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EXPLICIT FORMULA FOR COEFFICIENTS OF TODD SERIES OF LATTICE CONES

Hi-joon Chae, Byungheup Jun, and Jungyun Lee

Abstract. Todd series are associated to maximal non-degenerate lattice cones. The coefficients of Todd series of a particular class of lattice cones are closely related to generalized Dedekind sums of higher dimension. We generalize this construction and obtain an explicit formula for coefficients of the Todd series. It turns out that every maximal non-degenerate lattice cone, hence the associated Todd series can be obtained in this way.

1. Introduction

In [2, 3], we have defined the following *generalized Dedekind sums* of higher dimension and considered their properties including integrality, equidistribution and reciprocity: for $q \in \mathbb{Z}_{>0}$, $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ and $(i_1, \ldots, i_n) \in \mathbb{Z}^n_{\geq 0}$,

$$\sum_{(k_1,\ldots,k_n)} \widetilde{B}_{i_1}\left(\frac{k_1}{q}\right) \widetilde{B}_{i_2}\left(\frac{k_2}{q}\right) \cdots \widetilde{B}_{i_n}\left(\frac{k_n}{q}\right), \tag{1}$$

where the summation is taken over the set of n-tuples $(k_1, \ldots, k_n)$ of non-negative integers less that q such that $a_1k_1 + \cdots + a_nk_n \equiv 0 \mod q$. The $\widetilde{B}_k(x)$ denotes the k-th periodic Bernoulli function, which is equal to the k-th Bernoulli polynomial $B_k(x)$ on $[0, 1)$ (except $\widetilde{B}_1(0) = 0$ while $B_1(0) = -1/2$).

For the origin, development and applications of these Dedekind sums, we refer the introduction of [2, 3]. Let us just mention that the classical Dedekind sum, corresponding to $n = 2$, $(i_1, i_2) = (1, 1)$, appears in the modular transform of the logarithm of the Dedekind η-function [5], and the case $(i_1, \ldots, i_n) = (1, \ldots, 1)$ appears in the signature formula for some quotient manifolds [4, 6].

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We have called the sum obtained when periodic Bernoulli functions in (1) are replaced by Bernoulli polynomials, the *Todd coefficient* in [2, 3]. The multi-variable generating function of Todd coefficients for $(i_1, \ldots, i_n) \in \mathbb{Z}^n_{\geq 0}$ with $q, a_1, \ldots, a_n$ fixed is the Todd series of a lattice cone (hence the name).

A Todd series, whose precise definition is given in the next section, is defined for a cone generated by n linearly independent lattice vectors in an n-dimensional real vector space equipped with a lattice [1]. (Such a cone will be called a maximal non-degenerate lattice cone in this paper.) And the Dedekind sums (or Todd coefficients) for fixed $q, a_1, \ldots, a_n$ correspond to a particular lattice cone.

Our goal in this paper is to find a formula as explicit as (1) for coefficients of Todd series of arbitrary (maximal non-degenerate) lattice cone.

First, we generalize the construction in [3] of the lattice cone whose Todd series gives Dedekind sums of (1). For $q \in \mathbb{Z}_{>0}$ and $A = (a_{i,j}) \in M_{n\times m}(\mathbb{Z})$. we define an n-dimensional real vector space $V(q, A)$, a lattice $\Gamma(q, A)$ and a maximal non-degenerate lattice cone $C(q, A)$ in it. Then any maximal non-degenerate lattice cone is isomorphic to one constructed in this way (Theorem 3.5).

The coefficients of the Todd series of $C(q, A)$ are given in Theorem 4.1. The coefficient of the term of multidegree $(i_1, \ldots, i_n)$ is

$$(2) \quad (-1)^{|\mathbb{i}|}|N(q,A)|^{-1}\, \frac{q^{m-n+|\mathbb{i}|}}{i_1! \cdots i_n!} \sum_{(k_1,\ldots,k_n)} B_{i_1}\left(\frac{k_1}{q}\right) B_{i_2}\left(\frac{k_2}{q}\right) \cdots B_{i_n}\left(\frac{k_n}{q}\right),$$

where $N(q, A) = \{\mathbb{y} \in (\mathbb{Z}/q\mathbb{Z})^m \mid A\mathbb{y} \equiv 0 \mod q\}$ is the null space of $A \mod q$ and the sum is taken over the set of n-tuples $\mathbb{k} = (k_1, \ldots, k_n)$ of non-negative integers less than q such that $a_{1,j}k_1 + a_{2,j}k_2 + \cdots + a_{n,j}k_n \equiv 0 \mod q$ for $1 \leq j \leq m$ (i.e., the sum is over $N(q, A^T)$ where A^T is the transpose of A). Also we have put $|\mathbb{i}| = i_1 + \cdots + i_n$.

These results suggest the following definition (and the importance of the study of) for (more general form of) generalized Dedekind sums of higher dimension. We hope to extend the results of [2, 3] to these Dedekind sums.

Definition. Let $q \in \mathbb{Z}_{>0}$, $A = (a_{i,j}) \in M_{n\times m}(\mathbb{Z})$ and $\mathbb{i} = (i_1, \ldots, i_n) \in \mathbb{Z}^n_{\geq 0}$. The *generalized Dedekind sum* associated to (q, A), of dimension n and of degree $\mathbb{i}$ is

$$d_{\mathbb{i}}(q, A) := \sum_{(k_1,\ldots,k_n)} \widetilde{B}_{i_1}\left(\frac{k_1}{q}\right) \widetilde{B}_{i_2}\left(\frac{k_2}{q}\right) \cdots \widetilde{B}_{i_n}\left(\frac{k_n}{q}\right),$$

where the sum is taken over the same set as in (2).

A word on notations. Vector notations simplify formulas greatly: for example, for $\mathbb{k} = (k_1, \ldots, k_n)$, $\mathbb{i} = (i_1, \ldots, i_n)$, $\mathbb{x} = (x_1, \ldots, x_n)$, we put

$$\frac{q^{|\mathbb{i}|}}{\mathbb{i}!}\, B_{\mathbb{i}}(\mathbb{k})\, \mathbb{x}^{\mathbb{i}} = \frac{q^{i_1+\cdots+i_n}}{i_1! \cdots i_n!}\, B_{i_1}(k_1) \cdots B_{i_n}(k_n)\, x_1^{i_1} \cdots x_n^{i_n} \,.$$

2. Todd series of lattice cones

A *lattice* Γ in an n-dimensional real vector space V is a free abelian group of rank n which generates the vector space over $\mathbb{R}$. In other words, a lattice is the additive subgroup generated by a basis of the vector space. We may regard Γ as giving a "$\mathbb{Z}$-structure" on V. If we choose a basis for Γ, then the pair (V, Γ) is isomorphic to $(\mathbb{R}^n, \mathbb{Z}^n)$ in an obvious sense.

A *cone* $C = \mathrm{Cone}(v_1, \dots, v_n)$ is an ordered n-tuple of vectors $v_1, \dots, v_n \in V$. It is its underlying space $|C| := \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_n$ which is more commonly called a cone. It is called a *simplicial cone* if its underlying space (except the origin) is contained in a half space, i.e., if there exists $v^* \neq 0 \in V^*$ such that $v^*(v_i) > 0$ for $i = 1, \dots, n$. It is said to be *non-degenerate* if $v_1, \dots, v_n$ are linearly independent. Finally, it is called a *lattice cone* if $v_1, \dots, v_n \in \Gamma$.

Definition. Let $C = \mathrm{Cone}(v_1, \dots, v_n)$ be a non-degenerate (hence simplicial) lattice cone in (V, Γ) with $n = \dim V$. The *Todd series* of C is defined by

$$\mathrm{Todd}_C(\mathbb{x}) := \sum_{\gamma \in \Gamma/\Lambda_C} \prod_{k=1}^{n} \frac{x_k}{1 - e^{2\pi i \langle v_k^*, \gamma \rangle} e^{-x_k}} \,, \tag{3}$$

where $\Lambda_C := \mathbb{Z} v_1 + \cdots + \mathbb{Z} v_n$ and $\{v_1^*, \dots, v_n^*\}$ is the basis for V^* dual to $\{v_1, \dots, v_n\}$. Here and in the rest of the paper, $x_1, \dots, x_n$ are variables with $\mathbb{x} = (x_1, \dots, x_n)$.

It is most convenient to view these variables $x_1, \dots, x_n$ as coordinates on V with respect to the basis $\{v_1, \dots, v_n\}$. Then Todd_C is a meromorphic function on $V_{\mathbb{C}} := \mathbb{C} \otimes V$ which is analytic in a neighborhood of the origin. As such, it admits a power series expansion around the origin. We are most interested in the coefficients of this power series. The Todd series of a degenerate lattice cone is defined to be zero.

Expanding the denominators in (3) formally and summing over γ, we obtain

$$\mathrm{Todd}_C(\mathbb{x}) = |\Gamma/\Lambda_C| \; x_1 x_2 \cdots x_n \sum_{v^* \in |C^*| \cap \Gamma^*} e^{-\sum_{k=1}^{n} \langle v^*, v_k \rangle x_k} \,, \tag{4}$$

where $\Gamma^* = \mathrm{Hom}(\Gamma, \mathbb{Z})$ is the lattice in V^* dual to Γ and $|C^*| = \sum_{k=1}^{n} \mathbb{R}_{\geq 0} v_k^*$ is the underlying space of the dual cone $C^* = \mathrm{Cone}(v_1^*, v_2^*, \dots, v_n^*)$. This formal sum converges if all $x_k > 0$ (i.e., it converges inside $|C|$).

3. Construction of maximal non-degenerate lattice cones

We say a non-degenerate lattice cone $C = \mathrm{Cone}(u_1, \dots, u_n)$ in (V, Γ) is *maximal* if $n = \dim V$. So Todd series of the last section are associated to such cones. In this section, we give a construction which produces all maximal non-degenerate lattice cones up to isomorphism. Roughly speaking, we take a quotient of a standard cone in $(\mathbb{R}^m, \mathbb{Z}^m)$.

The image of a lattice in a quotient vector space is again a lattice if and only if the kernel is "defined over $\mathbb{Q}$". More precisely, we have the following.

Proposition 3.1. *Suppose (V, Γ) is a pair of a finite dimensional real vector space and a lattice in it. Let W be a subspace of V and $\pi : V \to V/W$ be the canonical map. Then the image $\pi(\Gamma)$ of Γ is a lattice in V/W if and only if* $\operatorname{rank}(\Gamma \cap W) = \dim W$.

Proof. Since $\pi(\Gamma)$ is a free abelian group which generates V/W over $\mathbb{R}$, it is enough to show that $\operatorname{rank} \pi(\Gamma) = \dim V/W$ if and only if $\operatorname{rank}(\Gamma \cap W) = \dim W$. Since $\pi(\Gamma)$ is free abelian, the short exact sequence of abelian groups

$$0 \to \Gamma \cap W \to \Gamma \to \pi(\Gamma) \to 0$$

splits. Thus we have $\Gamma \cong (\Gamma \cap W) \oplus \pi(\Gamma)$ and $\operatorname{rank} \pi(\Gamma) = \operatorname{rank} \Gamma - \operatorname{rank}(\Gamma \cap W)$. □

Remark 3.2. A subspace W of V satisfies $\operatorname{rank}(\Gamma \cap W) = \dim W$ if and only if it is generated (over $\mathbb{R}$) by a set of lattice vectors.

Let us say a non-degenerate lattice cone $C = \operatorname{Cone}(v_1, \ldots, v_n)$ in (V, Γ) is *admissible* if $v_1, \ldots, v_n \in \Gamma$ can be extended to a basis for Γ. An admissible lattice cone $C = \operatorname{Cone}(v_1, \ldots, v_n)$ in (V, Γ) is isomorphic to the cone $\operatorname{Cone}(e_1, \ldots, e_n)$ in $(\mathbb{R}^{n+m}, \mathbb{Z}^{n+m})$ in an obvious sense where $n+m = \dim V$ and $\{e_1, \ldots, e_{n+m}\}$ is the standard basis for $\mathbb{R}^{n+m}$. Let us call the last cone a *standard* admissible cone. It is easy to see that any lattice cone is the image of a standard admissible cone. More precisely, we have the following.

Proposition 3.3. *Let $C = \operatorname{Cone}(u_1, \ldots, u_n)$ be a lattice cone in (V, Γ). Then there exists a surjective map $\pi : (\mathbb{R}^{n+m}, \mathbb{Z}^{n+m}) \to (V, \Gamma)$ such that C is the image of $\operatorname{Cone}(e_1, \ldots, e_n)$. In other words, there exists a surjective linear map $\pi : \mathbb{R}^{n+m} \to V$ with $\Gamma = \pi(\mathbb{Z}^{n+m})$ such that $\pi(e_i) = u_i$ for $i = 1, \ldots, n$.*

Proof. Extend $\{u_1, \ldots, u_n\}$ to a set $\{u_1, \ldots, u_{n+m}\}$ of generators for Γ. Let $\pi : \mathbb{R}^{n+m} \to V$ be the linear map given by $e_i \mapsto u_i$ for $i = 1, \ldots, n+m$. □

Remark 3.4. We can choose m such that $m \leq \operatorname{rank} \Gamma = \dim V$.

Suppose $C = \operatorname{Cone}(u_1, \ldots, u_n)$ is a maximal non-degenerate lattice cone in (V, Γ). Extend $\{u_1, \ldots, u_n\}$ to a set $\{u_1, \ldots, u_{n+m}\}$ of generators for Γ. We claim that there exist $q \in \mathbb{Z}_{>0}$ and $A = (a_{i,j}) \in M_{n \times m}(\mathbb{Z})$ such that $q u_{n+j} = \sum_{i=1}^{n} a_{i,j} u_i$ for $j = 1, \ldots, m$. Really, since $\{u_1, \ldots, u_n\}$ is a basis for the vector space V, $u_{n+1}, \ldots, u_{n+m}$ are linear combinations of $u_1, \ldots, u_n$. And since they are lattice vectors, these coefficients belong to $\mathbb{Q}$. Conversely, we can reconstruct (V, Γ) and the cone C from these q and $A = (a_{i,j})$ as follows.

Let $q \in \mathbb{Z}_{>0}$ and $A = (a_{i,j}) \in M_{n \times m}(\mathbb{Z})$. Let $v_1, \ldots, v_{n+m}$ be column vectors of the following matrix, which are lattice vectors in $(\mathbb{R}^{n+m}, \mathbb{Z}^{n+m})$:

$$\begin{pmatrix} \mathbf{1}_n & A \\ \mathbf{0}_{m \times n} & q\mathbf{1}_m \end{pmatrix}, \tag{5}$$

where $\mathbf{1}$ and $\mathbf{0}$ denote the identity and the zero matrices of suitable size, respectively. Let W be the subspace of $\mathbb{R}^{n+m}$ generated by $v_{n+1}, \ldots, v_{n+m}$.

Then $\Gamma(q, A) = \pi(\mathbb{Z}^{n+m})$ is a lattice in $V(q, A) = \mathbb{R}^{n+m}/W$ by Remark 3.2 where $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}/W$ is the canonical map. And $C(q, A) = \mathrm{Cone}(\pi(v_1), \ldots, \pi(v_n))$ is a maximal non-degenerate lattice cone in $(V(q, A), \Gamma(q, A))$. It is clear that if (q, A) is obtained from a given maximal non-degenerate lattice cone C in (V, Γ) as in the last paragraph, then the linear map $\mathbb{R}^{n+m} \to V$ given by $e_i \mapsto u_i$ $(i = 1, \ldots, n + m)$ induces an isomorphism of triples $(V(q, A), \Gamma(q, A), C(q, A))$ and (V, Γ, C). Summing up, we have the following.

Theorem 3.5. *Let $q \in \mathbb{Z}_{>0}$ and $A = (a_{i,j}) \in M_{n\times m}(\mathbb{Z})$. The construction above produces a maximal non-degenerate lattice cone $C(q, A)$ in a vector space $(V(q, A), \Gamma(q, A))$ equipped with a lattice. Conversely, for any maximal non-degenerate lattice cone C in a vector space (V, Γ) equipped with a lattice, there exists (q, A) such that the triple (V, Γ, C) is isomorphic to $(V(q, A)$, $\Gamma(q, A)$, $C(q, A))$.*

4. Explicit formula for coefficients of Todd series

We give an explicit formula for coefficients of the Todd series of arbitrary maximal non-degenerate lattice cone. By Theorem 3.5, it is enough to consider $C(q, A)$ of the last section.

Theorem 4.1. *Let $q \in \mathbb{Z}_{>0}$ and $A = (a_{i,j}) \in M_{n\times m}(\mathbb{Z})$. Let $C = C(q, A)$ be the maximal non-degenerate lattice cone in $(V, \Gamma) = (V(q, A), \Gamma(q, A))$ constructed in the last section. Then for $\mathbb{i} = (i_1, \ldots, i_n) \in \mathbb{Z}^n_{\geq 0}$, the coefficient of $\mathbb{x}^{\mathbb{i}}$ in the Todd series $\mathrm{Todd}_C(\mathbb{x})$ of C is*

$$(-1)^{|\mathbb{i}|}|N(q, A)|^{-1}\,\frac{q^{m-n+|\mathbb{i}|}}{\mathbb{i}!}\sum_{\mathbb{k}} B_{\mathbb{i}}(q^{-1}\mathbb{k}),$$

where $N(q, A) = \{\mathbb{y} \in (\mathbb{Z}/q\mathbb{Z})^m \mid A\mathbb{y} \equiv 0 \mod q\}$ is the null space of $A \mod q$ and the sum is over the set of n-tuples $\mathbb{k} = (k_1, \ldots, k_n)$ of non-negative integers less than q such that $a_{1,j}k_1 + a_{2,j}k_2 + \cdots + a_{n,j}k_n \equiv 0 \mod q$ for $1 \leq j \leq m$ (i.e., the sum is over $N(q, A^T)$ where A^T is the transpose of A).

Proof. When $m = 1$, this is proved in [3]. We keep the notations of the last section. In particular, $v_1, \ldots, v_{n+m}$ are column vectors of (5), W is the span of $v_{n+1}, \ldots, v_{n+m}$ and $C = \mathrm{Cone}(u_1, \ldots, u_n)$ where u_i is the image of v_i $(i = 1, \ldots, n)$. Let $V_0 = \mathbb{R}^{n+m}$ and $\Gamma_0 = \mathbb{Z}^{n+m}$. We can identify the dual space V^* with the subspace $\{v^* \in V_0^* \mid v^* \equiv 0 \text{ on } W\}$ of V_0^* and the dual lattice $\Gamma^* = \mathrm{Hom}(\Gamma, \mathbb{Z})$ with $V^* \cap \Gamma_0^*$. Let $\{v_1^*, \ldots, v_{n+m}^*\}$ be the basis for V_0^* dual to $\{v_1, \ldots, v_{n+m}\}$ and similarly, let $\{u_1^*, \ldots, u_n^*\}$ be the dual basis for V^*. Under our identification $V^* \subset V_0^*$, we have $u_i^* = v_i^*$ for $i = 1, \ldots, n$. Consider the expansion of $\mathrm{Todd}_C(\mathbb{x})$ given in (4) :

$$\mathrm{Todd}_C(\mathbb{x}) = |\Gamma/\Lambda_C|\; x_1x_2\cdots x_n \sum_{v^*\in|C^*|\cap\Gamma^*} e^{-\sum_{j=1}^{n}\langle v^*,u_j\rangle x_j}\,. \tag{6}$$

The lattice vectors in the underlying space $|C^*|$ of $C^* = \mathrm{Cone}(u_1^*, \ldots, u_n^*)$ can be described as follows. First, note $v_1^*, \ldots, v_n^*$ are row vectors of the following matrix, the inverse of (5):

$$\begin{pmatrix} I_n & -q^{-1}A \\ 0_{m\times n} & q^{-1}I_m \end{pmatrix}.$$

Hence $qu_1^* = qv_1^*, \ldots, qu_n^* = qv_n^*$ are lattice vectors and we have

$$|C^*| \cap \Gamma^* = \{u^* + i_1 qu_1^* + i_2 qu_2^* + \cdots + i_n qu_n^* \,|\, u^* \in \mathcal{P} \cap \Gamma^*, (i_1, \ldots, i_n) \in \mathbb{Z}^n_{\geq 0}\},$$

where $\mathcal{P} = \{t_1 qu_1^* + t_2 qu_2^* + \cdots + t_n qu_n^* \,|\, 0 \leq t_1, \ldots, t_n < 1\}$ is the fundamental parallelepiped for the lattice generated by $\{qu_1^*, \ldots, qu_n^*\}$ in V^*. Then it is easy to see that $u^* = t_1 qu_1^* + t_2 qu_2^* + \cdots + t_n qu_n^* \in \mathcal{P}$ is a lattice vector if and only if $(t_1, \ldots, t_n) = (k_1/q, \ldots, k_n/q)$ with $(k_1, \ldots, k_n)$ as in the statement of the theorem. Now fix such $\mathbb{k}$, i.e., a lattice vector $u^* = k_1 u_1^* + k_2 u_2^* + \cdots + k_n u_n^* \in \mathcal{P} \cap \Gamma^*$ and consider the partial sum in (6) over $v^* = u^* + i_1 qu_1^* + i_2 qu_2^* + \cdots + i_n qu_n^*$ with $(i_1, \ldots, i_n) \in \mathbb{Z}^n_{\geq 0}$:

$$\begin{aligned}
&x_1 x_2 \cdots x_n \sum_{(i_1,\ldots,i_n)\in\mathbb{Z}^n_{\geq 0}} e^{-\sum_{j=1}^n \langle u^* + i_1 qu_1^* + i_2 qu_2^* + \cdots + i_n qu_n^*, u_j\rangle x_j} \\
&= \prod_{j=1}^n x_j \sum_{i_j=0}^{\infty} e^{-(k_j + q i_j)x_j} \\
&= \prod_{j=1}^n \frac{(-1)^{i_j}}{q} \sum_{i_j=o}^{\infty} \frac{1}{i_j!} B_{i_j}\left(\frac{k_j}{q}\right)(q x_j)^{i_j} \\
&= \sum_{\mathbb{i}\in\mathbb{Z}^n_{\geq 0}} \frac{(-1)^{|\mathbb{i}|} q^{-n+|\mathbb{i}|}}{\mathbb{i}!} B_{\mathbb{i}}(q^{-1}\mathbb{k})\mathbb{x}^{\mathbb{i}}.
\end{aligned}$$

It remains to prove $|\Gamma/\Lambda_C| = q^m |N(q, A)|^{-1}$. Consider the surjection $\Gamma_0/\Lambda_0 \to \Gamma/\Lambda_C$ where Λ_0 is the subgroup generated by $v_1, \ldots, v_{n+m}$. The kernel of this map is $\sum_{j=1}^n \mathbb{Z}e_j + W \cap \Gamma_0 \mod \Lambda_0$. It is isomorphic to $\pi_m(W \cap \Gamma_0)/\pi_m(\Lambda_0) = \pi_m(W \cap \Gamma_0)/q\mathbb{Z}^m$ where π_m denotes the projection onto the last m coordinates. For $\mathbb{y} \in \mathbb{R}^m$, we have $y_1 v_{n+1} + \cdots + y_m v_{n+m}$ is a lattice vector if and only if both $q\mathbb{y}$ and $A\mathbb{y}$ are lattice vectors, i.e., if and only if $q\mathbb{y} \in \mathbb{Z}^m$ such that $Aq\mathbb{y} \equiv 0 \mod q$. Since $\pi_m(y_1 v_{n+1} + \cdots + y_m v_{n+m}) = q\mathbb{y}$, the kernel of $\Gamma_0/\Lambda_0 \to \Gamma/\Lambda_C$ is isomorphic to $N(q, A)$. This completes the proof since $|\Gamma_0/\Lambda_0| = q^m$. □

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Hi-joon Chae
Department of Mathematics Education
Hongik University
Seoul 121-791, Korea
E-mail address: hchae@hongik.ac.kr

Byungheup Jun
Department of Mathematical Sciences
UNIST
Ulsan 689-798, Korea
E-mail address: bhjun@unist.ac.kr

Jungyun Lee
Department of Mathematics
Ewha Womans University
Seoul 120-750, Korea
E-mail address: lee9311@kias.re.kr