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A GRÜSS TYPE INTEGRAL INEQUALITY ASSOCIATED WITH GAUSS HYPERGEOMETRIC FUNCTION FRACTIONAL INTEGRAL OPERATOR

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Abstract. In this paper, we aim at establishing a generalized fractional integral version of Grüss type integral inequality by making use of the Gauss hypergeometric function fractional integral operator. Our main result, being of a very general character, is illustrated to specialize to yield numerous interesting fractional integral inequalities including some known results.

1. Introduction

The study of inequalities is an important research subject in mathematical analysis. The inequality technique is also one of the useful tools in the study of special functions and theory of approximations. Particularly, the fractional integral inequalities have many applications in numerical quadrature, transform theory, probability, and statistical problems. One of the most useful applications is to establish uniqueness of solutions in fractional boundary value problems. For detailed applications on the subject, one may refer to [1], [2], [3], [4], [5], [6], [18], [19], [23], [24], [31], [32], [33], [44], [46], [50], and the references cited therein.

In [27], Grüss proved a very useful and (now) well-known inequality, which establishes a connection between the integral of the product of two functions and the product of the integrals of individual functions. The Grüss inequality [27] (see also [37, p. 296]) is given as follows:

Let f and g be two continuous functions defined on $[a,b]$ such that $m \leq f(t) \leq M$ and $p \leq g(t) \leq P$ for all $t \in [a,b]$ and some real constants m, M, p,

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P. Then the following inequality holds true:

$$\left|\frac{1}{(b-a)}\int_a^b f(t)\,g(t)\,dt-\frac{1}{(b-a)^2}\int_a^b f(t)\,dt\int_a^b g(t)\,dt\right| \leq \frac{1}{4}(M-m)(P-p), \tag{1}$$

where the constant $1/4$ is the best possible.

In the theory of approximations, Chebyshev and Grüss inequalities (see, *e.g.*, [9], [10] and [37]) are useful to give a lower bound or an upper bound for certain functionals. Therefore, several generalizations of this type of integral inequality have been extensively addressed by researchers (see, *e.g.*, [16], [20], [21], [22], [25], [26], [29], [34], [35], [36], [39], [40], [41], [49]). Moreover, by applying fractional integral operators and fractional q-integral operators, many authors have obtained a lot of fractional integral inequalities, their q-analogues and applications (see, *e.g.*, [7], [11], [12], [13], [14], [17],[28], [38], [42], [43], [51], and the references cited therein).

Recently, Dahmani *et al.* [17] used Riemann-Liouville fractional integral operators to unify the Grüss integral inequality as follows:

Let f and g be two integrable functions defined on $[0,\infty)$ with constant bounds as follows:

$$m \leq f(t) \leq M \quad \text{and} \quad p \leq g(t) \leq P \quad (0 \leq t < \infty)$$

for some $m, M, p, P \in \mathbb{R}$. Then, for $\alpha > 0$,

$$\left|\frac{t^\alpha}{\Gamma(\alpha+1)} I^\alpha f(t)\,g(t) - I^\alpha f(t)\, I^\alpha g(t)\right| \leq \left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right)^2 (M-m)(P-p), \tag{2}$$

where $I^\alpha f(t)$ denotes the familiar Riemann-Liouville fractional integral operator of a function $f(t)$ and Γ is the familiar Gamma function.

In the sequel, by replacing the constants appeared as bounds of the functions f and g by four integrable functions, Tariboon *et al.* [47] and Baleanu *et al.* [8] investigated more general forms of the inequality (2), involving the Riemann-Liouville and Saigo fractional integral operators, respectively. In this paper, we aim at establishing a new integral inequality involving Gauss hypergeometric function fractional integral operators introduced by Curiel and Galué [15], which generalizes the Grüss integral inequality with integrable functions whose bounds are also four integrable functions. Some interesting special cases of our main result are also considered.

For our purpose, we begin by recalling basic definitions and notations of some well-known operators of fractional calculus. Let $\alpha > 0$, $\mu > -1$, $\beta, \eta \in \mathbb{R}$. Then a generalized fractional integral $I_t^{\alpha,\beta,\eta,\mu}$ of order α for a real-valued continuous function $f(t)$ is defined by [15] (see also [11, 12, 30]):

$$I_t^{\alpha,\beta,\eta,\mu}\{f(t)\} = \frac{t^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)} \int_0^t \tau^{\mu}(t-\tau)^{\alpha-1}\, {}_2F_1\left(\alpha+\beta+\mu, -\eta; \alpha; 1-\frac{\tau}{t}\right) f(\tau)d\tau, \tag{3}$$

where the function ${}_2F_1(\cdot)$ appearing as a kernel for the operator (3) is the familiar Gaussian hypergeometric function.

Following Curiel and Galué [15], the operator (3) would reduce immediately to the extensively investigated Saigo, Erdélyi-Kober and Riemann-Liouville type fractional integral operators given, respectively, as follows (see [30] and [45]; see also [11, 12]):

$$I_{0,t}^{\alpha,\beta,\eta}\{f(t)\} = I_t^{\alpha,\beta,\eta,0}\{f(t)\} = \frac{t^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\, {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\tau}{t}\right) f(\tau)\, d\tau \quad (\alpha>0,\ \beta,\eta\in\mathbb{R}), \tag{4}$$

$$I^{\alpha,\eta}\{f(t)\} = I_t^{\alpha,0,\eta,0}\{f(t)\} = \frac{t^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\, \tau^{\eta} f(\tau)\, d\tau \quad (\alpha>0,\ \eta\in\mathbb{R}) \tag{5}$$

and

$$\mathrm{R}^{\alpha}\{f(t)\} = I_t^{\alpha,-\alpha,\eta,0}\{f(t)\} = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau \quad (\alpha>0). \tag{6}$$

Further, for $f(t)=t^{\mu}$ in (3), we get (*cf.* [7])

$$I_t^{\alpha,\beta,\eta,\mu}\left\{t^{\lambda-1}\right\} = \frac{\Gamma(\mu+\lambda)\Gamma(\lambda-\beta+\eta)}{\Gamma(\lambda-\beta)\Gamma(\lambda+\mu+\alpha+\eta)}\ t^{\lambda-\beta-\mu-1}, \tag{7}$$

where $\alpha,\beta,\eta,\lambda\in\mathbb{R}$, $\mu>-1$, $\mu+\lambda>0$ and $\lambda-\beta+\eta>0$.

2. A generalized Grüss type integral inequality

Here we establish a generalized integral inequality involving fractional hypergeometric operators (3) which gives an estimation for the fractional integral of a product of two functions in terms of the product of the individual function fractional integrals. To do this, we first give a functional relation for the fractional hypergeometric operators associated with a bounded integrable function asserted by the following lemma.

Lemma. *Suppose f, φ_1 and φ_2 are integrable functions defined on $[0,\infty)$ such that*

$$\varphi_1(t) \le f(t) \le \varphi_2(t) \quad (t\in[0,\infty)). \tag{8}$$

Then the following relation holds true: *For* $t > 0$, $\alpha > \max\{0, -\beta-\mu\}$, $\mu > -1$, $\beta < 1$ and $\beta - 1 < \eta < 0$,

(9)
$$\begin{aligned}&\frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)} I_t^{\alpha,\beta,\eta,\mu} f^2(t) - \left(I_t^{\alpha,\beta,\eta,\mu} f(t)\right)^2 \\ &= (I_t^{\alpha,\beta,\eta,\mu}\varphi_2(t) - I_t^{\alpha,\beta,\eta,\mu} f(t))(I_t^{\alpha,\beta,\eta,\mu} f(t) - I_t^{\alpha,\beta,\eta,\mu}\varphi_1(t)) \\ &\quad - \frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)} I_t^{\alpha,\beta,\eta,\mu}(\varphi_2(t) - f(t))(f(t) - \varphi_1(t)) \\ &\quad + \frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)} I_t^{\alpha,\beta,\eta,\mu}\varphi_1(t) f(t) - I_t^{\alpha,\beta,\eta,\mu}\varphi_1(t) I_t^{\alpha,\beta,\eta,\mu} f(t) \\ &\quad + \frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)} I_t^{\alpha,\beta,\eta,\mu}\varphi_2(t) f(t) - I_t^{\alpha,\beta,\eta,\mu}\varphi_2(t) I_t^{\alpha,\beta,\eta,\mu} f(t) \\ &\quad + I_t^{\alpha,\beta,\eta,\mu}\varphi_1(t) I_t^{\alpha,\beta,\eta,\mu}\varphi_2(t) - \frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)} I_t^{\alpha,\beta,\eta,\mu}\varphi_1(t)\varphi_2(t).\end{aligned}$$

Proof. For the functions in (8), and any $\tau, \rho > 0$, we find that

(10)
$$\begin{aligned}&(\varphi_2(\rho) - f(\rho))(f(\tau) - \varphi_1(\tau)) + (\varphi_2(\tau) - f(\tau))(f(\rho) - \varphi_1(\rho)) \\ &\quad - (\varphi_2(\tau) - f(\tau))(f(\tau) - \varphi_1(\tau)) - (\varphi_2(\rho) - f(\rho))(f(\rho) - \varphi_1(\rho)) \\ &= f^2(\tau) + f^2(\rho) - 2f(\tau)f(\rho) + \varphi_2(\rho)f(\tau) + \varphi_1(\tau)f(\rho) - \varphi_1(\tau)\varphi_2(\rho) \\ &\quad + \varphi_2(\tau)f(\rho) + \varphi_1(\rho)f(\tau) - \varphi_1(\rho)\varphi_2(\tau) - \varphi_2(\tau)f(\tau) + \varphi_1(\tau)\varphi_2(\tau) \\ &\quad - \varphi_1(\tau)f(\tau) - \varphi_2(\rho)f(\rho) + \varphi_1(\rho)\varphi_2(\rho) - \varphi_1(\rho)f(\rho).\end{aligned}$$

Consider

(11)
$$\begin{aligned}F(t,\tau) &= \frac{t^{-\alpha-\beta-2\mu}\tau^{\mu}\,(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\;{}_2F_1\left(\alpha+\beta+\mu, -\eta; \alpha; 1-\frac{\tau}{t}\right) \\ &= \frac{\tau^{\mu}}{\Gamma(\alpha)}\;\frac{(t-\tau)^{\alpha-1}}{t^{\alpha+\beta+2\mu}} + \frac{\tau^{\mu}(\alpha+\beta+\mu)(-\eta)}{\Gamma(\alpha+1)}\;\frac{(t-\tau)^{\alpha}}{t^{\alpha+\beta+2\mu+1}} \\ &\quad + \frac{\tau^{\mu}(\alpha+\beta+\mu)(\alpha+\beta+\mu+1)(-\eta)(-\eta+1)}{2\Gamma(\alpha+2)}\;\frac{(t-\tau)^{\alpha+1}}{t^{\alpha+\beta+2\mu+2}} + \cdots,\end{aligned}$$

where $\tau \in (0, t)$ and $t > 0$. Multiplying both sides of (10) by $F(t,\tau)$ and integrating the resulting relation with respect to τ from 0 to t, and using (3), we get

(12)
$$\begin{aligned}&(\varphi_2(\rho) - f(\rho))(I_t^{\alpha,\beta,\eta,\mu} f(t) - I_t^{\alpha,\beta,\eta,\mu}\varphi_1(t)) \\ &\quad + (I_t^{\alpha,\beta,\eta,\mu}\varphi_2(t) - I_t^{\alpha,\beta,\eta,\mu} f(t))(f(\rho) - \varphi_1(\rho)) \\ &\quad - I_t^{\alpha,\beta,\eta,\mu}(\varphi_2(t) - f(t))(f(t) - \varphi_1(t)) \\ &\quad - (\varphi_2(\rho) - f(\rho))(f(\rho) - \varphi_1(\rho)) I_t^{\alpha,\beta,\eta,\mu}\,\{1\}\end{aligned}$$

$$
\begin{aligned}
&= I_t^{\alpha,\beta,\eta,\mu} f^2(t) + f^2(\rho) I_t^{\alpha,\beta,\eta,\mu}\{1\} - 2f(\rho) I_t^{\alpha,\beta,\eta,\mu} f(t) + \varphi_2(\rho) I_t^{\alpha,\beta,\eta,\mu} f(t) \\
&\quad + f(\rho) I_t^{\alpha,\beta,\eta,\mu} \varphi_1(t) - \varphi_2(\rho) I_t^{\alpha,\beta,\eta,\mu} \varphi_1(t) + f(\rho) I_t^{\alpha,\beta,\eta,\mu} \varphi_2(t) \\
&\quad + \varphi_1(\rho) I_t^{\alpha,\beta,\eta,\mu} f(t) - \varphi_1(\rho) I_t^{\alpha,\beta,\eta,\mu} \varphi_2(t) - I_t^{\alpha,\beta,\eta,\mu} \varphi_2(t) f(t) \\
&\quad + I_t^{\alpha,\beta,\eta,\mu} \varphi_1(t)\varphi_2(t) - I_t^{\alpha,\beta,\eta,\mu} \varphi_1(t) f(t) - \varphi_2(\rho) f(\rho) I_t^{\alpha,\beta,\eta,\mu}\{1\} \\
&\quad + \varphi_1(\rho)\varphi_2(\rho) I_t^{\alpha,\beta,\eta,\mu}\{1\} - \varphi_1(\rho) f(\rho) I_t^{\alpha,\beta,\eta,\mu}\{1\}.
\end{aligned}
$$

Next, multiplying both sides of (12) by $F(t,\rho)$ $(\rho \in (0,\, t)\,,\ t>0)$ in (11), and integrating with respect to ρ from 0 to t, we obtain

(13)

$$
\begin{aligned}
&2(I_t^{\alpha,\beta,\eta,\mu}\varphi_2(t) - I_t^{\alpha,\beta,\eta,\mu} f(t))(I_t^{\alpha,\beta,\eta,\mu} f(t) - I_t^{\alpha,\beta,\eta,\mu}\varphi_1(t)) \\
&\quad - 2I_t^{\alpha,\beta,\eta,\mu}(\varphi_2(t) - f(t))(f(t)-\varphi_1(t)) I_t^{\alpha,\beta,\eta,\mu}\{1\} \\
&= 2I_t^{\alpha,\beta,\eta,\mu}\{1\}\, I_t^{\alpha,\beta,\eta,\mu} f^2(t) - 2\left(I_t^{\alpha,\beta,\eta,\mu} f(t)\right)^2 + 2I_t^{\alpha,\beta,\eta,\mu}\varphi_1(t) I_t^{\alpha,\beta,\eta,\mu} f(t) \\
&\quad - 2I_t^{\alpha,\beta,\eta,\mu}\{1\}\, I_t^{\alpha,\beta,\eta,\mu}\varphi_1(t) f(t) + 2I_t^{\alpha,\beta,\eta,\mu}\varphi_2(t) I_t^{\alpha,\beta,\eta,\mu} f(t) \\
&\quad - 2I_t^{\alpha,\beta,\eta,\mu}\{1\}\, I_t^{\alpha,\beta,\eta,\mu}\varphi_2(t) f(t) - 2I_t^{\alpha,\beta,\eta,\mu}\varphi_1(t) I_t^{\alpha,\beta,\eta,\mu}\varphi_2(t) \\
&\quad + 2I_t^{\alpha,\beta,\eta,\mu}\{1\}\, I_t^{\alpha,\beta,\eta,\mu}\varphi_1(t)\varphi_2(t),
\end{aligned}
$$

which, upon using the image formula (7), leads to the desired result (9). □

Now, we give our main result satisfying the Cauchy-Schwarz type inequality asserted by the following theorem.

Theorem. *Assume that f and g are two integrable functions on $[0,\infty)$ and φ_1, φ_2, ψ_1 and ψ_2 are four integrable functions on $[0,\infty)$ such that*

(14) $$\varphi_1(t) \le f(t) \le \varphi_2(t),\ \ \psi_1(t) \le g(t) \le \psi_2(t) \quad (t \in [0,\infty)).$$

Then the following inequality holds true: For $t>0$, $\alpha > \max\{0, -\beta-\mu\}$, $\mu > -1$, $\beta < 1$ and $\beta - 1 < \eta < 0$,

(15)
$$
\begin{aligned}
&\left| \frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)} I_t^{\alpha,\beta,\eta,\mu} f(t)g(t) - I_t^{\alpha,\beta,\eta,\mu} f(t) I_t^{\alpha,\beta,\eta,\mu} g(t) \right| \\
&\quad \le \sqrt{\mathcal{T}(f,\varphi_1,\varphi_2)\ \mathcal{T}(g,\psi_1,\psi_2)},
\end{aligned}
$$

where

(16)
$$
\begin{aligned}
\mathcal{T}(u,v,w) := {} & (I_t^{\alpha,\beta,\eta,\mu} w(t) - I_t^{\alpha,\beta,\eta,\mu} u(t))(I_t^{\alpha,\beta,\eta,\mu} u(t) - I_t^{\alpha,\beta,\eta,\mu} v(t)) \\
& + \frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)} I_t^{\alpha,\beta,\eta,\mu} v(t)u(t) \\
& - I_t^{\alpha,\beta,\eta,\mu} v(t) I_t^{\alpha,\beta,\eta,\mu} u(t) \\
& + \frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)} I_t^{\alpha,\beta,\eta,\mu} w(t)u(t)
\end{aligned}
$$

$$- I_t^{\alpha,\beta,\eta,\mu}w(t) I_t^{\alpha,\beta,\eta,\mu}u(t) + I_t^{\alpha,\beta,\eta,\mu}v(t) I_t^{\alpha,\beta,\eta,\mu}w(t)$$
$$- \frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)} I_t^{\alpha,\beta,\eta,\mu}v(t)w(t).$$

Proof. We begin, for convenience, by defining a function $\mathcal{H}(\cdot,\cdot)$ by

$$\mathcal{H}(\tau,\rho) = (f(\tau) - f(\rho))(g(\tau) - g(\rho)) \quad (t > 0;\ 0 < \tau,\ \rho < t), \tag{17}$$

where f and g are two integrable functions on $[0,\infty)$ satisfying the inequality (14). Upon multiplying both sides of (17) by $F(t,\tau)\,F(t,\rho)$, where $F(t,\tau)$ and $F(t,\rho)$ are given by (11), and integrating with respect to τ and ρ, respectively, from 0 to t, we obtain

$$\begin{aligned} &\frac{t^{-2\alpha-2\beta-4\mu}}{2\Gamma^2(\alpha)} \int_0^t \int_0^t (t-\tau)^{\alpha-1}\ (t-\rho)^{\alpha-1} {}_2F_1\left(\alpha+\beta+\mu, -\eta; \alpha; 1-\frac{\tau}{t}\right) \\ &\times {}_2F_1\left(\alpha+\beta+\mu, -\eta; \alpha; 1-\frac{\rho}{t}\right) \tau^\mu \rho^\mu \mathcal{H}(\tau,\rho)\, d\tau d\rho \\ &= \frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)} I_t^{\alpha,\beta,\eta,\mu} f(t)g(t) - I_t^{\alpha,\beta,\eta,\mu} f(t) I_t^{\alpha,\beta,\eta,\mu} g(t). \end{aligned} \tag{18}$$

Now, by applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\left(\frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)} I_t^{\alpha,\beta,\eta,\mu} f(t)g(t) - I_t^{\alpha,\beta,\eta,\mu} f(t) I_t^{\alpha,\beta,\eta,\mu} g(t)\right)^2 \\ &\leq \left(\frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)} I_t^{\alpha,\beta,\eta,\mu} f^2(t) - \left(I_t^{\alpha,\beta,\eta,\mu} f(t)\right)^2\right) \\ &\times \left(\frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)} I_t^{\alpha,\beta,\eta,\mu} g^2(t) - \left(I_t^{\alpha,\beta,\eta,\mu} g(t)\right)^2\right). \end{aligned} \tag{19}$$

On the other hand, we observe that each term of the series in (11) is positive, and hence, the function $F(t,\tau)$ remains positive, for all $\tau \in (0,t)$ $(t>0)$. Therefore, under the hypothesis of Lemma, it is obvious to see that either if a function f is integrable and nonnegative on $[0,\infty)$, then $I_t^{\alpha,\beta,\eta,\mu} f(t) \geq 0$; or if a function f is integrable and nonpositive on $[0,\infty)$, then $I_t^{\alpha,\beta,\eta,\mu} f(t) \leq 0$.

Now, by noting the relation that, for all $t \in (0,\infty)$,

$$(\varphi_2(t) - f(t))(f(t) - \varphi_1(t)) \geq 0 \quad \text{and} \quad (\psi_2(t) - g(t))(g(t) - \psi_1(t)) \geq 0,$$

we have

$$\frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)} I_t^{\alpha,\beta,\eta,\mu} (\varphi_2(t) - f(t))(f(t) - \varphi_1(t)) \geq 0$$

and

$$\frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)} I_t^{\alpha,\beta,\eta,\mu} (\psi_2(t) - g(t))(g(t) - \psi_1(t)) \geq 0.$$

Thus, using Lemma, we obtain

(20)
$$\begin{aligned}&\frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)}I_t^{\alpha,\beta,\eta,\mu}f^2(t)-\left(I_t^{\alpha,\beta,\eta,\mu}f(t)\right)^2\\
&\leq (I_t^{\alpha,\beta,\eta,\mu}\varphi_2(t)-I_t^{\alpha,\beta,\eta,\mu}f(t))(I_t^{\alpha,\beta,\eta,\mu}f(t)-I_t^{\alpha,\beta,\eta,\mu}\varphi_1(t))\\
&\quad+\frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)}I_t^{\alpha,\beta,\eta,\mu}\varphi_1(t)f(t)-I_t^{\alpha,\beta,\eta,\mu}\varphi_1(t)I_t^{\alpha,\beta,\eta,\mu}f(t)\\
&\quad+\frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)}I_t^{\alpha,\beta,\eta,\mu}\varphi_2(t)f(t)-I_t^{\alpha,\beta,\eta,\mu}\varphi_2(t)I_t^{\alpha,\beta,\eta,\mu}f(t)\\
&\quad+I_t^{\alpha,\beta,\eta,\mu}\varphi_1(t)I_t^{\alpha,\beta,\eta,\mu}\varphi_2(t)-\frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)}I_t^{\alpha,\beta,\eta,\mu}\varphi_1(t)\varphi_2(t)\\
&=\mathcal{T}(f,\varphi_1,\varphi_2).\end{aligned}$$

A similar argument will give the following inequality:

$$\frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)}I_t^{\alpha,\beta,\eta,\mu}g^2(t)-\left(I_t^{\alpha,\beta,\eta,\mu}g(t)\right)^2\leq\mathcal{T}(g,\psi_1,\psi_2). \tag{21}$$

Finally, making use of the inequalities (19), (20) and (21), we are immediately led to the desired inequality (15). This completes the proof of Theorem. □

3. Consequent results and special cases

By virtue of the unified nature of the fractional hypergeometric operators (3), a large number of new and known integral inequalities involving Saigo, Erdélyi-Kober and Riemann-Liouville fractional integral operators are seen to follow as special cases of our main result. Indeed, by suitably specializing the values of parameters α, β, η and μ, the inequality (15) in Theorem would yield further Grüss type integral inequalities involving the above-mentioned integral operators. For example, if we set $\mu = 0$ in Theorem and use (4), the inequality (15) gives a known result involving Saigo's fractional integral operators, recently, introduced by Baleanu *et al.* [8]. For another example, if we put $\mu = 0$ and replace β by $-\alpha$ in Theorem, and make use of the relation (5), we find that the reduced final result is equal to the known result due to Tariboon *et al.* [47, p. 5, Theorem 9].

If we put $\varphi_1(t) = m$, $\varphi_2(t) = M$, $\psi_1(t) = p$ and $\psi_2(t) = P$, where $m, M, p, P \in \mathbb{R}$ and $t \in [0,\infty)$, we obtain a fractional integral inequality, which was obtained by Wang *et al.* [48] in a slightly different form, asserted by the following corollary.

Corollary 1. *Let f and g be two integrable functions on $[0, \infty)$ satisfying the following inequalities*

$$m \leq f(t) \leq M \quad \textit{and} \quad p \leq g(t) \leq P \quad (t \in [0,\infty)), \tag{22}$$

where m, M, p and P are real constants. Then the following inequality holds true: *For* $t > 0$, $\alpha > \max\{0, -\beta-\mu\}$, $\mu > -1$, $\beta < 1$ *and* $\beta - 1 < \eta < 0$,

$$\left|\frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)} I_t^{\alpha,\beta,\eta,\mu} f(t)g(t) - I_t^{\alpha,\beta,\eta,\mu} f(t) I_t^{\alpha,\beta,\eta,\mu} f(t)\right| \le \left(\frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)}\right)^2 (M-m)(P-p). \tag{23}$$

It is noted that setting $\mu = 0$ and replacing β by $-\alpha$ in (23), and making use of the relation (5) yields the known result due to Dahmani *et al.* [17].

Furthermore, if we take $\varphi_1(t) = t$, $\varphi_2(t) = t+1$, $\psi_1(t) = t-1$ and $\psi_2(t) = t$ in Theorem, and using formula (7), we obtain another fractional integral inequality asserted by the following corollary.

Corollary 2. *Assume that f and g be two integrable functions on* $[0, \infty)$ *such that*

$$t \le f(t) \le t+1 \quad \text{and} \quad t-1 \le g(t) \le t \quad (t \in [0,\infty)). \tag{24}$$

Then the following inequality holds true: *For* $t > 0$, $\alpha > \max\{0, -\beta-\mu\}$, $\mu > -1$, $\beta < 1$ *and* $\beta - 1 < \eta < 0$,

$$\left|\frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)} I_t^{\alpha,\beta,\eta,\mu} f(t)g(t) - I_t^{\alpha,\beta,\eta,\mu} f(t) I_t^{\alpha,\beta,\eta,\mu} g(t)\right| \le \sqrt{\mathcal{T}(f,t,t+1)\ \mathcal{T}(g,t-1,t)}, \tag{25}$$

where, for convenience,

$$\begin{aligned}
&\mathcal{T}(f,t,t+1)\\
&:= \left(\frac{\Gamma(2+\mu)\Gamma(2-\beta+\eta)t^{1-\beta-\mu}}{\Gamma(2-\beta)\Gamma(2+\mu+\alpha+\eta)} + \frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)} - I_t^{\alpha,\beta,\eta,\mu} f(t)\right)\\
&\quad \times \left(I_t^{\alpha,\beta,\eta,\mu} f(t) - \frac{\Gamma(2+\mu)\Gamma(2-\beta+\eta)t^{1-\beta-\mu}}{\Gamma(2-\beta)\Gamma(2+\mu+\alpha+\eta)}\right)\\
&\quad + \frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)} I_t^{\alpha,\beta,\eta,\mu} t f(t)\\
&\quad - \frac{\Gamma(2+\mu)\Gamma(2-\beta+\eta)t^{1-\beta-\mu}}{\Gamma(2-\beta)\Gamma(2+\mu+\alpha+\eta)} I_t^{\alpha,\beta,\eta,\mu} f(t)\\
&\quad + \frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)} I_t^{\alpha,\beta,\eta,\mu} (t+1) f(t)\\
&\quad - \left(\frac{\Gamma(2+\mu)\Gamma(2-\beta+\eta)t^{1-\beta-\mu}}{\Gamma(2-\beta)\Gamma(2+\mu+\alpha+\eta)} + \frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)}\right) I_t^{\alpha,\beta,\eta,\mu} f(t)\\
&\quad + \left(\frac{\Gamma(2+\mu)\Gamma(2-\beta+\eta)t^{1-\beta-\mu}}{\Gamma(2-\beta)\Gamma(2+\mu+\alpha+\eta)}\right)
\end{aligned}$$

$$\times\left(\frac{\Gamma(2+\mu)\Gamma(2-\beta+\eta)t^{1-\beta-\mu}}{\Gamma(2-\beta)\Gamma(2+\mu+\alpha+\eta)}+\frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)}\right)$$
$$-\frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)}$$
$$\times\left(\frac{\Gamma(3+\mu)\Gamma(3-\beta+\eta)t^{2-\beta-\mu}}{\Gamma(3-\beta)\Gamma(3+\mu+\alpha+\eta)}+\frac{\Gamma(2+\mu)\Gamma(2-\beta+\eta)t^{1-\beta-\mu}}{\Gamma(2-\beta)\Gamma(2+\mu+\alpha+\eta)}\right)$$

and

$$\begin{aligned}
&\mathcal{T}(g,t-1,t)\\
:=\ &\left(\frac{\Gamma(2+\mu)\Gamma(2-\beta+\eta)t^{1-\beta-\mu}}{\Gamma(2-\beta)\Gamma(2+\mu+\alpha+\eta)}-I_t^{\alpha,\beta,\eta,\mu}g(t)\right)\\
&\times\left(I_t^{\alpha,\beta,\eta,\mu}f(t)-\frac{\Gamma(2+\mu)\Gamma(2-\beta+\eta)t^{1-\beta-\mu}}{\Gamma(2-\beta)\Gamma(2+\mu+\alpha+\eta)}\frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)}\right)\\
&+\frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)}I_t^{\alpha,\beta,\eta,\mu}(t-1)g(t)\\
&-\left(\frac{\Gamma(2+\mu)\Gamma(2-\beta+\eta)t^{1-\beta-\mu}}{\Gamma(2-\beta)\Gamma(2+\mu+\alpha+\eta)}-\frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)}\right)I_t^{\alpha,\beta,\eta,\mu}g(t)\\
&+\frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)}I_t^{\alpha,\beta,\eta,\mu}tg(t)\\
&-\frac{\Gamma(2+\mu)\Gamma(2-\beta+\eta)t^{1-\beta-\mu}}{\Gamma(2-\beta)\Gamma(2+\mu+\alpha+\eta)}I_t^{\alpha,\beta,\eta,\mu}g(t)\\
&+\left(\frac{\Gamma(2+\mu)\Gamma(2-\beta+\eta)t^{1-\beta-\mu}}{\Gamma(2-\beta)\Gamma(2+\mu+\alpha+\eta)}+\frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)}\right)\\
&\times\left(\frac{\Gamma(2+\mu)\Gamma(2-\beta+\eta)t^{1-\beta-\mu}}{\Gamma(2-\beta)\Gamma(2+\mu+\alpha+\eta)}\right)-\frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)t^{-\beta-\mu}}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)}\\
&\times\left(\frac{\Gamma(3+\mu)\Gamma(3-\beta+\eta)t^{2-\beta-\mu}}{\Gamma(3-\beta)\Gamma(3+\mu+\alpha+\eta)}-\frac{\Gamma(2+\mu)\Gamma(2-\beta+\eta)t^{1-\beta-\mu}}{\Gamma(2-\beta)\Gamma(2+\mu+\alpha+\eta)}\right).
\end{aligned}$$

We conclude this paper by emphasizing, again, that our main result here, being of a very general nature, can be specialized to yield numerous interesting fractional integral inequalities including some known results.

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