

## PRECISE LARGE DEVIATIONS FOR AGGREGATE LOSS PROCESS IN A MULTI-RISK MODEL

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**ABSTRACT.** In this paper, we consider a multi-risk model based on the policy entrance process with  $n$  independent policies. For each policy, the entrance process of the customer is a non-homogeneous Poisson process, and the claim process is a renewal process. The loss process of the single-risk model is a random sum of stochastic processes, and the actual individual claim sizes are described as extended upper negatively dependent (EUND) structure with heavy tails. We derive precise large deviations for the loss process of the multi-risk model after giving the precise large deviations of the single-risk model. Our results extend and improve the existing results in significant ways.

### 1. Introduction

The classical risk model and its generations concentrate mainly on claim process under the assumptions of independent and identically distributed (i.i.d.) claim sizes and constant premium rate, however, those assumptions do not always hold. Since whenever the insurer issues a policy, he will have to burden the potential claims entitled by the policy. Based on this, Ng et al. [10] studied the precise large deviations for the prospective process of a standard customer-arrival-based insurance risk model, in which the customers' potential claim sizes are described as i.i.d. random variables (rv's) with heavy tails, and the customer-arrival process is an arbitrary nonnegative integer-valued process. Shen et al. [12] obtained the precise large deviations for the actual aggregate loss process of a nonstandard customer-arrival-based insurance risk model, in which customer's actual claim sizes are described as i.i.d. heavy tailed rv's multiplied with a generalized shot function, and the aggregate loss process can be treated as a Poisson shot noise process. Later, Yang et al. [18] investigated

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the precise large deviations for the aggregate loss process of a dependent compound customer-arrival-based insurance risk model, in which each customer purchases a random number of insurance contracts and the individual potential claim sizes are described as negatively dependent rv's with heavy tails. The research mentioned above arouses our interest and we focus on a more realistic customer-arrival-based risk model that was first developed by Li and Kong [7]. Here are some reasons why we are interested in this model. Firstly, the individual claim time is often later than the corresponding arrival time. This point has either never been mentioned in empirical research or vaguely demonstrated. Secondly, the insurance company provides  $n$  types of insurance contracts rather than one and each customer can claim more than once within the validity time. We state the details of the model as follows:

For the  $i$ th type of policy,  $i = 1, \dots, n$ , the arrival time of the  $j$ th customer is  $\sigma_j^i$  and that  $\{N_i(t); t \geq 0\}$  is a counting process associated with  $\{\sigma_j^i\}_{j=1}^\infty$ , i.e.,  $N_i(t) = \max\{j; \sigma_j^i \leq t\}$  is the number of the policies issued before  $t$ .  $T_{jk}^i$  denotes the duration time from  $\sigma_j^i$  to the  $k$ th claim time of the  $j$ th customer.  $\{M_j^i(t); t \geq 0\}$  is the counting process associated with  $\{T_{jk}^i\}_{k=1}^\infty$ , i.e.,  $M_j^i(t) = \max\{k; T_{jk}^i \leq t\}$ . The premium charged by the insurer and the validity time are supposed to be two constants, denoted by  $d_i$  and  $c_i$ , respectively. It is obvious that the  $j$ th customer can claim at most  $M_j^i(c_i)$  times for each  $i$ . The  $k$ th potential claim size of the  $j$ th customer is denoted by  $Y_{jk}^i$  and the corresponding actual claim size is  $Y_{jk}^i I\{T_{jk}^i \leq t - \sigma_j^i\}$  within the validity time. Hence, the total claim amount of the  $j$ th customer and the loss process of the model up to time  $t$  are, respectively,

$$(1.1) \quad H_j^i(t) = \sum_{k=1}^{M_j^i(c_i)} Y_{jk}^i I\{T_{jk}^i \leq t - \sigma_j^i\}$$

and

$$(1.2) \quad X_i(t) = \sum_{j=1}^{N_i(t)} (H_j^i(t) - d_i),$$

where  $I\{A\}$  is the indicator function of event  $A$ . Thus, the total loss process of the multi-risk model up to time  $t$  is

$$(1.3) \quad X(n, t) = \sum_{i=1}^n X_i(t).$$

*Remark 1.1.* Throughout the paper, a summation over an empty index set produces a value 0 by convention. Moreover, in (1.1),  $k = 0$  means that the customer has not claimed, namely  $Y_{j0}^i = 0$ . Note that  $X_i(t)$  can be considered as a special shot noise process.

We study the large deviations of  $X_i(t)$  and  $X(n, t)$  respectively. For classical works of large deviations with heavy tails, we refer to [5, 6, 9, 11, 14]. Recently,

more attention is paid to the large deviations of partial or random sums of rv's with dependent structures. Wang et al. [16] showed that the negatively associated (NA) structure has no effect on the asymptotic behavior of the tail of partial sums, and Tang [13] considered large deviations for partial sums with negatively dependent (ND) structure that is more verifiable than the commonly used notion of NA. Later, Liu [8] extended the study to the extended negatively dependent (END) structure. For more works on this topic, we refer to [2, 3, 17], among many others.

Throughout, for positive functions  $a(x)$  and  $b(x)$ , we write  $a(x) = o(b(x))$  if  $\lim_{x \rightarrow \infty} a(x)/b(x) = 0$ ;  $a(x) \lesssim b(x)$  if  $\limsup_{x \rightarrow \infty} a(x)/b(x) \leq 1$  or  $\liminf_{x \rightarrow \infty} b(x)/a(x) \geq 1$  and  $a(x) \sim b(x)$  if both  $a(x) \lesssim b(x)$  and  $a(x) \gtrsim b(x)$ . Furthermore, for two positive bivariate functions  $a(x; t)$  and  $b(x; t)$ , we say that  $a(x; t) \lesssim b(x; t)$  holds uniformly for  $x \in \Delta_t \neq \emptyset$  as  $t \rightarrow \infty$  if

$$\limsup_{t \rightarrow \infty} \sup_{x \in \Delta_t} \frac{a(x; t)}{b(x; t)} \leq 1.$$

The rest of this paper is organized as follows. Section 2 recalls the definition of heavy tails and the dependent structures of random variables, shows some notations used in our model and states our main results. The proofs of the main results are presented in Sections 3, 4 and 5.

### 2. Notations and main results

In this paper, we are interested in the heavy tailed claims. A distribution function  $F$  has dominated varying tails (denoted by  $\mathcal{D}$ ) if and only if

$$\limsup_{x \rightarrow \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty \text{ for any } y \in (0, 1) \text{ (or, equivalently, for } y = \frac{1}{2}\text{)}.$$

Closely related class is the long-tailed class (denoted by  $\mathcal{L}$ ). A distribution function  $F$  is in  $\mathcal{L}$  if and only if

$$\lim_{x \rightarrow \infty} \frac{\overline{F}(x + y)}{\overline{F}(x)} = 1 \text{ for any } y > 0.$$

Another important subclass of heavy-tailed distributions is the consistently varying class (denoted by  $\mathcal{C}$ ). A distribution function  $F$  is in  $\mathcal{C}$  if and only if

$$\lim_{y \searrow 1} \liminf_{x \rightarrow \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1, \text{ or, equivalently, } \lim_{y \nearrow 1} \limsup_{x \rightarrow \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1.$$

It is well known that  $\mathcal{C} \subset \mathcal{D} \cap \mathcal{L}$ . Set  $\overline{F}_*(y) = \liminf_{x \rightarrow \infty} \frac{\overline{F}(xy)}{\overline{F}(x)}$  and  $\mathbb{J}_F = \inf\{-\frac{\log \overline{F}_*(y)}{\log y} : y > 1\}$ , where  $\mathbb{J}_F$  is called the upper Matuszewska index of the distribution function  $F$ . For more details of the Matuszewska indices see [1].

Now, we introduce a dependent structure of rv's.

**Definition 2.1.** The rv's  $\{\xi_k; k = 1, 2, \dots\}$  are called extended negatively dependent (END) if there is some  $M > 0$  such that both

$$(2.1) \quad \Pr\left(\bigcap_{k=1}^m \{\xi_k \leq x_k\}\right) \leq M \prod_{k=1}^m \Pr(\xi_k \leq x_k)$$

and

$$(2.2) \quad \Pr\left(\bigcap_{k=1}^m \{\xi_k > x_k\}\right) \leq M \prod_{k=1}^m \Pr(\xi_k > x_k)$$

hold for  $m = 1, 2, \dots$  and all  $x_1, \dots, x_m$ .

*Remark 2.1.* Recall that  $\xi_1, \xi_2, \dots$  are called ELND if (2.1) holds and EUND if (2.2) holds. the rv's  $\xi_1, \xi_2, \dots$  are called negatively dependent (ND) with  $M = 1$ , and they are called positively dependent (PD) if the inequalities (2.1) and (2.2) hold both in the reverse direction. Obviously, an ND sequence must be an END sequence. Furthermore, for some PD sequences, it is possible to find a right positive constant  $M$  such that (2.1) and (2.2) hold. Therefore, the END structure is more general than the ND structure in that it can reflect not only a negative dependence structure but also a positive one to some extent. More details of ND can be found in Ebrahimi and Ghosh [4].

For model (1.3), we assume that the  $n$  kinds of policies are mutually independent. Besides this, we need some further assumptions on our model. For each  $i = 1, \dots, n$ , we suppose that:

**Assumption 2.1.**  $\{N_i(t); t \geq 0\}$  is a non-homogeneous Poisson process with finite intensity function  $\lambda_i(t)$ , and the accumulated intensity function  $\Lambda_i(t) = \int_0^t \lambda_i(s) ds$  satisfying  $\Lambda_i(t) \rightarrow \infty$  as  $t \rightarrow \infty$ .

**Assumption 2.2.**  $\{M_j^i(t); t \geq 0\}_j$  are independent ordinary renewal processes with mean function  $EM_j^i(t) = \nu_i(t)$ ,  $j \geq 1$ .

**Assumption 2.3.** The nonnegative rv's  $Y_{jk}^i$ ,  $j \geq 1$ ,  $k \geq 1$  are independent with a common distribution function  $F_i \in \mathcal{C}$  satisfying  $\mu_i = EY_{11}^i < \infty$ .

**Assumption 2.4.** The sequences  $\{Y_{jk}^i; j \geq 1, k \geq 1\}$ ,  $\{M_j^i(t); t \geq 0\}$  and  $\{N_i(t); t \geq 0\}$  are mutually independent.

*Remark 2.2.* By Assumptions 2.2 and 2.3, for  $i = 1, \dots, n$ , we can take  $d_i = (1 + \rho)\mu_i\nu_i(c_i)$  with  $\rho$  as the safety loading coefficient.

Under Assumption 2.1 and Lemma 8.1 of Mikosch and Nagaev [9], we can conclude that for every  $t > 0$  and  $i = 1, \dots, n$ ,

$$(2.3) \quad X_i(t) \stackrel{d}{=} \sum_{j=1}^{N_i(t)} \left( \sum_{k=1}^{M_j^i(c_i)} Y_{jk}^i I\{T_{jk}^i \leq t - U_j^i\} - d_i \right),$$

where  $\{U_j^i, j \geq 1\}$  is a sequence of i.i.d. rv's with a common distribution function  $\frac{\Lambda_i(s)}{\Lambda_i(t)}$  for  $0 \leq s \leq t$ , independent of the non-homogeneous Poisson processes  $N_i(t)$  and all other sources of randomness.

Now we introduce some notations that will be used later. For fixed  $t > 0$ ,  $i = 1, \dots, n$  and all  $j \geq 1, k \geq 1$ , write

$$Z_{jk}^i(t) = Y_{jk}^i I\{T_{jk}^i \leq t - U_j^i\} \quad \text{and} \quad h_j^i = \sum_{k=1}^{M_j^i(c_i)} Y_{jk}^i.$$

For easiness of notations, we use  $\widehat{H}_j^i(t)$  and  $\widehat{X}_i(t)$  to denote  $\sum_{k=1}^{M_j^i(c_i)} Z_{jk}^i(t)$  and

$$\sum_{j=1}^{N_i(t)} \left( \sum_{k=1}^{M_j^i(c_i)} Z_{jk}^i(t) - d_i \right), \text{ respectively.}$$

For each  $i = 1, \dots, n$ , by Assumption 2.2 and Assumption 2.3, it is easy to check that the rv's  $h_1^i, h_2^i, \dots$  are mutually independent and

$$(2.4) \quad \Pr(h_j^i > x) = \sum_{m=1}^{\infty} \Pr(M_j^i(c_i) = m) \Pr\left(\sum_{k=1}^m Y_{jk}^i > x\right) \\ \sim \nu_i(c_i) \overline{F}_i(x) \quad \text{as } x \rightarrow \infty.$$

Unfortunately, for fixed  $i$  and  $j$ , it seems not accurate to say that  $\{Z_{jk}^i(t)\}_k$  is a sequence of independent rv's since  $Z_{j1}^i(t), Z_{j2}^i(t), \dots$  contain a common  $U_j^i$ . Precisely, we assume that  $\{Z_{jk}^i(t)\}_k$  fulfill the following dependence structure.

**Assumption 2.5.** For fixed  $i, j$  and  $t > c_i$ , the sequence  $\{Z_{jk}^i(t)\}_k$  is EUND.

*Remark 2.3.* Assumption 2.5 is natural from the following discussions. For any  $x_l, x_k, 1 \leq k \neq l < \infty$ , by Assumption 2.2, we have

$$\Pr(Z_{jk}^i(t) > x_k, Z_{jl}^i(t) > x_l) \\ = \Pr(Y_{jk}^i > x_k, Y_{jl}^i > x_l) \Pr(T_{jk}^i \leq t - U_j, T_{jl}^i \leq t - U_j) \\ = \overline{F}_i(x_k) \overline{F}_i(x_l) \Pr(T_{j(k \vee l)}^i \leq t - U_j^i),$$

where  $k \vee l = \max\{k, l\}$ . Choosing some positive constant  $M$  satisfying  $M \geq \sup_k \frac{1}{\Pr(T_{jk}^i \leq t - U_j^i)}$  such that

$$\Pr(Z_{jk}^i(t) > x_k, Z_{jl}^i(t) > x_l) \leq M \Pr(Z_{jk}^i(t) > x_k) \Pr(Z_{jl}^i(t) > x_l).$$

Similarly, for any  $m \geq 2$  and all real number  $x_1, x_2, \dots, x_m$ , we derive

$$(2.5) \quad \Pr(Z_{j1}^i(t) > x_1, \dots, Z_{jm}^i(t) > x_m) \leq M \prod_{k=1}^m \Pr(Z_{jk}^i(t) > x_k)$$

with the coefficient  $M > 0$  satisfying  $M \geq \left[ \prod_{k=1}^{m-1} \Pr(T_{jk}^i \leq t - U_j^i) \right]^{-1}$ .

For a real number  $x$ ,  $[x]$  stands for its integer part. Now we state the main results of this paper.

**Theorem 2.1.** *Suppose that Assumptions 2.1-2.5 hold. Then, for each  $\gamma > 0$ , the relation*

$$(2.6) \quad \Pr\left(\sum_{j=1}^{[\Lambda_1(t)]} \widehat{H}_j^1(t) - [\Lambda_1(t)]\mu_1\nu_1(c_1) > x\right) \sim [\Lambda_1(t)]\nu_1(c_1)\overline{F}_1(x)$$

holds uniformly for  $x \geq \gamma\Lambda_1(t)$ ,  $t \rightarrow \infty$ .

**Theorem 2.2.** *Suppose that Assumptions 2.1-2.5 hold. Then, for each  $\gamma > 0$ , the relation*

$$(2.7) \quad \Pr\left(X_1(t) - \Lambda_1(t)(\mu_1\nu_1(c_1) - d_1) > x\right) \sim \Lambda_1(t)\nu_1(c_1)\overline{F}_1(x)$$

holds uniformly for  $x \geq \gamma\Lambda_1(t)$ ,  $t \rightarrow \infty$ .

**Theorem 2.3.** *Suppose that Assumptions 2.1-2.5 hold. Then, for each  $\gamma > 0$ , the relation*

$$(2.8) \quad \Pr\left(X(n, t) - \sum_{i=1}^n \Lambda_i(t)(\mu_i\nu_i(c_i) - d_i) > x\right) \sim \sum_{i=1}^n \Lambda_i(t)\nu_i(c_i)\overline{F}_i(x)$$

holds uniformly for  $x \geq \gamma\overline{\Lambda}_n(t)$ ,  $\overline{\Lambda}_n(t) = \max_{1 \leq i \leq n} \{\Lambda_i(t)\}$ ,  $t \rightarrow \infty$ .

### 3. Proof of Theorem 2.1

We start our proof with some famous results of heavy-tailed distributions. The first lemma below is from Tang and Tsitsiashvili [15].

**Lemma 3.1.** *If  $F \in \mathcal{D}$ , then*

(i) *for each  $p > \mathbb{J}_F$ , there exist positive constants  $x_0$  and  $B$  such that for all  $\theta \in (0, 1]$  and  $x \geq \theta^{-1}x_0$ ,*

$$\frac{\overline{F}(\theta x)}{\overline{F}(x)} \leq B\theta^{-p}.$$

(ii) *it holds for each constant  $p > \mathbb{J}_F$  such that  $x^{-p} = o(\overline{F}(x))$ , and  $\mathbb{J}_F \geq 1$  if the distribution function  $F(x) = F(x)I\{x \geq 0\}$  has a finite mean.*

**Lemma 3.2.** *Under Assumptions 2.1-2.5, for fixed  $i = 1, \dots, n$  and  $t > c_i$ , the rv's  $\{Z_{jk}^i(t); j \geq 1, k \geq 1\}$  belong to  $\mathcal{C}$ .*

*Proof.* Recall that  $F_i \in \mathcal{C}$ , for any fixed  $t > c_i$ , we have

$$\lim_{y \searrow 1} \liminf_{x \rightarrow \infty} \frac{\Pr(Z_{jk}^i(t) > xy)}{\Pr(Z_{jk}^i(t) > x)} = \lim_{y \searrow 1} \liminf_{x \rightarrow \infty} \frac{\overline{F}_i(xy)\Pr(T_{jk}^i \leq t - U_j^i)}{\overline{F}_i(x)\Pr(T_{jk}^i \leq t - U_j^i)} = 1.$$

Parallely,

$$\lim_{y \nearrow 1} \limsup_{x \rightarrow \infty} \frac{\Pr(Z_{jk}^i(t) > xy)}{\Pr(Z_{jk}^i(t) > x)} = 1. \quad \square$$

**Lemma 3.3.** *Suppose that Assumptions 2.2-2.5 hold. Then, for  $t \rightarrow \infty$ ,*

$$\frac{\sum_{j=1}^{[\Lambda_1(t)]} (\widehat{H}_j^1(t) - h_j^1)}{[\Lambda_1(t)]} \xrightarrow{P} 0.$$

*Proof.* Recall that  $\{M_j^1(t); t \geq 0\}$  is a renewal process for fixed  $j \geq 1$ ,  $c_1$  is a constant, hence,  $M_j^1(c_1) < \infty$ . For any  $0 < \beta \leq 1$ , by Minkowski inequality, we have

$$\begin{aligned} & \mathbb{E} \left| \widehat{H}_j^1(t) - h_j^1 \right|^\beta \\ &= \sum_{m=1}^{\infty} \Pr(M_j^1(c_1) = m) \mathbb{E} \left| \sum_{k=1}^m Y_{jk}^1 I\{T_{jk}^1 > t - U_j^1\} \right|^\beta \\ &\leq \frac{1}{\Lambda_1(t)} \sum_{m=1}^{\infty} \Pr(M_j^1(c_1) = m) \sum_{k=1}^m \mathbb{E} |Y_{jk}^1|^\beta \int_{t-c_1}^t \Pr(T_{jk}^1 > t-s) d\Lambda_1(s) \\ &\leq \frac{1}{\Lambda_1(t)} \nu_1(c_1) \mathbb{E} |Y_{11}^1|^\beta (\Lambda_1(t) - \Lambda_1(t - c_1)). \end{aligned}$$

Using Minkowski inequality again, it holds for  $t \rightarrow \infty$  that

$$\begin{aligned} \mathbb{E} \left| \frac{\sum_{j=1}^{[\Lambda_1(t)]} (\widehat{H}_j^1(t) - h_j^1)}{[\Lambda_1(t)]} \right|^\beta &\leq \frac{1}{[\Lambda_1(t)]^\beta} \sum_{j=1}^{[\Lambda_1(t)]} \mathbb{E} |\widehat{H}_j^1(t) - h_j^1|^\beta \\ &\leq \frac{1}{[\Lambda_1(t)]^\beta} \frac{[\Lambda_1(t)]}{\Lambda_1(t)} \nu_1(c_1) \mathbb{E} |Y_{11}^1|^\beta (\Lambda_1(t) - \Lambda_1(t - c_1)) \\ &\rightarrow 0. \end{aligned}$$

This ends the proof of Lemma 3.3. □

**Lemma 3.4.** *Under Assumptions 2.1-2.5, for any  $\delta > 0$ , it follows that*

$$\lim_{t \rightarrow \infty} \Pr \left( \left| \sum_{j=1}^{[\Lambda_1(t)]} \widehat{H}_j^1(t) - [\Lambda_1(t)] \mu_1 \nu_1(c_1) \right| > \delta \Lambda_1(t) \right) = 0.$$

*Proof.* For any  $\delta > 0$ ,

$$\begin{aligned} & \Pr \left( \left| \sum_{j=1}^{[\Lambda_1(t)]} \widehat{H}_j^1(t) - [\Lambda_1(t)] \mu_1 \nu_1(c_1) \right| > \delta \Lambda_1(t) \right) \\ &\leq \Pr \left( \left| \sum_{j=1}^{[\Lambda_1(t)]} (\widehat{H}_j^1(t) - h_j^1) \right| > \frac{\delta \Lambda_1(t)}{2} \right) + \Pr \left( \left| \sum_{j=1}^{[\Lambda_1(t)]} h_j^1 - [\Lambda_1(t)] \mu_1 \nu_1(c_1) \right| > \frac{\delta \Lambda_1(t)}{2} \right) \\ &= I_1 + I_2. \end{aligned}$$

By Lemma 3.3,  $I_1$  converges to zero as  $t \rightarrow \infty$ . Since

$$E\left[\sum_{j=1}^{[\Lambda_1(t)]} h_j^1\right] = [\Lambda_1(t)]\mu_1\nu_1(c_1),$$

by the law of large numbers for the partial sums  $\sum_{j=1}^{[\Lambda_1(t)]} h_j^1$ ,  $I_2$  converges to zero as  $t \rightarrow \infty$ . This ends the proof of Lemma 3.4.  $\square$

**Lemma 3.5.** *Under Assumptions 2.2-2.5, for any  $\delta > 0$  and  $x > 0$ , there exists some positive constant  $C_1 = C_1(\delta)$  such that*

$$Pr(h_j^1 > x) \leq \nu_1(c_1)\bar{F}_1(\delta x) + C_1x^{-\frac{1}{\delta}}.$$

*Proof.* By Lemma 2.4 of Ng et al. [11], there exist positive constants  $\delta$  and  $C = C(\delta)$  such that for all  $x > 0$  and  $m \geq 1$ ,

$$Pr\left(\sum_{k=1}^m Y_{jk}^1 > x\right) \leq m\bar{F}_1(\delta x) + Cm^{\frac{1}{\delta}}x^{-\frac{1}{\delta}}.$$

Since  $M_j^1(c_1) < \infty$ , for any  $j \geq 1$ , we have

$$\begin{aligned} Pr(h_j^1 > x) &= \sum_{m \geq 1} Pr(M_j^1(c_1) = m)Pr\left(\sum_{k=1}^m Y_{jk}^1 > x\right) \\ &\leq \sum_{m \geq 1} Pr(M_j^1(c_1) = m)[m\bar{F}_1(\delta x) + Cm^{\frac{1}{\delta}}x^{-\frac{1}{\delta}}] \\ &= \nu_1(c_1)\bar{F}_1(\delta x) + C_1x^{-\frac{1}{\delta}}, \end{aligned}$$

where  $C_1 = CE[M_j^1(c_1)]^{\frac{1}{\delta}} < \infty$ . This ends the proof of Lemma 3.5.  $\square$

**Lemma 3.6.** *Under Assumptions 2.1-2.5, for any  $t > 0$ ,  $x > 0$  and  $l \geq 1$ , there exist some positive constants  $v, \delta$  ( $v < \delta$ ),  $C_2 = C_2(v, \delta)$  and  $C_3 = C_3(v, \delta)$  such that*

$$(3.1) \quad Pr\left(\sum_{j=1}^l \widehat{H}_j^1(t) > x\right) \leq l\nu_1(c_1)\bar{F}_1(vx) + C_2lx^{-\frac{1}{v}} + C_3l^{\frac{1}{\delta}}x^{-\frac{1}{\delta}}.$$

*Proof.* Recall  $\widehat{H}_j^1(t)$  and  $h_j^1$ , denote  $\tilde{h}_j^1 = h_j^1I\{0 \leq h_j^1 \leq \delta x\} + \delta xI\{h_j^1 > \delta x\}$ . By employing the arguments in Lemma 2.3 of Chen et al. [2], we obtain, respectively, for  $r > 0$ ,

$$(3.2) \quad Pr\left(\sum_{j=1}^l \widehat{H}_j^1(t) > x\right) \leq Pr\left(\sum_{j=1}^l h_j^1 > x\right) \leq lPr(h_j^1 > \delta x) + Pr\left(\sum_{j=1}^l \tilde{h}_j^1 > x\right)$$



and

$$(3.3) \quad \Pr\left(\sum_{j=1}^l \tilde{h}_j^1 > x\right) \leq e^{-rx} \mathbb{E} e^{r \sum_{j=1}^l \tilde{h}_j^1} \\ \leq \exp\left\{-rx + l\left(\frac{e^{r\delta x} - 1}{\delta x} \mu_1 \nu_1(c_1) + (e^{r\delta x} - 1) \Pr(h_j^1 > \delta x)\right)\right\}.$$

Lemma 3.5 implies that for some  $0 < v < \delta$ , there exists some positive constant  $C_2 = C_2(v, \delta)$  such that

$$(3.4) \quad \Pr(h_j^1 > \delta x) \leq \nu_1(c_1) \bar{F}_1(vx) + C_2 x^{-\frac{1}{v}}.$$

By the finiteness of  $\mu_1$  and the fact of  $F_1 \in \mathcal{C} \subset \mathcal{D}$ , it holds for  $x \rightarrow \infty$  that

$$(3.5) \quad x \bar{F}_1(vx) \rightarrow 0.$$

Take  $r = \ln\left(\frac{x}{l\mu_1\nu_1(c_1)} + 1\right)/\delta x$ . For some small  $v > 0$  that satisfies  $1 - \frac{1}{v} < 0$ , it follows from (3.4) and (3.5) that

$$(3.6) \quad \Pr\left(\sum_{j=1}^l \tilde{h}_j^1 > x\right) \leq \exp\left\{-rx + \frac{1}{\delta} + \frac{x \bar{F}_1(vx)}{\mu_1} + C_2 \frac{x^{1-\frac{1}{v}}}{\mu_1 \nu_1(c_1)}\right\} \\ \leq C_3 l^{\frac{1}{\delta}} x^{-\frac{1}{\delta}},$$

where

$$C_3 = \sup_{x>0} \exp\left\{\frac{1}{\delta} + \frac{x \bar{F}_1(vx)}{\mu_1} + C_2 \frac{x^{1-\frac{1}{v}}}{\mu_1 \nu_1(c_1)}\right\} (\mu_1 \nu_1(c_1))^{\frac{1}{\delta}} < \infty.$$

Substituting (3.4) and (3.6) into (3.2) yields (3.1). □

**Lemma 3.7.** *Under Assumption 2.2-2.5, for any  $t > c_1$  and  $j \geq 1$ , it follows that*

$$\Pr\left(\sum_{k=1}^m Z_{jk}^1(t) > x\right) \sim \sum_{k=1}^m \Pr(Z_{jk}^1(t) > x) \quad \text{as } x \rightarrow \infty.$$

*Proof.* Observe that, for fixed  $t > c_1$  and  $j \geq 1$ , by Assumption 2.5, we have

$$\Pr\left(\sum_{k=1}^m Z_{jk}^1(t) > x\right) \geq \Pr\left(\max_{1 \leq k \leq m} Z_{jk}^1(t) > x\right) \\ = \sum_{k=1}^m \Pr(Z_{jk}^1(t) > x) - \sum_{1 \leq k < l \leq m} \Pr(Z_{jk}^1(t) > x, Z_{jl}^1(t) > x) \\ \geq \sum_{k=1}^m \Pr(Z_{jk}^1(t) > x) - M \left[\sum_{k=1}^m \Pr(Z_{jk}^1(t) > x)\right]^2.$$

It follows from (3.5) that  $\Pr(Z_{jk}^1(t) > x) = \bar{F}_1(x)\Pr(T_{jk}^1 \leq t - U_j^1) \rightarrow 0$  as  $x \rightarrow \infty$ . Thus,

$$\Pr\left(\sum_{k=1}^m Z_{jk}^1(t) > x\right) \gtrsim \sum_{k=1}^m \Pr(Z_{jk}^1(t) > x).$$

By a standard truncation argument we have

$$\begin{aligned} & \Pr\left(\sum_{k=1}^m Z_{jk}^1(t) > x\right) \\ &= \Pr\left(\sum_{k=1}^m Z_{jk}^1(t) > x, \max_{1 \leq k \leq m} Z_{jk}^1(t) > (1 - \delta)x\right) \\ & \quad + \Pr\left(\sum_{k=1}^m Z_{jk}^1(t) > x, \max_{1 \leq k \leq m} Z_{jk}^1(t) \leq (1 - \delta)x\right) \\ &\leq \sum_{k=1}^m \Pr(Z_{jk}^1(t) > (1 - \delta)x) + \sum_{k=1}^m \Pr\left(\sum_{1 \leq l \leq m; l \neq k} Z_{jl}^1(t) > \delta x, Z_{jk}^1(t) > \frac{x}{m}\right) \\ &\leq \sum_{k=1}^m \Pr(Z_{jk}^1(t) > (1 - \delta)x) + M \sum_{1 \leq l \leq m; l \neq k} \sum_{k=1}^m \Pr(Z_{jl}^1(t) > \frac{\delta x}{m}) \Pr(Z_{jk}^1(t) > \frac{x}{m}). \end{aligned}$$

By Lemma 3.1 and Lemma 3.2, for any  $j \geq 1, 1 \leq k \leq m$  and  $t > c_1$ , we have

$$\begin{aligned} (3.7) \quad \Pr(Z_{jk}^1(t) > \frac{x}{m}) &\leq I\{x \leq mx_0\} + \Pr(Z_{jk}^1(t) > \frac{x}{m})I\{x > mx_0\} \\ &\leq \left(\frac{mx_0}{x}\right)^p + Bm^p \Pr(Z_{jk}^1(t) > x) \\ &\leq C_4 m^p \Pr(Z_{jk}^1(t) > x), \end{aligned}$$

with a positive constant  $C_4 > B$ . Therefore, for any  $m \geq 1$ ,

$$\limsup_{x \rightarrow \infty} \frac{\sum_{1 \leq l \leq m; l \neq k} \sum_{k=1}^m \Pr(Z_{jl}^1(t) > \frac{\delta x}{m}) \Pr(Z_{jk}^1(t) > \frac{x}{m})}{\sum_{k=1}^m \Pr(Z_{jk}^1(t) > x)} = 0.$$

Thus, letting  $\delta \rightarrow 0$ , by Lemma 3.2, it holds for  $x \rightarrow \infty$  that

$$\Pr\left(\sum_{k=1}^m Z_{jk}^1(t) > x\right) \lesssim \sum_{k=1}^m \Pr(Z_{jk}^1(t) > x).$$

The proof of Lemma 3.7 is accomplished. □

**Lemma 3.8.** *Under the Assumptions 2.2-2.5, for any  $t > c_1$  and  $j \geq 1$ , the relation*

$$\Pr(\widehat{H}_j^1(t) > x) \gtrsim \nu_1(c_1)\bar{F}_1(x)\Pr(U_j^1 \leq t - c_1)$$

*holds for  $x \rightarrow \infty$ .*

*Proof.* For some  $0 < m_0 < \infty$ , we have

$$\begin{aligned} \Pr(\widehat{H}_j^1(t) > x) &= \sum_{m=1}^{\infty} \Pr(M_j^1(c_1) = m) \Pr\left(\sum_{k=1}^m Z_{jk}^1(t) > x\right) \\ &= \left(\sum_{m=1}^{m_0} + \sum_{m=m_0+1}^{\infty}\right) \Pr(M_j^1(c_1) = m) \Pr\left(\sum_{k=1}^m Z_{jk}^1(t) > x\right) \\ &= I_3 + I_4. \end{aligned}$$

For  $I_3$ . Recall that  $F_1 \in \mathcal{C}$ , for  $x \rightarrow \infty$ , we have

$$\begin{aligned} I_3 &\geq \sum_{m=1}^{m_0} \Pr(M_j^1(c_1) = m) \int_{0^-}^{t-c_1} \Pr\left(\sum_{k=1}^m Y_{jk}^1 > x\right) \Pr(U_j^1 \in ds) \\ &\sim \overline{F}_1(x) \frac{\Lambda_1(t-c_1)}{\Lambda_1(t)} \sum_{m=1}^{m_0} \Pr(M_j^1(c_1) = m) m \\ &= \overline{F}_1(x) \frac{\Lambda_1(t-c_1)}{\Lambda_1(t)} \mathbb{E}[M_j^1(c_1) I\{M_j^1(c_1) \leq m_0\}]. \end{aligned}$$

For  $I_4$ . By Kesten's inequality, it holds for every  $0 < \delta < 1$  and some  $C_5 > 0$  such that

$$\begin{aligned} I_4 &\leq \sum_{m=m_0+1}^{\infty} \Pr(M_j^1(c_1) = m) \Pr\left(\sum_{k=1}^m Y_{jk}^1 > x\right) \\ &\leq C_5 \overline{F}_1(x) \sum_{m=m_0+1}^{\infty} \Pr(M_j^1(c_1) = m) (1 + \delta)^m. \end{aligned}$$

Since  $M_j^1(c_1)$  has a finite moment generating function, we have

$$\limsup_{x \rightarrow \infty} \frac{I_4}{I_3} \leq \frac{C_5 \Lambda_1(t)}{\Lambda_1(t-c_1)} \frac{\sum_{m=m_0+1}^{\infty} \Pr(M_j^1(c_1) = m) (1 + \delta)^m}{\mathbb{E}[M_j^1(c_1) I\{M_j^1(c_1) \leq m_0\}]}.$$

Then, letting  $m_0$  sufficiently large yields that  $I_4 = o(I_3)$ . Thus, by Lemma 3.7, for  $x \rightarrow \infty$  and any  $t > c_1$ , it holds for sufficiently large  $m_0$  that,

$$\begin{aligned} \Pr(\widehat{H}_j^1(t) > x) &\sim \sum_{m=1}^{m_0} \Pr(M_j^1(c_1) = m) \sum_{k=1}^m \Pr(Z_{jk}^1(t) > x) \\ &\geq \sum_{m=1}^{m_0} \Pr(M_j^1(c_1) = m) \sum_{k=1}^m \Pr(Y_{jk}^1 I\{U_j^1 \leq t - c_1\} > x) \\ &= \overline{F}_1(x) \Pr(U_j^1 \leq t - c_1) \sum_{m=1}^{m_0} m \Pr(M_j^1(c_1) = m) \\ &\rightarrow \nu_1(c_1) \overline{F}_1(x) \Pr(U_j^1 \leq t - c_1). \end{aligned}$$

The proof of Lemma 3.8 is accomplished.  $\square$

*Proof of Theorem 2.1. The lower estimation.* For fixed  $t > 0$ , write

$$S_{[\Lambda_1(t)]}(t) = \sum_{j=1}^{[\Lambda_1(t)]} \widehat{H}_j^1(t), \overline{S}_{[\Lambda_1(t)]}(t) = S_{[\Lambda_1(t)]}(t) - [\Lambda_1(t)]\nu_1(c_1).$$

For any  $0 < \delta < 1$ , we have

$$\begin{aligned} (3.8) \quad & \Pr\left(\sum_{j=1}^{[\Lambda_1(t)]} \widehat{H}_j^1(t) - [\Lambda_1(t)]\nu_1(c_1)\mu_1 > x\right) \\ & \geq \Pr\left(\overline{S}_{[\Lambda_1(t)]}(t) > x, \bigcup_{1 \leq j \leq [\Lambda_1(t)]} \left\{\widehat{H}_j^1(t) > (1 + \delta)x, \max_{l \neq j} \widehat{H}_l^1(t) \leq (1 + \delta)x\right\}\right) \\ & = \sum_{j=1}^{[\Lambda_1(t)]} \Pr(\widehat{H}_j^1(t) > (1 + \delta)x) \\ & \quad \times \Pr(\overline{S}_{[\Lambda_1(t)]-1}(t) > -\delta x + \nu_1(c_1)\mu_1, \max_{1 \leq l \leq [\Lambda_1(t)]; l \neq j} \widehat{H}_l^1(t) \leq (1 + \delta)x). \end{aligned}$$

Recalling that  $F_1 \in \mathcal{C}$ , by Lemma 3.8 and the fact of  $\Pr(U_j^1 \leq t - c_1) = \frac{\Lambda_1(t-c_1)}{\Lambda_1(t)} \rightarrow 1$  as  $t \rightarrow \infty$ , it holds uniformly for  $x \geq \gamma\Lambda_1(t)$ ,  $t \rightarrow \infty$  that

$$(3.9) \quad \sum_{j=1}^{[\Lambda_1(t)]} \Pr(\widehat{H}_j^1(t) > (1 + \delta)x) \gtrsim [\Lambda_1(t)]\nu_1(c_1)\overline{F}_1(x).$$

Now we consider  $\Pr(\max_{l \neq j} \widehat{H}_l^1(t) \leq (1 + \delta)x)$ . By Lemma 3.8, for  $x \geq \gamma\Lambda_1(t)$ ,  $t \rightarrow \infty$ ,

$$\begin{aligned} (3.10) \quad & \Pr\left(\max_{1 \leq l \leq [\Lambda_1(t)]; l \neq j} \widehat{H}_l^1(t) \leq (1 + \delta)x\right) \\ & = \prod_{1 \leq l \leq [\Lambda_1(t)]; l \neq j} \Pr(\widehat{H}_l^1(t) \leq (1 + \delta)x) \\ & = \prod_{1 \leq l \leq [\Lambda_1(t)]; l \neq j} \left(1 - \Pr(\widehat{H}_l^1(t) > (1 + \delta)x)\right) \\ & \leq \left(1 - \nu_1(c_1)\overline{F}_1((1 + \delta)x)\right)^{[\Lambda_1(t)]-1} \rightarrow 1. \end{aligned}$$

Furthermore, Lemma 3.4 states that

$$(3.11) \quad \lim_{t \rightarrow \infty} \liminf_{x \geq \gamma\Lambda_1(t)} \Pr(\overline{S}_{[\Lambda_1(t)]-1}(t) > -\delta x + \nu_1(c_1)\mu_1) = 1.$$

Substituting (3.9), (3.10) and (3.11) into (3.8), it holds uniformly for  $x \geq \gamma\Lambda_1(t)$ ,  $t \rightarrow \infty$  that

$$(3.12) \quad \Pr\left(\sum_{j=1}^{[\Lambda_1(t)]} \widehat{H}_j^1(t) - [\Lambda_1(t)]\mu_1\nu_1(c_1) > x\right) \gtrsim [\Lambda_1(t)]\nu_1(c_1)\overline{F}_1(x).$$

**The upper estimation.** For  $0 < \delta < 1$ , denote  $\tilde{H}_j^1(t) = \hat{H}_j^1(t)I\{\hat{H}_j^1(t) \leq (1 - \delta)x\}$ . A standard truncation argument gives that

$$\begin{aligned}
 (3.13) \quad & \Pr\left(\sum_{j=1}^{[\Lambda_1(t)]} \hat{H}_j^1(t) - [\Lambda_1(t)]\mu_1\nu_1(c_1) > x\right) \\
 & \leq \sum_{j=1}^{[\Lambda_1(t)]} P(\hat{H}_j^1(t) > (1 - \delta)x) + \Pr\left(\sum_{j=1}^{[\Lambda_1(t)]} \tilde{H}_j^1(t) > x + [\Lambda_1(t)]\mu_1\nu_1(c_1)\right) \\
 & \leq \sum_{j=1}^{[\Lambda_1(t)]} P(h_j^1 > (1 - \delta)x) + \Pr\left(\sum_{j=1}^{[\Lambda_1(t)]} \tilde{H}_j^1(t) > x + [\Lambda_1(t)]\mu_1\nu_1(c_1)\right) \\
 & = (1 + o(1))[\Lambda_1(t)]\nu_1(c_1)\bar{F}_1(x) + I_5,
 \end{aligned}$$

where the last step is due to (2.4) and the arbitrariness of  $\delta$ . Set

$$a = \max\{-\ln([\Lambda_1(t)]\nu_1(c_1)\bar{F}_1(x)), 1\}, \quad r = \frac{a - p\tau \ln a}{(1 - \delta)x}$$

with  $\tau > 2$  and  $p > \mathbb{J}_F$ . Notice that  $a \rightarrow \infty$ ,  $r \rightarrow 0$  as  $x \geq \gamma\Lambda_1(t)$ ,  $t \rightarrow \infty$ . Using Markov's inequality yields that

$$\begin{aligned}
 (3.14) \quad & \frac{I_5}{[\Lambda_1(t)]\nu_1(c_1)\bar{F}_1(x)} \\
 & \leq \exp\{-r(x + [\Lambda_1(t)]\mu_1\nu_1(c_1)) + a\} \prod_{j=1}^{[\Lambda_1(t)]} \mathbb{E}e^{r\tilde{H}_j^1(t)}.
 \end{aligned}$$

By virtue of the inequality  $e^{ry} - 1 \leq rye^{ry}$ , for fixed  $t > 0$ , we have

$$\begin{aligned}
 (3.15) \quad & \mathbb{E}e^{r\tilde{H}_j^1(t)} \\
 & \leq \int_0^{(1-\delta)x} (e^{ry} - 1)\Pr(H_j^1(t) \in dy) + 1 \\
 & \leq \int_0^{\frac{(1-\delta)x}{a^\tau}} (e^{ry} - 1)\Pr(H_j^1(t) \in dy) + \int_{\frac{(1-\delta)x}{a^\tau}}^{(1-\delta)x} e^{ry}\Pr(H_j^1(t) \in dy) + 1 \\
 & \leq re^{r\frac{(1-\delta)x}{a^\tau}}\mu_1\nu_1(c_1) + e^{r(1-\delta)x}\Pr(h_j^1 > \frac{(1-\delta)x}{a^\tau}) + 1 \\
 & \leq re^{r\frac{(1-\delta)x}{a^\tau}} + (1 + \delta)e^{r(1-\delta)x}\nu_1(c_1)\bar{F}_1\left(\frac{(1-\delta)x}{a^\tau}\right) + 1 \\
 & \leq re^{a^{1-\tau}}\mu_1\nu_1(c_1) + C_4(1 + \delta)e^a\nu_1(c_1)\bar{F}_1((1 - \delta)x) + 1,
 \end{aligned}$$

where all the inequalities except the last one follows from (2.4) and the last one is obtained by (3.7). Substituting (3.15) into (3.14) yields that

$$\frac{I_5}{[\Lambda_1(t)]\nu_1(c_1)\bar{F}_1(x)}$$

$$\begin{aligned} &\leq \exp \left\{ -r(x + [\Lambda_1(t)]\mu_1\nu_1(c_1)) + a \right. \\ &\quad \left. + [\Lambda_1(t)](re^{a^{1-\tau}}\mu_1\nu_1(c_1) + C_4(1 + \delta)e^a\nu_1(c_1)\overline{F}_1((1 - \delta)x)) \right\} \\ &\leq \exp \left\{ -r(x + [\Lambda_1(t)]\mu_1\nu_1(c_1)) + a \right. \\ &\quad \left. + [\Lambda_1(t)]re^{a^{1-\tau}}\mu_1\nu_1(c_1) + \frac{C_4(1 + \delta)\overline{F}_1((1 - \delta)x)}{\overline{F}_1(x)} \right\}. \end{aligned}$$

Since  $a^{1-\tau} \rightarrow 0$  as  $t \rightarrow \infty$ , by a Taylor series expansion  $e^{a^{1-\tau}} = 1 + a^{1-\tau} + o(a^{1-\tau})$ , we get

(3.16)

$$\frac{I_5}{[\Lambda_1(t)]\nu_1(c_1)\overline{F}_1(x)} \leq C_6 \exp\left\{\left(1 - \frac{1}{1 - \delta}\right)a + a^{2-\tau}\frac{\mu_1\nu_1(c_1)}{\gamma(1 - \delta)} + o(a^{2-\tau})\right\} \rightarrow 0,$$

where  $C_6 = \sup_{x>0} \exp\left\{\frac{C_4(1+\delta)\overline{F}_1((1-\delta)x)}{\overline{F}_1(x)}\right\} < \infty$ . In view of (3.8)-(3.16), the proof of Theorem 2.1 is accomplished.  $\square$

#### 4. Proof of Theorem 2.2

By Lemma 1 of Yang et al. [18], one can immediately obtain the following result:

**Lemma 4.1.** *Under Assumption 2.1, it holds for any positive constants  $q$  and  $\delta$  that*

$$E[N_1(t)]^q I\{N_1(t) > (1 + \delta)\Lambda_1(t)\} = o(1), \quad t \rightarrow \infty.$$

*Proof of Theorem 2.2.* By the law of large numbers of Poisson process, there exists some small constant  $\delta > 0$  such that

$$\Pr(|N_1(t) - \Lambda_1(t)| \leq \delta\Lambda_1(t)) \rightarrow 1, \quad t \rightarrow \infty,$$

that is, for sufficiently large  $t$ ,

$$(4.1) \quad 1 - \delta \leq \Pr(|N_1(t) - \Lambda_1(t)| \leq \delta\Lambda_1(t)) \leq 1 + \delta.$$

We can split (2.7) into three parts as follows:

$$\begin{aligned} (4.2) \quad &\Pr\left(X_1(t) - \Lambda_1(t)(\mu_1\nu_1(c_1) - d_1) > x\right) \\ &= \Pr\left(\sum_{j=1}^{N_1(t)} (\widehat{H}_j^1(t) - d_1) - \Lambda_1(t)(\mu_1\nu_1(c_1) - d_1) > x\right) \\ &= \sum_{l=1}^{\infty} \Pr(N_1(t) = l)\Pr\left(\sum_{j=1}^l (\widehat{H}_j^1(t) - d_1) - \Lambda_1(t)(\mu_1\nu_1(c_1) - d_1) > x\right) \\ &= \left( \sum_{l < (1-\delta)\Lambda_1(t)} + \sum_{|l - \Lambda_1(t)| \leq \delta\Lambda_1(t)} + \sum_{l > (1+\delta)\Lambda_1(t)} \right) \end{aligned}$$

$$\begin{aligned} & \times \Pr(N_1(t) = l) \Pr\left(\sum_{j=1}^l (\hat{H}_j^1(t) - d_1) - \Lambda_1(t)(\mu_1\nu_1(c_1) - d_1) > x\right) \\ & = J_1 + J_2 + J_3. \end{aligned}$$

We first deal with the sum of  $J_1$  in (4.2). For some positive  $\delta$  small enough such that  $\gamma - \delta\rho\mu_1\nu_1(c_1) > 0$ , i.e.,  $x - \delta\rho\Lambda_1(t)\mu_1\nu_1(c_1) \geq \Lambda_1(t)(\gamma - \delta\rho\mu_1\nu_1(c_1)) > 0$  with  $\rho$  as the safety loading coefficient mentioned in Remark 2.2. Then, applying Theorem 2.1, uniformly for  $x \geq \gamma\Lambda_1(t)$ ,  $t \rightarrow \infty$ , we get

$$\begin{aligned} (4.3) \quad J_1 & \leq \Pr(N_1(t) < (1 - \delta)\Lambda_1(t)) \\ & \quad \times \Pr\left(\sum_{j=1}^{\lfloor (1-\delta)\Lambda_1(t) \rfloor} \hat{H}_j^1(t) - [(1 - \delta)\Lambda_1(t)]\mu_1\nu_1(c_1) > x - \delta\rho\Lambda_1(t)\mu_1\nu_1(c_1)\right) \\ & = o(1)[(1 - \delta)\Lambda_1(t)\nu_1(c_1)\bar{F}_1(x - \delta\rho\Lambda_1(t)\mu_1\nu_1(c_1))] \\ & = o\left(\Lambda_1(t)\nu_1(c_1)\bar{F}_1(x)\right), \end{aligned}$$

where the last step is obtained by  $\delta \rightarrow 0$  and  $F_1 \in \mathcal{C}$ .

Next, turn to  $J_2$ . Uniformly for  $x \geq \gamma\Lambda_1(t)$ ,  $t \rightarrow \infty$ , the following holds:

$$\begin{aligned} (4.4) \quad J_2 & \leq \Pr(|N_1(t) - \Lambda_1(t)| \leq \delta\Lambda_1(t)) \\ & \quad \times \Pr\left(\sum_{j=1}^{\lfloor (1+\delta)\Lambda_1(t) \rfloor} \hat{H}_j^1(t) - [(1 + \delta)\Lambda_1(t)]\mu_1\nu_1(c_1) > x + \delta\rho\Lambda_1(t)\mu_1\nu_1(c_1)\right) \\ & \leq (1 + \delta)[(1 + \delta)\Lambda_1(t)\nu_1(c_1)]\bar{F}_1(x + \delta\rho\Lambda_1(t)\mu_1\nu_1(c_1)) \\ & \sim \Lambda_1(t)\nu_1(c_1)\bar{F}_1(x), \end{aligned}$$

where the last step is due to  $\delta \rightarrow 0$  and  $F_1 \in \mathcal{C}$ . We now prove

$$J_2 \gtrsim \Lambda_1(t)\nu_1(c_1)\bar{F}_1(x).$$

By similar arguments used to estimate  $J_1$ , by Theorem 2.1, it holds uniformly for  $x \geq \gamma\Lambda_1(t)$ ,  $t \rightarrow \infty$  that

$$\begin{aligned} (4.5) \quad J_2 & \geq \Pr(|N_1(t) - \Lambda_1(t)| \leq \delta\Lambda_1(t)) \\ & \quad \times \Pr\left(\sum_{j=1}^{\lfloor (1-\delta)\Lambda_1(t) \rfloor} \hat{H}_j^1(t) - [(1 - \delta)\Lambda_1(t)]\mu_1\nu_1(c_1) > x - \delta\rho\Lambda_1(t)\mu_1\nu_1(c_1)\right) \\ & \geq (1 - \delta)[(1 - \delta)\Lambda_1(t)\nu_1(c_1)]\bar{F}_1(x - \delta\rho\Lambda_1(t)\mu_1\nu_1(c_1)) \\ & \sim \Lambda_1(t)\nu_1(c_1)\bar{F}_1(x). \end{aligned}$$

Finally, for  $J_3$ . By the arbitrariness of  $\delta$  and  $v$  in Lemma 3.6, there exist some  $\delta$  and  $v$  such that  $\frac{1}{v} > \frac{1}{\delta} > p$  for  $p > \mathbb{J}_F > 1$ . Then,

$$\begin{aligned} (4.6) \quad J_3 & \leq \sum_{l > (1+\delta)\Lambda_1(t)} \Pr(N_1(t) = l)[l\nu_1(c_1)\bar{F}_1(vx) + C_2lx^{-\frac{1}{v}} + C_3l^{\frac{1}{\delta}}x^{-\frac{1}{\delta}}] \\ & \leq \nu_1(c_1)\bar{F}_1(vx)\mathbb{E}N_1(t)I\{N_1(t) > (1 + \delta)\Lambda_1(t)\} \end{aligned}$$

$$\begin{aligned} &+ C_2 x^{-p} \mathbb{E} N_1(t) I\{N_1(t) > (1 + \delta)\Lambda_1(t)\} \\ &+ C_3 x^{-p} \mathbb{E} N_1^p(t) I\{N_1(t) > (1 + \delta)\Lambda_1(t)\} \\ &= o(\Lambda_1(t)\nu_1(c_1)\bar{F}_1(x)), \end{aligned}$$

where the last step is obtained by Lemma 3.1 and Lemma 4.1. Precisely, Lemma 3.1 shows that  $x^{-p} = o(\bar{F}_1(x))$  for  $p > \mathbb{J}_F > 1$ , and Lemma 4.1 states that  $\mathbb{E} N_1^q(t) I\{N_1(t) > (1 + \delta)\Lambda_1(t)\} = o(1)$  for any  $q > 0$ , specifically,  $q = 1$  and  $q = p$  in (4.6), respectively. In view of (4.2)-(4.6), the proof is accomplished.  $\square$

**5. Proof of Theorem 2.3**

For fixed  $t > 0$ , write  $\bar{X}_i(t) = \hat{X}_i(t) - \Lambda_i(t)(\mu_i\nu_i(c_i) - d_i)$ ,  $i = 1, \dots, n$ . Firstly, we show three useful relations (5.1), (5.2) and (5.3) before proving (2.8). By Theorem 2.2, for  $x \geq \gamma\bar{\Lambda}_n(t)$  and sufficiently large  $t$ , we obtain that

$$\begin{aligned} (5.1) \quad &(1 - \delta)\Lambda_i(t)\nu_i(c_i)\bar{F}_i(x) \leq \Pr(\bar{X}_i(t) > x) \\ &\leq (1 + \delta)\Lambda_i(t)\nu_i(c_i)\bar{F}_i(x), \quad i = 1, \dots, n. \end{aligned}$$

Furthermore, for  $i = 1, \dots, n$  and sufficiently large  $t$ , it follows from (4.1) that

$$(5.2) \quad \Pr(|N_i(t) - \Lambda_i(t)| < \delta\Lambda_i(t)) \geq 1 - \delta.$$

Hence, for any  $i = 1, \dots, n$ , by Lemma 3.4 and (5.2), it holds uniformly for  $x \geq \gamma\bar{\Lambda}_n(t)$  and sufficiently large  $t$  that

$$\begin{aligned} (5.3) \quad &\Pr(\bar{X}_i(t) > -\delta x) \\ &\geq \Pr(|N_i(t) - \Lambda_i(t)| < \delta\Lambda_i(t)) \\ &\quad \times \Pr\left(\sum_{j=1}^{[(1-\delta)\Lambda_i(t)]} \hat{H}_j^i(t) - [(1-\delta)\Lambda_i(t)]\mu_i\nu_i(c_i) > -\delta x - \rho\delta\Lambda_i(t)\mu_i\nu_i(c_i)\right) \\ &\geq (1 - \delta)\Pr\left(\sum_{j=1}^{[(1-\delta)\Lambda_i(t)]} \hat{H}_j^i(t) - [(1-\delta)\Lambda_i(t)]\mu_i\nu_i(c_i) > -\delta\Lambda_i(t)(\gamma + \rho\mu_i\nu_i(c_i))\right) \\ &\geq (1 - \delta)^2. \end{aligned}$$

Now, we prove (2.8). Let us proceed the proof by induction. For the case in which  $n = 2$  we first show the lower estimation. For any  $\delta > 0$ , it holds uniformly for  $x \geq \gamma\bar{\Lambda}_2(t)$  and sufficiently large  $t$  that

$$\begin{aligned} (5.4) \quad &\Pr(X(2, t) - \sum_{i=1}^2 \Lambda_i(t)(\mu_i\nu_i(c_i) - d_i) > x) \\ &= \Pr(\bar{X}_1(t) + \bar{X}_2(t) > x) \\ &\geq \Pr(\bar{X}_1(t) > (1 + \delta)x)\Pr(\bar{X}_2(t) > -\delta x) \\ &\quad + \Pr(\bar{X}_2(t) > (1 + \delta)x)\Pr(\bar{X}_1(t) > -\delta x) \\ &\quad - \Pr(\bar{X}_1(t) > (1 + \delta)x)\Pr(\bar{X}_2(t) > (1 + \delta)x). \end{aligned}$$



$$\begin{aligned} &\geq (1 - \delta)^3 \Lambda_1(t) \nu_1(c_1) \overline{F}_1((1 + \delta)x) + (1 - \delta)^3 \Lambda_2(t) \nu_2(c_2) \overline{F}_2((1 + \delta)x) \\ &\quad - (1 - \delta)^2 \Lambda_1(t) \nu_1(c_1) \overline{F}_1((1 + \delta)x) \Lambda_2(t) \nu_2(c_2) \overline{F}_2((1 + \delta)x), \end{aligned}$$

where the last step is obtained by (5.1) and (5.3). On the other hand, since  $\mu_i < \infty$ , it holds uniformly for  $x \geq \gamma \overline{\Lambda}_2(t)$ ,  $t \rightarrow \infty$  that

$$(5.5) \quad \Lambda_i(t) \nu_i(c_i) \overline{F}_i((1 + \delta)x) \leq \gamma^{-1} x \nu_i(c_i) \overline{F}_i((1 + \delta)x) \rightarrow 0.$$

Thus, for any  $\delta > 0$ ,

$$(5.6) \quad \lim_{t \rightarrow \infty} \liminf_{x \geq \gamma \overline{\Lambda}_2(t)} \frac{\Lambda_1(t) \nu_1(c_1) \overline{F}_1((1 + \delta)x) \Lambda_2(t) \nu_2(c_2) \overline{F}_2((1 + \delta)x)}{\Lambda_1(t) \nu_1(c_1) \overline{F}_1((1 + \delta)x) + \Lambda_2(t) \nu_2(c_2) \overline{F}_2((1 + \delta)x)} = 0.$$

By (5.5) and (5.6) and letting  $\delta \rightarrow 0$  yields that

$$(5.7) \quad \lim_{\delta \rightarrow 0} \lim_{t \rightarrow \infty} \liminf_{x \geq \gamma \overline{\Lambda}_2(t)} \frac{\Pr(X(2, t) - \sum_{i=1}^2 \Lambda_i(t)(\mu_i \nu_i(c_i) - d_i) > x)}{\sum_{i=1}^2 \Lambda_i(t) \nu_i(c_i) \overline{F}_i(x)} \geq 1.$$

Now, we consider the upper estimation. For any  $\delta > 0$ , it holds uniformly for  $x \geq \gamma \overline{\Lambda}_2(t)$  and sufficiently large  $t$  that

$$\begin{aligned} (5.8) \quad &\Pr(X(2, t) - \sum_{i=1}^2 \Lambda_i(t)(\mu_i \nu_i(c_i) - d_i) > x) \\ &\leq \Pr\left(\max_{1 \leq i \leq 2} \overline{X}_i(t) \geq (1 - \delta)x\right) + \Pr\left(\overline{X}_1(t) + \overline{X}_2(t) > x, \max_{1 \leq i \leq 2} \overline{X}_i(t) \leq (1 - \delta)x\right) \\ &\leq \sum_{i=1}^2 \Pr(\overline{X}_i(t) > (1 - \delta)x) + \sum_{i,l=1,2; i \neq l} \Pr(\overline{X}_i(t) > \delta x, \overline{X}_l(t) > \frac{x}{2}) \\ &= \sum_{i=1}^2 \Pr(\overline{X}_i(t) > (1 - \delta)x) + \sum_{i,l=1,2; i \neq l} P(\overline{X}_i(t) > \delta x) \Pr(\overline{X}_l(t) > \frac{x}{2}) \\ &\leq (1 + \delta) \sum_{i=1}^2 \Lambda_i(t) \nu_i(c_i) \overline{F}_i((1 - \delta)x) \\ &\quad + (1 + \delta)^2 \Lambda_1(t) \nu_1(c_1) \Lambda_2(t) \nu_2(c_2) \left(\overline{F}_1(\delta x) \overline{F}_2\left(\frac{x}{2}\right) + \overline{F}_2(\delta x) \overline{F}_1\left(\frac{x}{2}\right)\right), \end{aligned}$$

where at the last step we use (5.1) three times.

To estimate (5.8), we should point out that

$$\begin{aligned} (5.9) \quad &\frac{\Lambda_1(t) \nu_1(c_1) \Lambda_2(t) \nu_2(c_2) \left(\overline{F}_1(\delta x) \overline{F}_2\left(\frac{x}{2}\right) + \overline{F}_2(\delta x) \overline{F}_1\left(\frac{x}{2}\right)\right)}{\sum_{i=1}^2 \Lambda_i(t) \nu_i(c_i) \overline{F}_i((1 - \delta)x)} \\ &\leq \max \left\{ \frac{\Lambda_1(t) \nu_1(c_1) \Lambda_2(t) \nu_2(c_2) \overline{F}_1(\delta x) \overline{F}_2\left(\frac{x}{2}\right)}{\Lambda_1(t) \nu_1(c_1) \overline{F}_1((1 - \delta)x)}, \frac{\Lambda_1(t) \nu_1(c_1) \Lambda_2(t) \nu_2(c_2) \overline{F}_2(\delta x) \overline{F}_1\left(\frac{x}{2}\right)}{\Lambda_2(t) \nu_2(c_2) \overline{F}_2((1 - \delta)x)} \right\} \\ &= \max \left\{ \frac{\Lambda_2(t) \nu_2(c_2) \overline{F}_2\left(\frac{x}{2}\right) \overline{F}_1(\delta x)}{\overline{F}_1((1 - \delta)x)}, \frac{\Lambda_1(t) \nu_1(c_1) \overline{F}_1\left(\frac{x}{2}\right) \overline{F}_2(\delta x)}{\overline{F}_2((1 - \delta)x)} \right\}. \end{aligned}$$

Furthermore, since  $F_i \in \mathcal{C} \subset \mathcal{D}$ , for any  $\delta \in (0, \frac{1}{2})$ , it follows from (5.5) that

$$(5.10) \quad \sup_{t \rightarrow \infty} \sup_{x \geq \gamma \bar{\Lambda}_2(t)} \frac{\Lambda_i(t) \nu_i(c_i) \bar{F}_i(\frac{x}{2}) \bar{F}_j(\delta x)}{\bar{F}_j((1-\delta)x)} \leq \delta.$$

Plugging (5.9) and (5.10) into (5.8) and letting  $\delta \rightarrow 0$  yields that, uniformly for  $x \geq \gamma \bar{\Lambda}_2(t)$ ,

$$(5.11) \quad \Pr(X(2, t) - \sum_{i=1}^2 \Lambda_i(t) (\mu_i \nu_i(c_i) - d_i) > x) \lesssim \sum_{i=1}^2 \Lambda_i(t) \nu_i(c_i) \bar{F}_i(x).$$

A combination of (5.7) and (5.11) shows that (2.8) holds for  $n = 2$ .

Now suppose that (2.8) holds for  $n - 1$ , that is, for  $x \geq \gamma \bar{\Lambda}_{n-1}(t)$  and sufficiently large  $t$ ,

$$(5.12) \quad \Pr(X(n-1, t) - \sum_{i=1}^{n-1} \Lambda_i(t) (\mu_i \nu_i(c_i) - d_i) > x) \geq (1-\delta) \sum_{i=1}^{n-1} \Lambda_i(t) \nu_i(c_i) \bar{F}_i(x).$$

Then, we prove that (2.8) holds for  $n$ . As for the case (5.4), by (5.12), it holds uniformly for  $x \geq \gamma \bar{\Lambda}_n(t)$  and sufficiently large  $t$  that

$$(5.13) \quad \begin{aligned} & \Pr(X(n, t) - \sum_{i=1}^n \Lambda_i(t) (\mu_i \nu_i(c_i) - d_i) > x) \\ &= \Pr(\sum_{i=1}^n \bar{X}_i(t) > x) \\ &\geq \Pr(\sum_{i=1}^{n-1} \bar{X}_i(t) > (1+\delta)x) \Pr(\bar{X}_n(t) > -\delta x) \\ &\quad + \Pr(\bar{X}_n(t) > (1+\delta)x) \Pr(\sum_{i=1}^{n-1} \bar{X}_i(t) > -\delta x) \\ &\quad - \Pr(\sum_{i=1}^{n-1} \bar{X}_i(t) > (1+\delta)x) \Pr(\bar{X}_n(t) > (1+\delta)x) \\ &\geq (1-\delta)^3 \sum_{i=1}^{n-1} \Lambda_i(t) \nu_i(c_i) \bar{F}_i((1+\delta)x) + (1-\delta)^{n+1} \Lambda_n(t) \nu_n(c_n) \bar{F}_n((1+\delta)x) \\ &\quad - (1-\delta)^2 \sum_{i=1}^{n-1} \Lambda_i(t) \nu_i(c_i) \bar{F}_i((1+\delta)x) \Lambda_n(t) \nu_n(c_n) \bar{F}_n((1+\delta)x). \end{aligned}$$

In (5.13), it is necessary to mention that

$$(5.14) \quad \Pr(\sum_{i=1}^n \bar{X}_i(t) > -\delta x) \geq (1-\delta)^{n+1}.$$

Actually, by (5.3), the assertion holds for  $n = 1$ . By the induction hypothesis we assume that (5.14) holds for  $n - 1$  and we prove it for  $n$ . For some positive constant  $v(v < \delta)$ , it holds that

$$\begin{aligned} \Pr\left(\sum_{i=1}^n \bar{X}_i(t) > -\delta x\right) &\geq \Pr\left(\sum_{i=1}^{n-1} \bar{X}_i(t) > (v - \delta)x\right)\Pr(\bar{X}_n(t) > -vx) \\ &\quad + \Pr\left(\sum_{i=1}^{n-1} \bar{X}_i(t) > -vx\right)\Pr(\bar{X}_n(t) > (v - \delta)x) \\ &\quad - \Pr\left(\sum_{i=1}^{n-1} \bar{X}_i(t) > (v - \delta)x\right)\Pr(\bar{X}_n(t) > (v - \delta)x) \\ &\geq (1 - \delta)^{n+1}. \end{aligned}$$

Therefore, letting  $\delta \rightarrow 0$ , it holds uniformly for  $x \geq \gamma\bar{\Lambda}_n(t)$ ,  $t \rightarrow \infty$  that

$$(5.15) \quad \Pr(X(n, t) - \sum_{i=1}^n \Lambda_i(t)(\mu_i\nu_i(c_i) - d_i) > x) \gtrsim \sum_{i=1}^n \Lambda_i(t)\nu_i(c_i)\bar{F}_i(x).$$

On the other hand, as for the case (5.8), it holds uniformly for  $x \geq \gamma\bar{\Lambda}_n(t)$ ,  $t \rightarrow \infty$  that

$$\begin{aligned} (5.16) \quad \Pr(X(n, t) - \sum_{i=1}^n \Lambda_i(t)(\mu_i\nu_i(c_i) - d_i) > x) &= \Pr\left(\sum_{i=1}^n \bar{X}_i(t) > x\right) \\ &\leq \sum_{i=1}^n \Pr(\bar{X}_i(t) > (1 - \delta)x) + \sum_{i=1}^n \sum_{j \neq i} \Pr(\bar{X}_i(t) > \frac{x}{n})\Pr(\bar{X}_j(t) > \frac{\delta x}{n}) \\ &\leq (1 + \delta) \sum_{i=1}^n \Lambda_i(t)\nu_i(c_i)\bar{F}_i((1 - \delta)x) \\ &\quad + (1 + \delta)^2 \sum_{i=1}^n \Lambda_i(t)\nu_i(c_i)\bar{F}_i\left(\frac{x}{n}\right) \sum_{j \neq i} \Lambda_j(t)\nu_j(c_j)\bar{F}_j\left(\frac{\delta x}{n - 1}\right) \\ &\sim \sum_{i=1}^n \Lambda_i(t)\nu_i(c_i)\bar{F}_i(x), \end{aligned}$$

where all the inequalities except the last two are obtained by (5.3) and the last step is due to Lemma 3.1 and the arbitrariness of  $\delta$ . Combining (5.15) and (5.16), the proof of Theorem 2.3 is accomplished.

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