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CERTAIN CURVATURE CONDITIONS OF REAL HYPERSURFACES IN A COMPLEX HYPERBOLIC SPACE

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Abstract. The purpose of this paper is to study real hypersurfaces immersed in a complex hyperbolic space CH^n and especially to investigate certain curvature conditions for such real hypersurfaces to be the model hypersurfaces in classification theorem (said to be Theorem M-R) given by Montiel and Romero ([4]) in Section 3.

1. introduction

Let CH^n be an n-dimensional complex hyperbolic space with Bergmann metric of constant holomorphic sectional curvature -4 and let M be a real hypersurface of CH^n. Then M has an almost contact metric structure (ϕ, U, u, g) induced from the complex structure of CH^n (cf. [3, 4]). On a real hypersurface we can consider two structures, namely, an almost contact structure ϕ and a submanifold structure represented by the second fundamental form H. In this point of view many differential geometers have investigated real hypersurfaces under some conditions concerning those structures (cf. [1, 3, 4, 5]). In particular, Montiel and Romero ([4]) have classified the real hypersurface M of CH^n which satisfies the commutativity condition such that

$$\phi H = H\phi \tag{1.1}$$

by using the S^1-fibration $\widetilde{\pi} : H_1^{2n+1} \longrightarrow CH^n$ of the anti-de Sitter space H_1^{2n+1} over CH^n and obtained a classification theorem (see Theorem M-R in Section 3).

In this paper we investigate certain curvature conditions for real hypersurfaces to be the model hypersurfaces given in Theorem M-R.

All manifolds, submanifolds and geometric objects will be assumed to be connected, differentiable and of class C^∞, and all maps also be of class C^∞ if not stated otherwise.

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2. Fundamental equations for hypersurfaces of CH^n

Let M be a real hypersurface of a complex hyperbolic space CH^n. Denote by (J,G) the Kähler structure of CH^n and g the induced metric on M from G. We also denote by ξ the unit vector field normal to M. For any vector field X tangent to M, we have the following decomposition in tangential and normal components:

$$JX = \phi X + u(X)\xi, \tag{2.1}$$

$$J\xi = -U, \tag{2.2}$$

where ϕ is a tensor field of type (1,1), u a 1-form and U a vector field on M. Since the structure (J,G) is Hermitian and $J^2 = -I$, it follows from (2.1) and (2.2) that for any tangent vector fields X, Y to M the following equations are established

$$\phi^2 X = -X + u(X)U,\ \phi U = 0,\ u(U) = 1, \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X,Y) - u(X)u(Y),\ g(U,X) = u(X). \tag{2.4}$$

The equations (2.3) and (2.4) tell us that the aggregate (ϕ, U, u, g) defines an almost contact metric structure on M.

Now let $\overline{\nabla}$ and ∇ be the Levi-Civita connections on CH^n and M, respectively. Then Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.5}$$

$$\overline{\nabla}_X \xi = -HX \tag{2.6}$$

for vector fields X and Y tangent to M. Here and in the sequel h and H denote the second fundamental form and the shape operator corresponding to the unit normal vector field ξ, respectively. It is clear that h and H are related by

$$h(X,Y) = g(HX,Y)\xi.$$

On the other hand, since the ambient manifold is Kählerian manifold, differentiating (2.1) and (2.2) covariantly and using (2.5) and (2.6) and thus comparing with tangential and normal parts respectively, we have

$$(\nabla_X \phi)Y = u(Y)HX - g(HY,X)U, \tag{2.7}$$

$$(\nabla_X u)Y = g(\phi HX, Y),\ \nabla_X U = \phi HX \tag{2.8}$$

for any vector fields X, Y tangent to M.

Moreover, since the ambient manifold CH^n is of constant holomorphic sectional curvature -4, its Riemannian curvature tensor $\overline{R}$ satisfies

$$\begin{aligned}\overline{R}_{\bar{X}\bar{Y}}\overline{Z} = & -\{G(\overline{Y},\overline{Z})\overline{X} - G(\overline{X},\overline{Z})\overline{Y} \\ & + G(J\overline{Y},\overline{Z})J\overline{X} - G(J\overline{X},\overline{Z})J\overline{Y} - 2G(J\overline{X},\overline{Y})J\overline{Z}\}\end{aligned}$$

for any vector fields $\overline{X},\ \overline{Y},\ \overline{Z}$ tangent to CH^n. Hence equations of Gauss and Codazzi imply

$$
\begin{aligned}
(2.9) \qquad R_{XY}Z = & -\{g(Y,Z)X - g(X,Z)Y \\
& + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\} \\
& + \{g(HY,Z)HX - g(HX,Z)HY\},
\end{aligned}
$$

$$
(2.10)\ (\nabla_X H)(Y) - (\nabla_Y H)(X) = -\{g(X,U)\phi Y - g(Y,U)\phi X - 2g(\phi X,Y)U\}
$$

for any vector fields X, Y, Z tangent to M, where R denotes the Riemannian curvature tensor of M (cf. [2]).

3. Fibrations and immersions

Let H_1^{2n+1} be an anti-de Sitter space

$$
(3.1) \qquad H_1^{2n+1} = \{z \in C^{n+1} \mid F(z,z) = -1\},
$$

where F is a Hermitian form in C^{n+1} defined by

$$
(3.2) \qquad F(z,z) = -z_0\overline{w}_0 + \sum_{k=1}^{n} z_k \overline{w}_k
$$

for $z = (z_0, z_1, \ldots, z_n)$, $w = (w_0, w_1, \ldots, w_n) \in C^{n+1}$. If $z \in H_1^{2n+1}$, then it follows that

$$
(3.3) \qquad T_z H_1^{2n+1} = \{w \in C^{n+1} \mid \mathrm{Re}F(z,w) = 0\}.
$$

The restriction $\widetilde{g}$ of ReF on H_1^{2n+1} is a Lorenzian structure with constant sectional curvature -1. Let

$$
\widetilde{\pi}\ :\ H_1^{2n+1} \longrightarrow CH^n
$$

be the natural projection of H_1^{2n+1} onto CH^n defined by the Hopf-fibration $S^1 \longrightarrow H_1^{2n+1} \longrightarrow CH^n$. As is well known that it is a Riemannian submersion with fundamental tensor J and time-like totally geodesic fibers. If $z \in H_1^{2n+1}$, putting $V = Jz \in T_z H_1^{2n+1}$ and then we have the following orthogonal decomposition

$$
(3.4) \qquad T_z H_1^{2n+1} = T_{\pi(z)} CH^n \oplus \mathrm{span}\{V\}.
$$

For a real hypersurface M of CH^n, we can construct a Lorentzian hypersurface $M' = \widetilde{\pi}^{-1}(M)$ of H_1^{2n+1} which is a principal S^1-bundle over M with time-like totally geodesic fibers and projection $\pi : M' \longrightarrow M$ (cf. [2, 4]). Moreover, the diagram

$$
\begin{array}{ccc}
M' & \xrightarrow{\ i'\ } & H_1^{2n+1} \\
{\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{\pi}} \\
M & \xrightarrow{\ i\ } & CH^n
\end{array}
$$

is commutative where i and i' are immersions, respectively. In this way, if $z \in M'$, then we can put

$$T_z M' = T_{\pi'(z)} M \oplus \mathrm{span}\{V\}. \tag{3.5}$$

Given integers $p,\ q$ with $p + q = n - 1$ and $r \in R$ with $0 < r < 1$, let $M_{2p+1,2q+1}(r)$ be the Lorentz hypersurface of H_1^{2n+1} defined by the following equations

$$-|z_0|^2 + \sum_{k=1}^{n} |z_k|^2 = -1,\ r(-|z_0|^2 + \sum_{k=1}^{p} |z_k|^2) = -\sum_{k=p+1}^{n} |z_k|^2,$$

where $z = (z_0, z_1, \ldots, z_n) \in C^{n+1}$. Moreover $M_{2p+1,2q+1}(r)$ is isometric to the product

$$H_1^{2p+1}(1/(r-1)) \times S_1^{2q+1}(r/(1-r)),$$

where $1/(r-1)$ and $r/(1-r)$ denote the square of the radii and each is embedded in H_1^{2n+1} in a totally umbilical way. Since $M_{2p+1,2q+1}(r)$ is S^1-invariant, $M^h_{2p+1,2q+1}(r) := \pi(M_{2p+1,2q+1}(r))$ is the real hypersurface of CH^n which is complete and satisfies the condition (1.1).

As already mentioned in Section 1, Montiel and Romero ([4]) have classified real hypersurfaces of CH^n which satisfy the commutativity condition (1.1), and thus obtained the following classification theorem:

Theorem M-R. *Let M be a complete real hypersurface of CH^n which satisfies the condition* (1.1). *Then we have the following*:

(1) *M has three constant principal curvatures* $\tanh\theta,\ \coth\theta,\ 2\coth 2\theta$ *with multiplicities $2p,\ 2q,\ 1$, respectively, $p + q = n - 1$. Moreover M is congruent to* $M^h_{2p+1,2q+1}(\tanh^2\theta)$.

(2) *M has two constant principal curvatures $\lambda_1,\ \lambda_2$ with multiplicities $2n-1$ and 1, respectively.*
   - (a) *If $\lambda_1 > 1$, then $\lambda_1 = \coth\theta,\ \lambda_2 = \coth 2\theta$ with $\theta > 0$ and M is congruent to a geodesic hypersphere* $M^h_{1,2n-1}(\tanh^2\theta)$*,*
   - (b) *If $\lambda_1 < 1$, then $\lambda_1 = \tanh\theta,\ \lambda_2 = 2\coth 2\theta$ with $\theta > 0$ and M is congruent to a geodesic hypersphere* $M^h_{2n-1,1}(\tanh^2\theta)$*,*
   - (c) *If $\lambda_1 = 1$, then $\lambda_2 = 2$ and M is congruent to a horosphere.*

Now let ξ be a local unit vector field normal to M defined near $\pi(z)$. We also denote ξ its lift by π, which is a local unit vector field normal to M' near to z. We denote by X^* the horizontal lift of a vector field X tangent to M. (In what follows we shall delete the i' and i'_* in our notations.) Then the following fundamental equations called co-Gauss and co-Codazzi formulae for the submersion π are given by

$$'\nabla_{X^*} Y^* = (\nabla_X Y)^* + g'((\phi X)^*, Y^*)V, \quad '\nabla_V X^* = {}'\nabla_{X^*} V = (\phi X)^*, \tag{3.6}$$

where g' denotes the Riemannian metric of $\pi^{-1}(M)$ induced from $\widetilde{g}$ of H_1^{2n+1} and $'\nabla$ the Levi-Civita connection with respect to g'. The similar equations are valid for the submersion $\widetilde{\pi}$ by replacing ϕ with J.

4. Main results

In this section we will investigate certain curvature conditions for the real hypersurface M of a complex hyperbolic space form CH^n which imply condition (1.1). Moreover we will obtain six curvature conditions which satisfy $\phi H = H\phi$, that is, the necessary condition of Theorem M-R in Section 3. Consequently we will provide six classification theorems for real hypersurfaces of CH^n to be the model hypersurfaces in Theorem M-R.

It follows from (3.6) that

$$\begin{aligned}(4.1)\qquad {'\nabla}_{X^*}{'\nabla}_{Y^*}Z^* = {} & (\nabla_X\nabla_Y Z)^* + g'((\phi Y)^*, Z^*)(\phi X)^* \\ & + \{g'((\phi Y)^*, (\nabla_X Z)^*) + g'((\phi X)^*, (\nabla_Y Z)^*)\}V \\ & + \{g'((\phi X)^*, (\nabla_Y Z)^*)\}V,\end{aligned}$$

$$(4.2)\qquad [X^*, Y^*] = [X, Y]^* + 2g'((\phi X)^*, Y^*)V,\ [X^*, V] = 0.$$

Using the above equations and taking account of (2.9) and (2.10) with $c = -4$, we can easily see that

$$\begin{aligned}(4.3)\qquad {'R}(X^*, Y^*)Z^* = {} & g(X, Z)^*Y^* - g(Y, Z)^*X^* \\ & + g(HY, Z)^*HX^* - g(HX, Z)^*HY^* \\ & + \{g(Y, U)^*g(HX, Z)^* - g(X, U)^*g(HY, Z)^*\}V,\end{aligned}$$

where ${'R}$ denotes the Riemannian curvature tensor of ${'M}$. Therefore we have:

Theorem 4.1. *Let M be a real hypersurface of a complex hyperbolic space CH^n. If $({'\nabla}_V{'R})(X^*, Y^*)U^* = 0$ and $g(HU, U) \neq 0$ at least one point of M, then $\phi H = H\phi$.*

Proof. Since ${'\nabla}_V U^* = (\phi U)^* = 0$, we get

$$\begin{aligned}({'\nabla}_V{'R})(X^*, Y^*)U^* = {} & {'\nabla}_V({'R}(X^*, Y^*)U^*) \\ & - {'R}({'\nabla}_V X^*, Y^*)U^* - {'R}(X^*, {'\nabla}_V Y^*)U^*,\end{aligned}$$

which combined with the fact that $({'\nabla}_V{'R})(X^*, Y^*)U^* = 0$ implies

$${'\nabla}_V({'R}(X^*, Y^*)U^*) = {'R}({'\nabla}_V X^*, Y^*)U^* + {'R}(X^*, {'\nabla}_V Y^*)U^*.$$

From (4.3), we obtain

$$\begin{aligned}(4.4)\qquad {'R}(X^*, Y^*)U^* = {} & g(X, U)^*Y^* - g(Y, U)^*X^* \\ & + g(HY, U)^*HX^* - g(HX, U)^*HY^* \\ & + \{g(Y, U)^*g(HX, U)^* - g(X, U)^*g(HY, U)^*\}V,\end{aligned}$$

which together with (2.3) gives

$$\begin{aligned}(4.5)\qquad {'\nabla}_V({'R}(X^*, Y^*)U^*) = {} & g(X, U)^*(\phi Y)^* - g(Y, U)^*(\phi X)^* \\ & + g(HY, U)^*(\phi HX)^* - g(HX, U)^*(\phi HY)^*,\end{aligned}$$

which joined with (4.3) yields

$$
\begin{aligned}
(4.6) \qquad & {}'R({}'\nabla_V X^*, Y^*)U^* + {}'R(X^*, {}'\nabla_V Y^*)U^* \\
&= g(X,U)^*(\phi Y)^* - g(Y,U)^*(\phi X)^* \\
&\quad + g(HY,U)^*(H\phi X)^* - g(HX,U)^*(H\phi Y)^* \\
&\quad + g(H\phi Y,U)^*(HX)^* - g(H\phi X,U)^*(HY)^* \\
&\quad + \{g(Y,U)^* g(H\phi X,U)^* - g(X,U)^* g(H\phi Y,U)^*\}V.
\end{aligned}
$$

Comparing with vertical parts in (4.5) and (4.6) respectively, we get

$$
(4.7) \qquad g(Y,U)g(H\phi X,U) - g(X,U)g(H\phi Y,U) = 0.
$$

Also comparing with horizontal parts in (4.5) and (4.6) respectively, we obtain

$$
\begin{aligned}
(4.8) \qquad & g(HY,U)(\phi H - H\phi)X - g(HX,U)(\phi H - H\phi)Y \\
&= g(H\phi Y,U)HX - g(H\phi X,U)(HY).
\end{aligned}
$$

Putting $Y = U$ into (4.7), we can see that $\phi HU = 0$, thus we have $HU = \alpha U$, where $\alpha = g(HU,U)$. By putting $Y = U$ into (4.8), we get $\alpha(\phi H - H\phi)X = 0$.

On the other hand, differentiating $HU = \alpha U$ covariantly along M and making use of (2.8), we have

$$
g(U,(\nabla_X H)Y) + g(H\phi HX, Y) = (X\alpha)g(U,Y) + \alpha g(\phi HX, Y),
$$

where we have used the fact ∇_X is skew-symmetric. Taking the skew-symmetric part of the last equation and using the Codazzi equation (2.10), we get

$$
\begin{aligned}
(4.9) \qquad & -2g(X,\phi Y) + 2g(\phi HX, HY) \\
&= (X\alpha)g(Y,U) - (Y\alpha)g(X,U) + \alpha g((\phi H + H\phi)X, Y).
\end{aligned}
$$

Similarly, putting $Y = U$ into (4.9), we have

$$
(4.10) \qquad X\alpha = \beta g(X,U),
$$

where $\beta = U\alpha$. Combining (4.9) with (4.10) imply

$$
(4.11) \qquad -2g(X,\phi Y) + 2g(\phi HX, HY) = \alpha g((\phi H + H\phi)X, Y).
$$

Differentiating (4.10) covariantly along M and using (2.8), we have

$$
YX\alpha = (Y\beta)g(X,U) + \beta\{g(\phi HY, X) + g(\nabla_Y X, U)\}.
$$

Taking the skew-symmetric part of the last equation and making use of (4.10), we obtain

$$
(Y\beta)g(X,U) - (X\beta)g(Y,U) + \beta g((\phi H + H\phi)Y, X) = 0
$$

and putting $Y = U$ into the last equation, we get

$$
X\beta = (U\beta)g(X,U)
$$

thus consequently we have

$$
(4.12) \qquad \beta(\phi H + H\phi)X = 0.
$$

Therefore $\phi H + H\phi = 0$ on the open set $S = \{x \in M \,|\, \beta(x) \neq 0\}$. Owing to the fact that $\alpha(\phi H - H\phi)X = 0$, we have $\alpha HX = 0$ on S, which gives

$$\alpha g(\phi X, Y) = 0$$

on S. Hence $\alpha = 0$ on S and consequently we get $\beta = 0$ on S, which is a contradiction. Thus we have $\beta = 0$ identically on M, so α is constant. Therefore, if $\alpha = g(HU, U) \neq 0$ at least one point of M, then we conclude $\phi H - H\phi = 0$. □

Remark. If $\phi H - H\phi = 0$ and $n > 1$, then $\alpha \neq 0$.

Indeed, if we set

$$V := \nabla_U U + (\mathrm{div} U)U,$$

then we can see that

$$\mathrm{div} V = \frac{1}{2}||\phi H - H\phi||^2 + g(HU, U)(\mathrm{tr} H) - \mathrm{tr} H^2 - 2(n-1).$$

Since the commutativity condition such that $\phi H - H\phi = 0$ implies $HU = \alpha U$, consequently we conclude $V = 0$. Thus, if $\alpha = 0$, then

$$\mathrm{tr} H^2 + 2(n-1) = 0.$$

This yields $n = 1$, which is a contradiction.

Owing to Theorem 4.1, we have:

Theorem 4.2. *Let M be a real hypersurface of a complex hyperbolic space CH^n. If $('\nabla_V {}'R)(Y^*, U^*)X^* = 0$ and $g(HU, U) \neq 0$ at least one point of M, then $\phi H = H\phi$.*

Proof. Since $'\nabla_V U^* = (\phi U)^* = 0$, we get

$$\begin{aligned}('\nabla_V {}'R)(Y^*, U^*)X^* &= {}'\nabla_V ({}'R(Y^*, U^*)X^*) \\ &\quad - {}'R('\nabla_V Y^*, U^*)X^* - {}'R(Y^*, U^*)'\nabla_V X^*,\end{aligned}$$

which combined with the fact that $('\nabla_V {}'R)(Y^*, U^*)X^* = 0$ implies

$$'\nabla_V ({}'R(Y^*, U^*)X^*) = {}'R('\nabla_V Y^*, U^*)X^* + {}'R(Y^*, U^*)'\nabla_V X^*.$$

Then from (4.3), the last equation reduces to

$$\begin{aligned}{}'R(Y^*, U^*)X^* &= g(Y, X)^* U^* - g(U, X)^* Y^* \\ &\quad + g(HU, X)^* HY^* - g(HY, X)^* HU^* \\ &\quad + \{g(HY, X)^* - g(Y, U)^* g(HU, X)^*\}V,\end{aligned} \tag{4.13}$$

which together with (2.3) gives

$$\begin{aligned}{}'\nabla_V ({}'R(Y^*, U^*)X^*) = &- g(X, U)^*(\phi Y)^* + g(HU, X)^*(\phi HY)^* \\ &- g(HY, X)^*(\phi HU)^*,\end{aligned} \tag{4.14}$$

and by means of (4.3), the last equation (4.14) yields

$$'R('\nabla_V Y^*, U^*)X^* + {}'R(Y^*, U^*)'\nabla_V X^* \tag{4.15}$$

$$\begin{aligned}= &-g(X,U)^*(\phi Y)^* + g(HU,X)^*(H\phi Y)^* - g(H\phi Y,X)^*(HU)^* \\ &+ g(HU,\phi X)^*(HY)^* - g(HY,\phi X)^*(HU)^* \\ &+ \{g(H\phi Y,X)^* + g(HY,\phi X)^* - g(Y,U)^* g(HU,\phi X)^*\}V.\end{aligned}$$

Comparing with vertical parts in (4.14) and (4.15) respectively, we get

$$g(H\phi Y,X) + g(HY,\phi X) - g(Y,U)g(HU,\phi X) = 0. \tag{4.16}$$

Putting $X = U$ into (4.16), we get ϕHU=0, thus we have $HU = \alpha U$, where $\alpha = g(HU,U)$. Also comparing with horizontal parts in (4.14) and (4.15) respectively, we get

$$\begin{aligned}&g(HU,X)(\phi HY) - g(HY,X)(\phi HU) \\ &= g(HU,\phi X)(HY) - g(HY,\phi X)(HU) \\ &\quad + g(HU,X)(H\phi Y) - g(H\phi Y,X)(HU).\end{aligned} \tag{4.17}$$

Putting $X = U$ into the above equation (4.17) and making use of the fact that $HU = \alpha U$, we obtain $\alpha(\phi H - H\phi)Y = 0$. Owing to similar method as it in the proof of Theorem 4.1, we conclude $\phi H - H\phi = 0$. □

From (2.4) and (3.6) it follows that

$$'\nabla_{Y^*}{}'\nabla_{X^*}V = (\nabla_Y(\phi X))^* + g(\phi Y,\phi X)^*V, \tag{4.18}$$

$$'\nabla_V{}'\nabla_{Y^*}X^* = (\phi(\nabla_Y X))^*, \tag{4.19}$$

$$'\nabla_{Y^*}{}'\nabla_V X^* = (\nabla_Y(\phi X))^* + g(\phi Y,\phi X)^*V, \tag{4.20}$$

$$'\nabla_V{}'\nabla_{X^*}V = -X^* + g(X,U)^*U^*, \tag{4.21}$$

which together with (4.2) and the fact that fibre is totally geodesic imply

$$'R(Y^*,X^*)V = ((\nabla_Y\phi)X)^* - ((\nabla_X\phi)Y)^*, \tag{4.22}$$

$$\begin{aligned}'R(Y^*,V)X^* = &\, g(X,U)^*(HY)^* - g(HY,X)^*U^* \\ &+ g(X,Y)^*V - g(X,U)^*g(Y,U)^*V,\end{aligned} \tag{4.23}$$

$$'R(Y^*,V)V = Y^* - g(Y,U)^*U^*. \tag{4.24}$$

Therefore we get:

Theorem 4.3. *Let M be a real hypersurface of a complex hyperbolic space CH^n. If $('\nabla_V{}'R)(Y^*,X^*)V = 0$, then $\phi H = H\phi$.*

Proof. Since $'\nabla_V V = 0$, we have

$$\begin{aligned}('\nabla_V{}'R)(Y^*,X^*)V = &\, '\nabla_V('R(Y^*,X^*)V) \\ &- {}'R('\nabla_V Y^*,X^*)V - {}'R(Y^*,{}'\nabla_V X^*)V,\end{aligned}$$

which combined with the fact that $('\nabla_V{}'R)(Y^*,U^*)X^* = 0$ implies

$$'\nabla_V('R(Y^*,X^*)V) = {}'R('\nabla_V Y^*,X^*)V + {}'R(Y^*,{}'\nabla_V X^*)V. \tag{4.25}$$

Because of (2.7) and (4.22), we obtain

$$'R(Y^*,X^*)V = g(X,U)^*(HY)^* - g(Y,U)^*(HX)^*, \tag{4.26}$$

which together with (4.25) reduces to

$$g(X,U)\phi HY - g(Y,U)\phi HX = g(X,U)H\phi Y - g(Y,U)H\phi X. \tag{4.27}$$

Putting $Y = U$ into (4.27) and taking inner product with U, we get $\phi HU = 0$. Consequently from (4.27) it follows that $(\phi H - H\phi)X = 0$. □

Thus owing to Theorem 4.3, we have also:

Theorem 4.4. *Let M be a real hypersurface of a complex hyperbolic space CH^n. If $('\nabla_V {}'R)(Y^*, V)X^* = 0$, then $\phi H = H\phi$.*

Proof. Since $'\nabla_V V = 0$, we get

$$\begin{aligned}('\nabla_V {}'R)(Y^*, V)X^* &= {}'\nabla_V('R(Y^*, V)X^*)\\ &\quad - {}'R('\nabla_V Y^*, V)X^* - {}'R(Y^*, V)'\nabla_V X^*,\end{aligned}$$

which together with the fact that $('\nabla_V {}'R)(Y^*, V)X^* = 0$ implies

$${}'\nabla_V('R(Y^*, V)X^*) = {}'R('\nabla_V Y^*, V)X^* + {}'R(Y^*, V)'\nabla_V X^*. \tag{4.28}$$

Using (4.23), we obtain

$${}'\nabla_V('R(Y^*, V)X^*) = g(X,U)^*(\phi HY)^*. \tag{4.29}$$

Since

$$\begin{aligned}&{}'R('\nabla_V Y^*, V)X^* + {}'R(Y^*, V)'\nabla_V X^*\\ &= g(X,U)^*(H\phi Y)^* - g(H\phi Y, X)^*U^* - g(HY, \phi X)^*U^*,\end{aligned} \tag{4.30}$$

we get

$$g(X,U)\phi HY = g(X,U)(H\phi Y) - g(H\phi Y, X)U - g(HY, \phi X)U.$$

Putting $X = Y = U$ into (4.30), we get $\phi HU = 0$, thus we have $HU = \alpha U$, where $\alpha = g(HU, U)$. Also putting $X = U$ into (4.30), consequently we get $(\phi H - H\phi)X = 0$. □

Therefore similarly we have:

Theorem 4.5. *Let M be a real hypersurface of a complex hyperbolic space CH^n. If $('\nabla_{Z^*} {}'R)(Y^*, V)X^* = 0$, then $\phi H = H\phi$.*

Proof. Since

$$\begin{aligned}('\nabla_{Z^*} {}'R)(Y^*, V)X^* &= {}'\nabla_{Z^*}('R(Y^*, V)X^*)\\ &\quad - {}'R('\nabla_{Z^*} Y^*, V)X^* - {}'R(Y^*, {}'\nabla_{Z^*} V)X^*\\ &\quad - {}'R(Y^*, V)'\nabla_{Z^*} X^*,\end{aligned}$$

we have $('\nabla_{Z^*} {}'R)(Y^*, V)X^* = 0$, which implies

$$\begin{aligned}&{}'\nabla_{Z^*}('R(Y^*, V)X^*)\\ &= {}'R('\nabla_{Z^*} Y^*, V)X^* + {}'R(Y^*, {}'\nabla_{Z^*} V)X^* + {}'R(Y^*, V)('\nabla_{Z^*} X^*).\end{aligned}$$

Taking use of (3.6) and (4.23), we get

$$
\begin{aligned}
(4.31) \qquad & {}'\nabla_{Z^*}({}'R(Y^*,V)X^*) \\
&= g(\nabla_Z X,U)^*(HY)^* + g(X,\phi HZ)^*(HY)^* + g(X,U)^*(\nabla_Z(HY))^* \\
&\quad - g(\nabla_Z X,HY)^*U^* - g(X,\nabla_Z(HZ))^*U^* - g(X,HY)^*(\phi HZ)^* \\
&\quad + g(X,Y)^*(\phi Z)^* - g(X,U)^*g(Y,U)^*(\phi Z)^* \\
&\quad + \{g(X,U)^*g(\phi Z,HY)^* + g(\nabla_Z X,Y)^* + g(X,\nabla_Z Y)^* \\
&\quad - g(\nabla_Z X,U)^*g(Y,U)^* - g(X,\phi HZ)^*g(Y,U)^* \\
&\quad - g(X,U)^*g(\nabla_Z Y,U)^* - g(X,U)^*g(Y,\phi HZ)^*\}V.
\end{aligned}
$$

Taking account of (3.6) and (4.23) and the fact that ${}'R(\cdot,\cdot)$ is skew symmetric, we obtain

$$
\begin{aligned}
(4.32) \qquad {}'R({}'\nabla_{Z^*}Y^*,V)X^* &= g(X,U)^*(H(\nabla_Z Y))^* - g(X,H(\nabla_Z Y))^*U^* \\
&\quad + \{g(X,\nabla_Z Y)^* - g(X,U)^*g(\nabla_Z Y,U)^*\}V.
\end{aligned}
$$

From (4.3) it follows that

$$
\begin{aligned}
(4.33) \qquad {}'R(Y^*,{}'\nabla_{Z^*}V)X^* &= g(Y,X)^*(\phi Z)^* - g(\phi Z,X)^*Y^* \\
&\quad + g(H\phi Z,X)^*(HY)^* - g(HY,X)^*(H\phi Z)^* \\
&\quad - g(Y,U)^*g(H\phi Z,X)^*V.
\end{aligned}
$$

By means of (3.6), (4.23) and (4.24), we obtain

$$
\begin{aligned}
(4.34) \qquad {}'R(Y^*,V)({}'\nabla_{Z^*}X^*) &= g(\nabla_Z X,U)^*(HY)^* - g(\nabla_Z X,HY)^*U^* \\
&\quad + g(\phi Z,X)^*Y^* - g(\phi Z,X)^*g(Y,U)^*U^* \\
&\quad + \{g(\nabla_Z X,Y)^* - g(\nabla_Z X,U)^*g(Y,U)^*\}V.
\end{aligned}
$$

Thus from (4.32), (4.33) and (4.34), it follows that

$$
\begin{aligned}
(4.35) \qquad & {}'R({}'\nabla_{Z^*}Y^*,V)X^* + {}'R(Y^*,{}'\nabla_{Z^*}V)X^* + {}'R(Y^*,V)({}'\nabla_{Z^*}X^*) \\
&= g(X,U)^*(H(\nabla_Z Y))^* - g(X,H(\nabla_Z Y))^*U^* \\
&\quad + \{g(X,\nabla_Z Y)^* - g(X,U)^*g(\nabla_Z Y,U)^*\}V \\
&\quad + g(Y,X)^*(\phi Z)^* - g(\phi Z,X)^*Y^* + g(H\phi Z,X)^*(HY)^* \\
&\quad - g(HY,X)^*(H\phi Z)^* - g(Y,U)^*g(H\phi Z,X)^*V \\
&\quad + g(\nabla_Z X,U)^*(HY)^* - g(\nabla_Z X,HY)^*U^* \\
&\quad + g(\phi Z,X)^*Y^* - g(\phi Z,X)^*g(Y,U)^*U^* \\
&\quad + \{g(\nabla_Z X,Y)^* - g(\nabla_Z X,U)^*g(Y,U)^*\}V.
\end{aligned}
$$

Comparing with vertical parts in (4.31) and (4.35) respectively, we get

$$
\begin{aligned}
(4.36) \qquad & g(X,U)g(\phi Z,HY) - g(X,\phi HZ)g(Y,U) - g(X,U)g(Y,\phi HZ) \\
&= -g(Y,U)g(H\phi Z,X).
\end{aligned}
$$

Putting $X = U$ into (4.36), we have

$$g(\phi Z, HY) - g(Y, \phi HZ) = -g(Y, U)g(H\phi Z, U). \tag{4.37}$$

Also putting $Y = U$ into (4.36), we obtain $\phi HU = 0$, which together with (4.37) reduces to $\phi H - H\phi = 0$. □

Finally owing to Theorem 4.5, we get:

Theorem 4.6. *Let M be a real hypersurface of a complex hyperbolic space CH^n. If $({}'\nabla_{Z^*}{}'R)(V, X^*)V = 0$, then $\phi H = H\phi$.*

Proof. Since

$$\begin{aligned}({}'\nabla_{Z^*}{}'R)(V, X^*)V = {}&{}'\nabla_{Z^*}({}'R(V, X^*)V)\\ &- {}'R({}'\nabla_{Z^*}V, X^*)V - {}'R(V, {}'\nabla_{Z^*}X^*)V\\ &- {}'R(V, X^*){}'\nabla_{Z^*}V,\end{aligned}$$

we have $({}'\nabla_{Z^*}{}'R)(Y^*, V)X^* = 0$, which implies

$${}'\nabla_{Z^*}({}'R(V, X^*)V) = {}'R({}'\nabla_{Z^*}V, X^*)V + {}'R(V, {}'\nabla_{Z^*}X^*)V + {}'R(V, X^*)({}'\nabla_{Z^*}V).$$

From (3.6) and (4.24) it follows that

$$\begin{aligned}&{}'\nabla_{Z^*}({}'R(V, X^*)V)\\ &= -(\nabla_Z X)^* + g(\nabla_Z X, U)^*U^* + g(X, \phi HZ)^*U^*\\ &\quad + g(X, U)^*(\phi HZ)^* + g(\phi X, Z)^*V.\end{aligned} \tag{4.38}$$

By means of (3.6), (4.22), (4.23) and (4.24), we obtain

$$\begin{aligned}&{}'R({}'\nabla_{Z^*}V, X^*)V + {}'R(V, {}'\nabla_{Z^*}X^*)V + {}'R(V, X^*)({}'\nabla_{Z^*}V)\\ &= g(X, U)^*(H\phi Z)^* - (\nabla_Z X)^* + g(\nabla_Z X, U)^*U^*\\ &\quad + g(HX, \phi Z)^*U^* + g(\phi X, Z)^*V.\end{aligned} \tag{4.39}$$

Comparing with horizontal parts in (4.38) and (4.39) respectively, we get

$$g(X, \phi HZ)U + g(X, U)\phi HZ = g(X, H\phi Z)U + g(X, U)H\phi Z. \tag{4.40}$$

Putting $X = Z = U$ into (4.40), we get $\phi HU = 0$. Also putting $X = U$ into (4.40), we conclude $\phi H - H\phi = 0$. □

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