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SECOND ORDER PARALLEL TENSORS AND RICCI SOLITONS ON $(LCS)_n$-MANIFOLDS

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Abstract. The object of the present paper is to study the second order parallel symmetric tensors and Ricci solitons on $(LCS)_n$-manifolds. We found the conditions of Ricci soliton on $(LCS)_n$-manifolds to be shrinking, steady and expanding respectively.

1. Introduction

In 2003 Shaikh [19] introduced the notion of Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifolds) with an example, which generalizes the notion of LP-Sasakian manifolds introduced by Matsumoto [13] and also by Mihai and Rosca [14]. Then Shaikh and Baishya ([22], [23]) investigated the applications of $(LCS)_n$-manifolds to the general theory of relativity and cosmology. The $(LCS)_n$-manifolds are also studied by Atceken et al. ([3], [4], [10]), Narain and Yadav [16], Prakasha [18], Shaikh and his co-authors ([20], [21], [24]–[27]) and many others.

In [30] Sharma studied the Ricci solitons in contact geometry. Thereafter Ricci solitons in contact metric manifolds have been studied by various authors such as Bagewadi et al. ([1], [2], [5], [11]), Bejan and Crasmareanu [6], Chen and Deshmukh [8], Nagaraja and Premalatta [15], Tripathi [32] and many others.

A Ricci soliton (g, V, λ) on a Riemannian manifold (M, g) is a generalization of Einstein metric such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1.1}$$

where S is the Ricci tensor and $\pounds_V$ is the Lie derivative along the vector field V on M and λ is a real number. The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectively.

Motivated by the above studies the object of the present paper is to study Ricci soliton on $(LCS)_n$-manifolds. The paper is organized as follows. Section 2 is concerned with preliminaries. In 1926, Levy [12] proved that a second order parallel symmetric non-singular tensor in real space forms is proportional to

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the metric tensor. Then Sharma ([28], [29]) studied second order parallel tensor in Kaehler space of constant holomorphic sectional curvature as well as contact manifolds. Second order parallel tensor have been studied by various authors in different structure of manifolds. In Section 3 of this paper we have studied second order parallel tensors on $(LCS)_n$-manifolds and the Ricci soliton on such a manifold. It is shown that in a $(LCS)_n$-manifold with $\alpha^2 - \rho \neq 0$, if the $(0, 2)$ type tensor field $\pounds_V g + 2S$ is parallel for any vector field V, then (g, V, λ) yields a Ricci soliton of that $(LCS)_n$-manifold. Also we obtain the conditions of Ricci soliton of ϕ-pseudo Ricci symmetric $(LCS)_n$-manifold to be shrinking, steady and expanding respectively.

2. Preliminaries

An n-dimensional Lorentzian manifold M is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type $(0, 2)$ such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate inner product of signature $(-, +, \ldots, +)$, where T_pM denotes the tangent vector space of M at p and $\mathbb{R}$ is the real number space. A non-zero vector $v \in T_pM$ is said to be timelike (resp., non-spacelike, null, spacelike) if it satisfies $g_p(v, v) < 0$ (resp, ≤ 0, $= 0$, > 0) [17].

Definition 2.1 ([33])**.** In a Lorentzian manifold (M, g) a vector field P defined by

$$g(X, P) = A(X)$$

for any $X \in \Gamma(TM)$, the section of all smooth tangent vector fields on M, is said to be a concircular vector field if

$$(\nabla_X A)(Y) = \alpha\{g(X, Y) + \omega(X)A(Y)\},$$

where α is a non-zero scalar and ω is a closed 1-form and ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

Let M be an n-dimensional Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the characteristic vector field of the manifold. Then we have

$$g(\xi, \xi) = -1. \tag{2.1}$$

Since ξ is a unit concircular vector field, it follows that there exists a non-zero 1-form η such that for

$$g(X, \xi) = \eta(X), \tag{2.2}$$

the equation of the following form holds

$$(\nabla_X \eta)(Y) = \alpha\{g(X, Y) + \eta(X)\eta(Y)\}, \quad \alpha \neq 0 \tag{2.3}$$

that is,

$$\nabla_X \xi = \alpha[X + \eta(X)\xi]$$

for all vector fields X, Y, where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfies

$$\nabla_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X), \tag{2.4}$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. If we put

$$\phi X = \frac{1}{\alpha}\nabla_X \xi, \tag{2.5}$$

then from (2.3) and (2.5) we have

$$\phi X = X + \eta(X)\xi, \tag{2.6}$$

from which it follows that ϕ is a symmetric (1, 1) tensor and called the structure tensor of the manifold. Thus the Lorentzian manifold M together with the unit timelike concircular vector field ξ, its associated 1-form η and an (1, 1) tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly, $(LCS)_n$-manifold) [20]. Especially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto [13]. In a $(LCS)_n$-manifold ($n > 2$), the following relations hold ([20], [22], [23], [24]):

$$\eta(\xi) = -1,\ \phi\xi = 0,\ \eta(\phi X) = 0,\ g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.7}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{2.8}$$

$$S(X,\xi) = (n-1)(\alpha^2 - \rho)\eta(X), \tag{2.9}$$

$$R(X,Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \tag{2.10}$$

$$R(\xi,Y)Z = (\alpha^2 - \rho)[g(Y,Z)\xi - \eta(Z)Y], \tag{2.11}$$

$$(\nabla_X \phi)Y = \alpha\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \tag{2.12}$$

$$(X\rho) = d\rho(X) = \beta\eta(X), \tag{2.13}$$

$$R(X,Y)Z = \phi R(X,Y)Z + (\alpha^2 - \rho)\big\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\big\}\xi, \tag{2.14}$$

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)(\alpha^2 - \rho)\eta(X)\eta(Y) \tag{2.15}$$

for any vector fields X, Y, Z on M and $\beta = -(\xi\rho)$ is a scalar function, where R is the curvature tensor and S is the Ricci tensor of the manifold.

Let (g, ξ, λ) be a Ricci soliton on a $(LCS)_n$-manifold M. From (2.3), we get

$$\frac{1}{2}(\pounds_\xi g)(X,Y) = \alpha\{g(X,Y) + \eta(X)\eta(Y)\}. \tag{2.16}$$

From (1.1) and (2.16) we have

$$S(X,Y) = -(\alpha + \lambda)g(X,Y) - \alpha\eta(X)\eta(Y), \tag{2.17}$$

which yields

$$QX = -(\alpha + \lambda)X - \alpha\eta(X)\xi, \tag{2.18}$$

$$S(X,\xi) = -\lambda\eta(X), \tag{2.19}$$

$$r = -\lambda n - (n-1)\alpha, \tag{2.20}$$

where Q is the Ricci operator, i.e., $g(QX,Y) = S(X,Y)$ for all X, Y and r is the scalar curvature of M.

3. Second order parallel symmetric tensors and Ricci solitons on $(LCS)_n$-manifolds

Definition 3.1. A tensor h of second order is said to be a parallel tensor if $\nabla h = 0$, where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g.

Let h be a (0, 2) symmetric tensor field on an $(LCS)_n$-manifold M such that $\nabla h = 0$. Then applying the Ricci identity [28]

$$\nabla^2 h(X,Y;Z,W) - \nabla^2 h(X,Y;W,Z) = 0, \tag{3.1}$$

we obtain

$$h\big(R(X,Y)Z,W\big) + h\big(Z,R(X,Y)W\big) = 0 \tag{3.2}$$

for arbitrary vector fields X, Y, Z on M. Putting $X = Z = W = \xi$ in (3.2) and since h is symmetric, we get

$$h\big(\xi, R(\xi,Y)\xi\big) = 0. \tag{3.3}$$

Using (2.11) in (3.3) we get

$$(\alpha^2 - \rho)\big[h(Y,\xi) + \eta(Y)h(\xi,\xi)\big] = 0. \tag{3.4}$$

Since $\alpha^2 - \rho \neq 0$ we have

$$h(Y,\xi) + \eta(Y)h(\xi,\xi) = 0. \tag{3.5}$$

Differentiating (3.5) covariantly along X, we get

$$\begin{aligned} &g(\nabla_X Y,\xi)h(\xi,\xi) + g(Y,\nabla_X\xi)h(\xi,\xi) + 2g(Y,\xi)h(\nabla_X\xi,\xi) \\ &+h(\nabla_X Y,\xi) + h(Y,\nabla_X\xi) = 0. \end{aligned} \tag{3.6}$$

Putting $Y = \nabla_X Y$ in (3.5) we obtain

$$g(\nabla_X Y,\xi)h(\xi,\xi) + h(\nabla_X Y,\xi) = 0. \tag{3.7}$$

In view of (3.7) it follows from (3.6) that

$$g(Y,\nabla_X\xi)h(\xi,\xi) + 2g(Y,\xi)h(\nabla_X\xi,\xi) + h(Y,\nabla_X\xi) = 0. \tag{3.8}$$

Using (2.5) in (3.8) we get

$$g(Y,\phi X)h(\xi,\xi) + 2\eta(Y)h(\phi X,\xi) + h(Y,\phi X) = 0, \quad \text{since} \;\; \alpha \neq 0. \tag{3.9}$$

Replacing X by ϕX in (3.9) and then using (2.8) and (3.5) we obtain

$$h(X,Y) = -h(\xi,\xi)g(X,Y). \tag{3.10}$$

Differentiating (3.10) covariantly along any vector field on M, it can be easily shown that $h(\xi,\xi)$ is constant. This leads to the following:

Theorem 3.1. *A second order parallel symmetric tensor on a $(LCS)_n$-manifold with $\alpha^2 - \rho \neq 0$, is a constant multiple of the metric tensor.*

Suppose that the (0, 2) type symmetric tensor field $\pounds_V g + 2S$ is parallel for any vector field V on a $(LCS)_n$-manifold M. Then Theorem 3.1 yields $\pounds_V g + 2S$ is a constant multiple of the metric tensor g, i.e., $\pounds_V g + 2S = -2\lambda g$ for all X, Y on M, where λ is a constant. Hence the relation (1.1) holds. This implies that (g, V, λ) yields a Ricci soliton. Thus we can state the following:

Theorem 3.2. *If the tensor field $\pounds_V g+2S$ on a $(LCS)_n$-manifold with $\alpha^2-\rho \neq 0$ is parallel for any vector field V, then (g, V, λ) is a Ricci soliton.*

A Lorentzian manifold M is said to be Ricci symmetric if its Ricci tensor S of type (0, 2) satisfies $\nabla S = 0$. As a generalization of Ricci symmetric manifold, Chaki [7] introduced the notion of pseudo Ricci symmetric manifold for the Riemannian case. A Riemannian manifold (M, g) is said to be pseudo Ricci symmetric [7] if its Ricci tensor $S = g(Q.,.)$ of type (0, 2) is not identically zero and satisfies the condition

$$(\nabla_X Q)(Y) = 2A(X)Q(Y) + A(Y)Q(X) + S(Y, X)\sigma, \tag{3.11}$$

where σ is the vector field associated to the nowhere vanishing 1-form A such that $A(X) = g(X, \sigma)$.

As a weaker version of local symmetry, the notion of locally ϕ-symmetric Sasakian manifolds was introduced by Takahashi [31]. In the context of Lorentzian geometry, Shaikh et al. ([24], [25]) studied locally ϕ-symmetric and locally ϕ- recurrent $(LCS)_n$-manifolds. Motivated by the above studies Hui [9] studied ϕ-pseudo Ricci symmetric $(LCS)_n$-manifolds and obtained the form of Ricci tensor S as

$$S(X, Y) = \frac{(n-1)\alpha(\alpha^2-\rho)}{\alpha + A(\xi)} g(X, Y) + \frac{(n-1)(\alpha^2-\rho)A(\xi)}{\alpha + A(\xi)} \eta(X)\eta(Y), \tag{3.12}$$

provided $\alpha + A(\xi) \neq 0$.

Suppose that h is a (0, 2) symmetric parallel tensor field on a $(LCS)_n$-manifold such that

$$h(X, Y) = (\pounds_\xi g)(X, Y) + 2S(X, Y). \tag{3.13}$$

Using (3.12) and (2.16) in (3.13) we get

$$\begin{aligned} h(X, Y) = {} & 2\big[\frac{(n-1)\alpha(\alpha^2-\rho)}{\alpha + A(\xi)} + \alpha\big] g(X, Y) \\ & + 2\big[\frac{(n-1)(\alpha^2-\rho)A(\xi)}{\alpha + A(\xi)} + \alpha\big] \eta(X)\eta(Y). \end{aligned} \tag{3.14}$$

Putting $X = Y = \xi$ in (3.14) we get

$$h(\xi, \xi) = \frac{2(n-1)(\alpha^2-\rho)[A(\xi) - \alpha]}{\alpha + A(\xi)}. \tag{3.15}$$

If (g, ξ, λ) is a Ricci soliton on a $(LCS)_n$-manifold M, then from (1.1) we have

$$h(X, Y) = -2\lambda g(X, Y) \tag{3.16}$$

and hence

$$h(\xi, \xi) = 2\lambda. \tag{3.17}$$

From (3.15) and (3.17) we get

$$\lambda = \frac{(n-1)(\alpha^2 - \rho)[A(\xi) - \alpha]}{\alpha + A(\xi)}. \tag{3.18}$$

Since $n > 1$ and $\alpha^2 - \rho \neq 0$ and $\alpha + A(\xi) \neq 0$, we have $\lambda > 0$ or $= 0$ or < 0 according as $\frac{(\alpha^2-\rho)[A(\xi)-\alpha]}{\alpha+A(\xi)} > 0$ or $\alpha = A(\xi)$ or $\frac{(\alpha^2-\rho)[A(\xi)-\alpha]}{\alpha+A(\xi)} < 0$. Thus we can state the following:

Theorem 3.3. *If the tensor field $\pounds_\xi + 2S$ on a ϕ-pseudo Ricci symmetric $(LCS)_n$-manifold with $\alpha^2 - \rho \neq 0$ is parallel, then the Ricci soliton (g, ξ, λ) is shrinking, steady and expanding according as $\frac{(\alpha^2-\rho)[A(\xi)-\alpha]}{\alpha+A(\xi)} < 0$, $\alpha = A(\xi)$ and $\frac{(\alpha^2-\rho)[A(\xi)-\alpha]}{\alpha+A(\xi)} > 0$ respectively.*

Corollary 3.1. *If the tensor field $\pounds_\xi + 2S$ on a ϕ-Ricci symmetric $(LCS)_n$-manifold with $\alpha^2 - \rho \neq 0$ is parallel, then the Ricci soliton (g, ξ, λ) is shrinking and expanding according as $\alpha^2 - \rho > 0$ and $\alpha^2 - \rho < 0$ respectively.*

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