'' = -\frac{g(a\lambda + bc)}{(\lambda + cg^2)^2}. \tag{3.5}$$

Let us compare these expressions of f'' and g'' with the ones that appeared in (3.3) and replace the value of f'^2 and g'^2 obtained in (3.4). After some manipulations, we get

$$(bc + a(\lambda - 2))\,(\lambda - 1 + bf^2) = 0,$$
$$(bc + a\lambda)\,(\lambda - 1 + cg^2) = 0.$$

We discuss all possibilities.

(a) If $bc + a(\lambda - 2) = bc + a\lambda = 0$, then $a = 0$ and $bc = 0$. Then (3.5) gives $f'' = 0$ or $g'' = 0$, a contradiction.

(b) If $bc + a(\lambda - 2) = 0$ and $c = \lambda - 1 = 0$, we obtain $a = 0$. From (3.5), we get $g'' = 0$, a contradiction.

(c) If $bc + a\lambda = 0$ and $b = \lambda - 1 = 0$, then $a = 0$ and (3.5) gives $f'' = 0$, a contradiction.

(d) If $b = c = 0$ and $\lambda = 1$, from the expressions of f'^2 and g'^2 in (3.4), we deduce $f'^2 = af^2$ and $g'^2 = -ag^2$, that is, $a = 0$. Then (3.5) gives $f' = g' = 0$, a contradiction again.

4. Proof of Theorem 1.3

The computation of K for the surface $X(x, y) = (x, y, f(x)g(y))$ gives

$$K = \frac{fgf''g'' - f'^2g'^2}{(1 + f'^2g^2 + f^2g'^2)^2}. \tag{4.1}$$

4.1. Case $K = 0$

If $K = 0$, then

$$ff''gg'' = f'^2g'^2. \tag{4.2}$$

Since the roles of the functions f and g are symmetric in (4.2), we discuss the different cases according to the function f.

(1) Case $f' = 0$. Then f is a constant function $f(x) = x_0$ and the parametrization of the surface writes as $X(x, y) = (0, y, x_0g(y)) + x(1, 0, 0)$. This means that S is a cylindrical surface whose directrix lies in the yz-plane and the rulings are parallel to the x-axis.

(2) Case $f'' = 0$ and $f', g' \neq 0$. Now $f(x) = ax + b$, $a, b \in \mathbb{R}$, $a \neq 0$. Moreover, (4.2) gives $g' = 0$ and $g(y) = y_0$ is a constant function. Now S is the plane of equation $z = x_0(ax + b)$.

(3) Case $f'' \neq 0$. By the symmetry on the arguments, we also suppose $g'' \neq 0$. Equation (4.2) writes as

$$\frac{ff''}{f'^2} = \frac{g'^2}{gg''}.$$

As in each side of this equation we have a function depending only on x and other depending only on y, there exists $a \in \mathbb{R}$, $a \neq 0$, such that

$$\frac{ff''}{f'^2} = a = \frac{g'^2}{gg''}.$$

A direct integration implies that there exist $b, c > 0$ such that

$$f' = bf^a, \quad g' = cg^{1/a}.$$

(a) Case $a = 1$. Then

$$f(x) = pe^{bx}, g(y) = qe^{cy}, \quad p, q > 0.$$

(b) Case $a \neq 1$. Then

$$f(x) = ((1-a)bx + p)^{\frac{1}{1-a}}, \quad g(y) = \left(\frac{a-1}{a}cy + q\right)^{\frac{a}{a-1}}$$

for $p, q \in \mathbb{R}$. This concludes the case $K = 0$.

4.2. Case $K \neq 0$

The proof is by contradiction. We assume the existence of a homothetical surface S with constant Gauss curvature $K \neq 0$. Let us observe the symmetry of the expression (4.1) on f and g. If $f = 0$ or $f' = 0$, then (4.1) implies $K = 0$, which is not our case. If $f'' = 0$, then $f(x) = ax + b$, for some constants a, b, $a \neq 0$. Then (4.1) writes as

$$K(1 + a^2g^2 + (ax+b)^2g'^2)^2 + a^2g'^2 = 0.$$

This is a polynomial equation on x of degree 4 because $K \neq 0$. Then the leading coefficient, namely $Ka^4g'^4$, must vanish. This means $g' = 0$ and (4.1) gives now $K = 0$: contradiction.

Thus $f'' \neq 0$. Interchanging the argument with g, we also suppose $g'' \neq 0$. In particular, $fgf'g' \neq 0$.

We write (4.1) as

$$K(1 + f'^2g^2 + f^2g'^2)^2 - fgf''g'' + f'^2g'^2 = 0. \tag{4.3}$$

Then

$$\log\left((1 + f'^2g^2 + f^2g'^2)^2 + \frac{1}{K}f'^2g'^2\right) = \log\left(\frac{1}{K}fgf''g''\right)$$

and so

$$\frac{\partial^2}{\partial x \partial y}\log\left((1 + f'^2g^2 + f^2g'^2)^2 + \frac{1}{K}f'^2g'^2\right) = 0.$$

This implies

(4.4)

$$(f'^2g'^2 + KD^2)\left(f''g'' + 2K\left(D(f''g + fg'') + (fg'^2 + f''g^2)(f'^2g + f^2g'')\right)\right)$$
$$- \left(f''g'^2 + 2KD(fg'^2 + f''g^2)\right)\left(f'^2g'' + 2KD(f'^2g + f^2g'')\right) = 0,$$

where $D = 1 + {f'}^2 g^2 + f^2 {g'}^2$. On the other hand, we take the derivative in (4.3) with respect to x and obtain

$$4Kf'D(fg'^2 + f''g^2) + 2f'f''g'^2 - (f'f'' + ff''')gg'' = 0. \tag{4.5}$$

Next, from equation (4.3) we obtain g'' as

$$g'' = \frac{KD^2 + f'^2 g'^2}{ff''g}$$

and we replace it first in equation (4.4) and then in equation (4.5), obtaining two equations $P_1(f, f', f'', f''', g, g') = 0$ and $P_2(f, f', f'', f''', g, g') = 0$. We see both expressions as two polynomials in g'. As they have a common solution for g', then their resultant will vanish. The computation for their resultant gives a polynomial in g, with coefficients depending on f and its first, second and third derivatives. Taking the coefficients identically zero, we obtain a system of equations for f and its derivatives. We are only interested in the leading coefficient, namely, the one for g^{28}, which must vanish. This is equivalent to

$$K^{16} f^{16} f'^{20} (f'^2 - ff'')^{14} = 0.$$

This implies $f'^2 - ff'' = 0$ and leads to $f(x) = ce^{dx}$ for c, d positive constants. Finally, we will prove that this gives a contradiction. For this value of f, we substitute f into (4.3), obtaining

$$K + c^2 \left(2d^2 K g^2 + (d^2 + 2K) g'^2 - d^2 g g''\right) e^{2dx} + c^4 K (d^2 g^2 + g'^2)^2 e^{4dx} = 0.$$

This expression is a polynomial equation on e^{dx} and so, the coefficients vanish. This implies $K = 0$, a contradiction.

5. The Lorentzian case

We consider the Lorentzian-Minkowski space $\mathbb{L}^3$, that is, $\mathbb{R}^3$ endowed with the metric $(dx)^2 + (dy)^2 - (dz)^2$. A surface immersed in $\mathbb{L}^3$ is said non-degenerate if the induced metric on S is not degenerate. The induced metric can only be of two types: positive definite and the surface is called spacelike, or a Lorentzian metric, and the surface is called timelike. For both types of surfaces, the mean curvature H and the Gauss curvature K are defined and they have the following expressions in local coordinates $X = X(s, t)$:

$$H = \epsilon\, \frac{1}{2} \frac{lG - 2mF + nE}{EG - F^2}, \quad K = \epsilon\, \frac{ln - m^2}{EG - F^2},$$

where $\epsilon = -1$ if S is spacelike and $\epsilon = 1$ if S is timelike. Here $\{E, F, G\}$ and $\{l, m, n\}$ are the coefficients of the first and second fundamental forms with respect to X, respectively. See [6] for more details. Again we ask for those translation and homothetical surfaces in $\mathbb{L}^3$ with constant mean curvature and constant Gauss curvature. Recall that the property of a surface to be a translation surface or a homothetical surface is not metric but it is given by the affine structure of $\mathbb{R}^3$ and the multiplication of real functions of $\mathbb{R}$.

We generalize the results obtained in the previous sections for non-degenerate surfaces of $\mathbb{L}^3$. The proofs are similar, and we omit the details.

(1) Extension of Theorem 1.1. Assume that S is a translation surface. The computation of K gives

$$K=-\frac{(f_2''-f_1''g_2'+g_1'(f_1''f_2'-f_1'f_2''))\,(g_2''-f_2'g_1''+f_1'(g_1''g_2'-g_1'g_2''))}{\left((1+f_1'^2-f_2'^2)(1+g_1'^2-g_2'^2)-(f_1'+g_1'-f_2'g_2')^2\right)^2}.$$

If $K=0$, then the numerator coincides with the one in (2.1) and the conclusion is that S is a cylindrical surface. In the case $K\neq 0$, the result asserts that, under the same hypothesis, there are no further examples. We discuss the cases when α and β lies in two non-orthogonal planes and when one curve is planar. In the former case, the expression of K is

$$K=-\frac{\cos^2\theta f''g''}{(-f'^2-g'^2+\cos^2\theta+2\sin\theta f'g')^2}.$$

The proof works in the same way. In the second case,

$$K=-\frac{f''(g_2''-f'g_1'')}{(1-g_2'^2-f'^2-f'^2g_1'^2+2f'g_1'g_2')^2}.$$

Again, the proof is similar because we can move f'' to the left hand-side, differentiate with respect to t and observe that there appears an expression which is a polynomial on the function f'.

(2) Extension of Theorem 1.2. As we have pointed out, this result was proved in [9].

(3) Extension of Theorem 1.3. Assume now that S is a homothetical surface and we study those surfaces with constant Gauss curvature. If S is spacelike, then the surface is locally a graph on the xy-plane and S writes as $z=f(x)g(y)$. The expression of K is

$$K=-\frac{fgf''g''-f'^2g'^2}{(1-f'^2g^2-f^2g'^2)^2},\ \text{with } 1-f'^2g^2-f^2g'^2>0.$$

If S is timelike, then the surface is locally a graph on the xz-plane or on the yz-plane. Without loss of generality, we assume that the surface writes as $x=f(y)g(z)$. Now the Gauss curvature K is

$$K=-\frac{fgf''g''-f'^2g'^2}{(1+f^2g'^2-f'^2g^2)^2},\ \text{with } 1+f^2g'^2-f'^2g^2<0.$$

Because both expressions are the same as in (4.1) and the arguments are the same as in Euclidean space, we only give the statements. If $K\neq 0$, then there does not exist homothetical (spacelike or timelike) surfaces with constant Gauss curvature K. If $K=0$, then $fgf''g''-f'^2g'^2=0$, which is the same as (4.2). They are

(a) The surface is a plane or a cylindrical surface whose directrix is contained in one of the three coordinates planes and the rulings are orthogonal to this plane, or

(b) The function $z = f(x)g(y)$ agrees with Theorem 1.3, items 1) and 2).

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