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UNIQUENESS OF ENTIRE FUNCTIONS CONCERNING DIFFERENTIAL POLYNOMIALS

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Abstract. In this paper, we study the uniqueness of entire functions concerning differential polynomials and deficient value. The results extend and improve Theorem 2 in Yi [13].

1. Introduction and main results

Let f be a nonconstant meromorphic function in the whole complex plane $\mathbf{C}$, we will use the standard notations of Nevanlinna's value distribution theory such as $T(r,f)$, $N(r,f)$, $\bar{N}(r,f)$, $m(r,f)$ and so on, as found in [11]. In particular, we denote by $S(r,f)$ any function satisfying $S(r,f)=o(T(r,f))$ as $r\to\infty$, possibly outside a set of r of finite linear measure. For $a\in\mathbf{C}\cup\{\infty\}$, we set $E(a,f)=\{z\,|\,f(z)-a=0,\text{counting multiplicities}\}$ and $\bar{E}(a,f)=\{z\,|\,f(z)-a=0,\text{ignoring multiplicities}\}$ respectively.

Let f and g be two nonconstant meromorphic functions, we say that f and g share the value a CM (IM) provided that $E(a,f)=E(a,g)(\bar{E}(a,f)=\bar{E}(a,g))$.

The quantity $\lambda(f)=\overline{\lim\limits_{r\to\infty}}\frac{\log^{+}T(r,f)}{\log r}$ is called the order of $f(z)$. Also

$$\delta(a,f)=\underline{\lim_{r\to\infty}}\frac{m(r,\frac{1}{f-a})}{T(r,f)}=1-\overline{\lim_{r\to\infty}}\frac{N(r,\frac{1}{f-a})}{T(r,f)}$$

is called the deficiency of a with respect to $f(z)$. If $\delta(a,f)>0$, then the complex number a is named a deficient value of $f(z)$.

In 1976, Yang [8] posed the following question:

What can be said about the relationship between two nonconstant entire functions f and g if f and g share the value 0 CM and f' and g' share the value 1 CM?

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The above problem has been studied by K. Shibazaki [7], Yi [12, 13], Yang-Yi [10], Hua [2], Muse-Reinders [6] and I. Lahiri [3]. And Yi [13] has proved the following theorem.

Theorem 1.1 ([13, Theorem 2]). *Let f and g be two nonconstant entire functions and let k be a nonnegative integer. If f and g share the value 0 CM, $f^{(k)}$ and $g^{(k)}$ share the value 1 CM and $\delta(0,f) > \frac{1}{2}$, then $f \equiv g$ unless $f^{(k)} \cdot g^{(k)} \equiv 1$.*

Let h be a nonconstant meromorphic function. We denote by $P(h) = h^{(k)} + a_1 h^{(k-1)} + a_2 h^{(k-2)} + \cdots + a_{k-1}h' + a_k h$ the differential polynomial of h, where $a_1, a_2, \ldots, a_k$ are finite complex numbers and k is a positive integer.

Remark 1.2. The following example shows that in Theorem 1.1 the functions $f^{(k)}$ and $g^{(k)}$ cannot be replaced by $P(f)$ and $P(g)$. Let $f = \frac{1}{2}e^{-2z}$ and $g = e^{-2z}$. Then f and g share the value 0 CM, $f'' + 2f'$ and $g'' + 2g'$ share the value 1 CM and $\delta(0,f) > \frac{1}{2}$, but $f \neq g$ and $(f'' + 2f')(g'' + 2g') \neq 1$.

In this paper, we shall prove the following general results which extend and improve Theorem 1.1.

Theorem 1.3. *Let f and g be two nonconstant entire functions. Suppose that f and g share the value 0 CM, $P(f)$ and $P(g)$ share the value 1 CM and $\delta(0,f) > \frac{1}{2}$. If $\lambda(f) \neq 1$, then $f \equiv g$ unless $P(f) \cdot P(g) \equiv 1$.*

Theorem 1.4. *Let f and g be two nonconstant entire functions. Suppose f and g share the value 0 CM, $P(f)$ and $P(g)$ share the value 1 IM and $\delta(0,f) > \frac{4}{5}$. If $\lambda(f) \neq 1$, then $f \equiv g$ unless $P(f) \cdot P(g) \equiv 1$.*

2. Some lemmas

Lemma 2.1 ([5]). *Let f be a nonconstant meromorphic function and let k be a nonnegative integer. Then*

$$T(r, P(f)) \leq T(r,f) + k\bar{N}(r,f) + S(r,f). \tag{1}$$

Lemma 2.2. *Suppose that $f(z)$ is a nonconstant meromorphic function in the complex plane and $a(z)$ is a small function of $f(z)$, that is, $T(r,a) = S(r,f)$. If $f(z)$ is not a polynomial, then*

$$N(r, \frac{1}{P(f) - P(a)}) \leq T(r, P(f)) - T(r,f) + N(r, \frac{1}{f-a}) + S(r,f) \tag{2}$$

and

$$N(r, \frac{1}{P(f) - P(a)}) \leq N(r, \frac{1}{f-a}) + k\bar{N}(r,f) + S(r,f). \tag{3}$$

Proof. By the Nevanlinna's first fundamental theorem and the lemma of logarithmic derivatives, we have

$$T(r,f) - N(r, \frac{1}{f-a}) = m(r, \frac{1}{f-a}) + S(r,f)$$

$$\leq m(r, \frac{1}{P(f)-P(a)}) + m(r, \frac{P(f-a)}{f-a}) + S(r,f)$$
$$= T(r, P(f)) - N(r, \frac{1}{P(f)-P(a)}) + S(r,f).$$

We get (2) by transposition. And we obtain (3) combined with (1) and (2), which proves this lemma. □

Next, we introduce some notations.

Let F and G be two nonconstant meromorphic functions such that F and G share the value 1 IM. We denote by $\bar{N}_L(r, \frac{1}{F-1})$ the reduced counting function for zeros of both $F-1$ and $G-1$ about which $F-1$ has lager multiplicity than $G-1$, $N_E^{1)}(r, \frac{1}{F-1})$ the counting function for common simple zeros of both $F-1$ and $G-1$, and $\bar{N}_E^{(2}(r, \frac{1}{F-1})$ the reduced counting function for common multiple zeros of both $F-1$ and $G-1$. In the same way, we can define $\bar{N}_L(r, \frac{1}{G-1})$,$N_E^{1)}(r, \frac{1}{G-1})$ and $\bar{N}_E^{(2}(r, \frac{1}{G-1})$. Also we denote by $N_{1)}(r, \frac{1}{F})$ the counting function for simple zeros of F, and $\bar{N}_{(2}(r, \frac{1}{F})$ the reduced counting function for multiple zeros of F.

Lemma 2.3. *Let F and G be two nonconstant meromorphic functions such that F and G share the value* 1 *IM. Let*

$$H = \frac{F''}{F'} - \frac{2F'}{F-1} - \frac{G''}{G'} + \frac{2G'}{G-1}. \tag{4}$$

If $H \not\equiv 0$, then

$$T(r,F) \leq N(r, \frac{1}{F}) + 2\bar{N}(r,F) + N(r, \frac{1}{G}) + 2\bar{N}_L(r, \frac{1}{F-1}) \tag{5}$$
$$+ 2\bar{N}(r,G) + \bar{N}_L(r, \frac{1}{G-1}) + S(r,F) + S(r,G).$$

Proof. Let z_0 be a common simple zero of $F-1$ and $G-1$. By (4), we have $H(z_0) = 0$ and $m(r,H) = S(r,F) + S(r,G)$, then

$$N_E^{1)}(r, \frac{1}{F-1}) \leq N(r, \frac{1}{H}) \leq T(r,H) + O(1)$$

and

$$N_E^{1)}(r, \frac{1}{F-1}) \leq N(r,H) + S(r,F) + S(r,G). \tag{6}$$

By the Nevanlinna's second fundamental theorem, we have

$$T(r,F) + T(r,G) \leq \bar{N}(r, \frac{1}{F}) + \bar{N}(r, \frac{1}{F-1}) + \bar{N}(r,F) - N_0(r, \frac{1}{F'}) \tag{7}$$
$$+ S(r,F) + \bar{N}(r, \frac{1}{G}) + \bar{N}(r, \frac{1}{G-1})$$
$$+ \bar{N}(r,G) - N_0(r, \frac{1}{G'}) + S(r,G),$$

where $N_0(r, 1/F')$ denotes the counting function corresponding to the zeros of F' that are not zeros of F and $F-1$ and $N_0(r, 1/G')$ denotes the counting function corresponding to the zeros of G' that are not zeros of G and $G-1$. Since F and G share the value 1 IM, we get

$$\begin{aligned}\bar{N}(r, \frac{1}{F-1}) &= N_E^{1)}(r, \frac{1}{F-1}) + \bar{N}_L(r, \frac{1}{F-1}) + \bar{N}_L(r, \frac{1}{G-1}) \\ &\quad + \bar{N}_E^{(2}(r, \frac{1}{G-1}) + S(r,F) + S(r,G) \\ &= \bar{N}(r, \frac{1}{G-1}) + S(r,F) + S(r,G).\end{aligned}$$

Then

$$\begin{aligned}\bar{N}(r, \frac{1}{F-1}) + \bar{N}(r, \frac{1}{G-1}) &= N_E^{1)}(r, \frac{1}{F-1}) + \bar{N}_L(r, \frac{1}{F-1}) \\ &\quad + \bar{N}_L(r, \frac{1}{G-1}) + \bar{N}_E^{(2}(r, \frac{1}{G-1}) \\ &\quad + \bar{N}(r, \frac{1}{G-1}) + S(r,F) + S(r,G) \\ &\leq N_E^{1)}(r, \frac{1}{F-1}) + \bar{N}_L(r, \frac{1}{F-1}) \\ &\quad + N(r, \frac{1}{G-1}) + S(r,F) + S(r,G) \\ &\leq N_E^{1)}(r, \frac{1}{F-1}) + \bar{N}_L(r, \frac{1}{F-1}) \\ &\quad + T(r,G) + S(r,F) + S(r,G).\end{aligned} \tag{8}$$

From (7) and (8), we obtain

$$\begin{aligned}T(r,F) &\leq \bar{N}(r, \frac{1}{F}) + \bar{N}(r,F) + \bar{N}(r, \frac{1}{G}) + \bar{N}(r,G) + N_E^{1)}(r, \frac{1}{F-1}) \\ &\quad + \bar{N}_L(r, \frac{1}{F-1}) - N_0(r, \frac{1}{F'}) - N_0(r, \frac{1}{G'}) + S(r,F) + S(r,G).\end{aligned} \tag{9}$$

By (4), we get

$$\begin{aligned}N(r,H) &\leq \bar{N}_{(2}(r, \frac{1}{F}) + \bar{N}(r,F) + \bar{N}_{(2}(r, \frac{1}{G}) + \bar{N}(r,G) \\ &\quad + \bar{N}_L(r, \frac{1}{F-1}) + \bar{N}_L(r, \frac{1}{G-1}) + N_0(r, \frac{1}{F'}) + N_0(r, \frac{1}{G'}) \\ &\quad + S(r,F) + S(r,G).\end{aligned} \tag{10}$$

Combine (6), (9) and (10), we have

$$\begin{aligned}T(r,F) &\leq \bar{N}(r, \frac{1}{F}) + \bar{N}_{(2}(r, \frac{1}{F}) + 2\bar{N}(r,F) + \bar{N}(r, \frac{1}{G}) \\ &\quad + \bar{N}_{(2}(r, \frac{1}{G}) + 2\bar{N}(r,G) + 2\bar{N}_L(r, \frac{1}{F-1})\end{aligned} \tag{11}$$

$$+\bar{N}_L(r,\frac{1}{G-1})+S(r,F)+S(r,G).$$

It is obvious that

$$\bar{N}(r,\frac{1}{F})+\bar{N}_{(2}(r,\frac{1}{F})\le N(r,\frac{1}{F}),\tag{12}$$

$$\bar{N}(r,\frac{1}{G})+\bar{N}_{(2}(r,\frac{1}{G})\le N(r,\frac{1}{G}).\tag{13}$$

From (11), (12) and (13), we get (5), which completes the proof. □

Lemma 2.4 ([9])**.** *Suppose f_j $(j=1,2,\dots,m+1)$ and g_j $(j=1,2,\dots,m)$ are entire functions satisfying the following conditions*:

- $\sum_{j=1}^{m} f_j(z)e^{g_j(z)}\equiv f_{m+1}(z)$;
- *The order of $f_j(z)$ is less than the order of $e^{g_k(z)}$ for $1\le j\le m+1$, $1\le k\le m$; And furthermore, the order of $f_j(z)$ is less than the order of $e^{g_l(z)-g_k(z)}$ for $m\ge 2$ and $1\le j\le m+1$, $1\le l,k\le m$, $l\ne k$.*

Then $f_j\equiv 0$ $(j=1,2,\dots,m+1)$.

3. Proof of Theorem 1.4

We just prove Theorem 1.4, and the proof of Theorem 1.3 is similar. Next we consider two cases.

Case 1. Assume that $P(f),P(g)\not\equiv c$, where c is a finite complex constant.

Since f and g share the value 0 CM and $P(f)$ and $P(g)$ share the value 1 IM, by Milloux's basic result we have

$$\begin{aligned}T(r,f)&\le \bar{N}(r,f)+N(r,\frac{1}{f})+\bar{N}(r,\frac{1}{P(f)-1})+S(r,f)\\&=N(r,\frac{1}{g})+\bar{N}(r,\frac{1}{P(g)-1})+S(r,f)\\&\le T(r,g)+T(r,P(g))+S(r,f).\end{aligned}$$

By Lemma 2.1, we get

$$T(r,f)\le (k+2)T(r,g)+S(r,f)+S(r,g).\tag{14}$$

Similarly we can get

$$T(r,g)\le (k+2)T(r,f)+S(r,f)+S(r,g).\tag{15}$$

Then

$$S(r,f)=S(r,g).\tag{16}$$

Let $F=P(f)$, $G=P(g)$ and let H be defined by (4), then F and G share the value 1 IM. If $H\not\equiv 0$, then by Lemma 2.3 we have

$$T(r,F)\le N(r,\frac{1}{F})+N(r,\frac{1}{G})+2\bar{N}_L(r,\frac{1}{F-1})\tag{17}$$

$$+ \bar{N}_L(r, \frac{1}{G-1}) + S(r,F) + S(r,G).$$

From (3), we obtain

$$(18) \qquad \bar{N}_L(r, \frac{1}{F-1}) \le N(r, \frac{1}{F'}) \le N(r, \frac{1}{F}) + \bar{N}(r,F) + S(r,F),$$
$$\bar{N}_L(r, \frac{1}{G-1}) \le N(r, \frac{1}{G'}) \le N(r, \frac{1}{G}) + \bar{N}(r,G) + S(r,G).$$

Substituting (18) into (17), we deduce that

$$(19) \qquad T(r,F) \le 3N(r, \frac{1}{F}) + 2N(r, \frac{1}{G}) + S(r,F) + S(r,G).$$

By Lemma 2.2 and (19), we have

$$(20) \qquad T(r,P(f)) \le T(r,P(f)) - T(r,f) + N(r, \frac{1}{f}) + 2N(r, \frac{1}{f})$$
$$+ 2N(r, \frac{1}{g}) + S(r,f) + S(r,g).$$

Noting that f and g share the value 0 CM, by (16) and (20) we get $T(r,f) \le 5N(r, \frac{1}{f}) + S(r,f)$, a contradiction to the condition $\delta(0,f) > \frac{4}{5}$. Thus $H \equiv 0$. Solving this equation, we get

$$(21) \qquad F = \frac{AG+B}{CG+D} \qquad (AD - BC \ne 0),$$

where A, B, C and D are finite complex constants. Next we consider three subcases.

Subcase 1.1. Assume that $AC \ne 0$. From (21), we know that $\frac{A}{C}$ is a Picard exceptional value of F. By the Nevanlinna's second fundamental theorem, we have

$$(22) \qquad T(r,F) \le N(r, \frac{1}{F}) + N(r, \frac{1}{F - \frac{A}{C}}) + N(r,F) + S(r,F)$$
$$= N(r, \frac{1}{F}) + S(r,F).$$

From (3) and (22), we get

$$T(r,P(f)) \le T(r,P(f)) - T(r,f) + N(r, \frac{1}{f}) + S(r,f),$$

that is, $T(r,f) \le N(r, \frac{1}{f}) + S(r,f)$, which contradicts the condition $\delta(0,f) > \frac{4}{5}$.

Subcase 1.2. Assume that $A \ne 0$ and $C = 0$. Then $F = \frac{A}{D}G + \frac{B}{D}$. If $B \ne 0$, then $N(r, \frac{1}{F - \frac{B}{D}}) = N(r, \frac{1}{G})$. By the Nevanlinna's second fundamental theorem, we have

$$(23) \qquad T(r,F) \le N(r, \frac{1}{F}) + N(r, \frac{1}{F - \frac{B}{D}}) + N(r,F) + S(r,F)$$
$$= N(r, \frac{1}{F}) + N(r, \frac{1}{G}) + S(r,F).$$

From Lemma 2.3 and (23), we obtain

$$T(r,P(f)) \leq T(r,P(f)) - T(r,f) + N(r,\frac{1}{f}) + N(r,\frac{1}{g}) + S(r,f) + S(r,g). \tag{24}$$

By (16) and (24), we have

$$T(r,f) \leq N(r,\frac{1}{f}) + N(r,\frac{1}{g}) + S(r,f) = 2N(r,\frac{1}{f}) + S(r,f),$$

a contradiction to the condition $\delta(0,f) > \frac{4}{5}$. Thus $B = 0$, that is, $F = \frac{A}{D}G$. If 1 is a Picard exceptional value of F, then $\frac{A}{D} = 1$. Otherwise, $\frac{A}{D}$ is a Picard exceptional value of F that is different from 1, which contradicts the Deficiency Theorem [11]. Thus $F \equiv G$. If 1 is not a Picard exceptional value of F, then there is a complex number z_0 such that $F(z_0) = G(z_0) = 1$. Therefore, $\frac{A}{D} = 1$, that is, $F \equiv G$.

Subcase 1.3. Assume that $A = 0$ and $C \neq 0$. Proceeding as in the proof of subcase 1.2 we can get $F \cdot G \equiv 1$.

In conclusion, we know that $F \equiv G$ unless $F \cdot G \equiv 1$. If $F \cdot G \equiv 1$, that is, $P(f) \cdot P(g) \equiv 1$, then the result of theorem 1.4 is true. If the former is established, that is, $P(f - g) \equiv 0$, solving this equation (see [1, 4]) we get

$$f - g = \sum_{j=1}^{m} p_j(z) e^{\alpha_j z}, \tag{25}$$

where $m(\leq k)$ is a positive integer, α_j $(j = 1, \ldots, m)$ are distinct complex constants and $p_j(z)$ $(j = 1, \ldots, m)$ are polynomials. Next we prove that if $\lambda(f) \neq 1$, then $f \equiv g$. We distinguish two cases below.

Case I. Assume that $\lambda(f) < 1$. By (14) and (15), we know that $\lambda(f) = \lambda(g)$. Since f and g share the value 0 CM, we can get $\frac{f}{g} = e^{h(z)}$, where $h(z)$ is an entire function. Then

$$\lambda(e^{h(z)}) = \lambda(\frac{f}{g}) \leq \max\{\lambda(f),\ \lambda(\frac{1}{g})\} < 1.$$

Thus $e^{h(z)} \equiv c_0$, where c_0 is a finite complex constant. We obtain $f \equiv c_0 g$, then $P(f) \equiv c_0 P(g)$. By $P(f) \equiv P(g)$, we can get $c_0 = 1$, that is, $f \equiv g$.

Case II. Assume that $\lambda(f) > 1$. By the Weierstrass's factorization theorem, we have

$$f(z) = \pi(z) e^{l_1(z)}, \quad g(z) = \pi(z) e^{l_2(z)},$$

where $\pi(z)$ is canonical product formed with common zeros of f and g and $l_1(z)$ and $l_2(z)$ are entire functions.

If $l_1 \equiv l_2$, then $f \equiv g$. If $l_1 \not\equiv l_2$, since $\lambda(\pi)$ is equal to $\tau(f)$ which is the exponent of convergence of zeros of $f(z)$ and $\tau(f) \leq \tau(f - g) \leq \lambda(f - g)$, by

(25) we have

$$\lambda(\pi) \le \lambda(f-g) = \lambda(\sum_{j=1}^{m} p_j(z)e^{\alpha_j z}) \le 1.$$

Since $\lambda(f) = \lambda(g) > 1$ and $f - g = (e^{l_1-l_2} - 1)g$, we can get that $\lambda(e^{l_1(z)}) > 1$, $\lambda(e^{l_2(z)}) > 1$ and $\lambda(e^{l_1(z)-l_2(z)}) > 1$. By $\pi(z)e^{l_1(z)} - \pi(z)e^{l_2(z)} = \sum_{j=1}^{m} p_j(z)e^{\alpha_j z}$ and Lemma 2.4 we know that $\sum_{j=1}^{m} p_j(z)e^{\alpha_j z} \equiv 0$ and $\pi(z) \equiv 0$. Then $f(z) \equiv 0$, a contradiction.

Case 2. Assume that $P(f) \equiv c$, where c is a finite complex constant.

We can know that $f \equiv c_1 + \sum_{j=1}^{m} q_j(z)e^{\beta_j z}$, where c_1 is finite complex constant, q_j $(j = 1, 2, \ldots, m)$ are polynomials and β_j $(j = 1, 2, \ldots, m)$ are distinct finite complex constants. Since $\lambda(f) \neq 1$, we get $\lambda(f) < 1$. Then $f \equiv c_1 + \sum_{j=1}^{m} q_j(z)$, that is, f is a polynomial. Suppose the degree of f is n. Then

$$N(r, \frac{1}{f}) = n \log r \quad \text{and} \quad T(r, f) = n \log r + O(1).$$

Therefore, $\delta(0, f) = 1 - \overline{\lim_{r\to\infty}} \frac{N(r,\frac{1}{f})}{T(r,f)} = 0 < \frac{4}{5}$, which is a contradiction.

This completes the proof of Theorem 1.4.

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