한국경영과학회지
제40권 제1호
2015년 2월
http://dx.doi.org/10.7737/JKORMS.2015.40.1.001

GI/G/1 대기행렬 대기시간 분포의 새로운 유도방법

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A Heuristic Derivation of the Waiting Time Distribution of a GI/G/1 Queue

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■ Abstract ■

This paper presents a heuristic approach to derive the Laplace-Stieltjes transform (LST) and the probability generating function (PGF) of the waiting time distributions of a continuous- and a discrete-time GI/G/1 queue, respectively. This is a new idea to derive the well-known results, the waiting time distribution of GI/G/1 queue, in a different way.

Keywords : GI/G/1 Queue, Discrete-Time Queue, Waiting Time Distribution, Pollaczek-Khinchin Formula

논문접수일 : 2014년 09월 18일 논문게재확정일 : 2014년 11월 03일
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1. Introduction

For the waiting time distribution of a GI/G/1 queueing system, Lindley's integral equation [2] appears to be one of the most widely known approaches. It starts with the following elementary equation,

$$W_{n+1} = \begin{cases} W_n + v_n - u_n, & \text{if } W_n + v_n - u_n \geq 0, \\ 0, & \text{if } W_n + v_n - u_n < 0, \end{cases} \quad (1)$$

where W_n, u_n and v_n denote the waiting time of the nth arriving customer C_n in the queue (excluding service time), the inter-arrival time between C_n and C_{n+1}, and the service time of C_n, respectively. The LST of the waiting time distribution of a continuous GI/G/1 queue according to Lindley's equation is given as [3, 4],

$$W_Q^*(s) = \frac{a_0\{1 - I^*(-s)\}}{1 - X^*(s)}. \quad (2)$$

It is also called the generalized Pollaczek-Khinchin formula. $W_Q^*(s)$, $I^*(s)$ and $X^*(s)$ in (2) are defined as follows :

$$W_Q^*(s) = \lim_{n\to\infty} E[e^{-sW_n}] = \lim_{n\to\infty} E[e^{-sW_{n+1}}],$$
$$I^*(s) = \lim_{n\to\infty} E[e^{-sI_n}],$$
$$X^*(s) = \lim_{n\to\infty} E[e^{-sX_n}],$$

where $X_n \equiv v_n - u_n$ and $I_n = -\min(0,\ W_n + v_n - u_n)$ denotes the length of the idle period preceding the $(n+1)$th arriving customer. $E[e^{-sW_n}]$, $E[e^{-sI_n}]$ and $E[e^{-sX_n}]$ are respectively the LSTs of W_n, I_n, and X_n. The relationship between W_n, W_{n+1}, X_n and I_n is depicted in [Figure 1]. Finally, a_0 is the probability that an arriving customer finds no customer in the system, in other words, that the system is idle.

2. Continuous-Time GI/G/1 Queue

At this point, a heuristic approach to derive GI/G/1 $W_Q^*(s)$ is presented. The derivation starts with equation (1) again. Note that $X_n \equiv v_n - u_n$. The two-sided LSTs of W_{n+1} and $(W_n + X_n)$ can be written as follows :

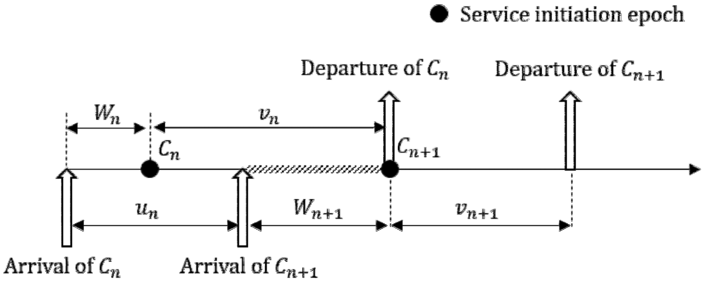
$$\begin{aligned} E\left[e^{-sW_{n+1}}\right] = {} & \Pr\{W_n + X_n < 0\} \times e^{-s\cdot 0} \\ & + \Pr\{W_n + X_n \geq 0\} \\ & \times E\left[e^{-s(W_n+X_n)} \middle| W_n + X_n \geq 0\right], \end{aligned} \quad (3)$$

$$\begin{aligned} E\left[e^{-s(W_n+X_n)}\right] = {} & \Pr\{W_n + X_n < 0\} \\ & \times E\left[e^{-s(W_n+X_n)} \middle| W_n + X_n < 0\right] \\ & + \Pr\{W_n + X_n \geq 0\} \\ & \times E\left[e^{-s(W_n+X_n)} \middle| W_n + X_n \geq 0\right]. \end{aligned} \quad (4)$$

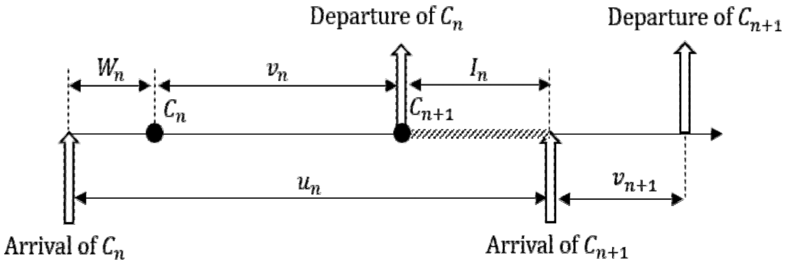
Subtracting (4) from (3) gives

$$\begin{aligned} & E\left[e^{-sW_{n+1}}\right] - E\left[e^{-s(W_n+X_n)}\right] \\ & = \Pr\{W_n + X_n < 0\} \\ & \times \left\{1 - E\left[e^{-s(W_n+X_n)} \middle| W_n + X_n < 0\right]\right\}. \end{aligned} \quad (5)$$

By assuming a limit of $n \to \infty$ on both sides of (5), it is evident that the left-hand side of (5) becomes $\left(W_Q^*(s) - W_Q^*(s)X^*(s)\right)$. Note that W_n and X_n are independent since the inter-arrival time between two consecutive customers (v_n), the service and the waiting time of nth customer (u_n and W_n) are independent each other. In addition, $\Pr\{W_n + X_n < 0\}$ is equal to a_0 because $W_n + X_n < 0$ denotes that an arriving customer finds the system idle. As depicted in [Figure 1], $W_n + X_n = -I_n$ when $W_n + X_n < 0$. Thus, $\lim_{n\to\infty} E\left[e^{-s(W_n+X_n)} \middle| W_n + X_n\right.$



(a) Case 1 : An arriving customer finds the system busy $(W_n + X_n \geqq 0)$



(b) Case 2 : An arriving customer finds the system idle $(W_n + X_n < 0)$

[Figure 1] The Relationship between W_n, W_{n+1}, X_n and I_n

$<0] = I^*(-s)$. By substituting these items, LST equal to (2) is obtained :

$$W_Q^*(s) = \frac{a_0\{1 - I^*(-s)\}}{1 - X^*(s)}. \qquad (6)$$

3. Discrete-Time GI/G/1 Queue

In this section, the PGF of the waiting time distribution of a discrete-time GI/G/1 queue is derived. The PGF has already been summarized by Hunter [1]. The derivation in this section is done in a less rigorous manner, and it is similar to a continuous case.

First, the equation (1) is revisited. The same notation used in the continuous-time case is used here as well, except that random variables are discrete and LSTs are replaced by PGFs. The discrete versions of (3) and (4) can be represented as follows :

$$E\left[z^{W_{n+1}}\right] = \Pr\{W_n + X_n < 0\} \times z^0 + \Pr\{W_n + X_n \geqq 0\} \times E\left[z^{W_n + X_n} \middle| W_n + X_n \geqq 0\right], \qquad (7)$$

$$E\left[z^{W_n + X_n}\right] = \Pr\{W_n + X_n < 0\} \times E\left[z^{W_n + X_n} \middle| W_n + X_n < 0\right] + \Pr\{W_n + X_n \geqq 0\} \times E\left[z^{W_n + X_n} \middle| W_n + X_n \geqq 0\right]. \qquad (8)$$

$E[z^{W_n}] (= W_Q^*(z))$ and $E[z^{X_n}] (= X^*(z))$ are defined as the PGFs of W_n and X_n, respectively. Subtracting (8) from (7) gives

$$E\left[z^{W_{n+1}}\right] - E\left[z^{W_n + X_n}\right] = \Pr\{W_n + X_n < 0\} \times \left\{1 - E\left[z^{W_n + X_n} \middle| W_n + X_n < 0\right]\right\}. \qquad (9)$$

Assuming that the limit is $n \rightarrow \infty$ on the left side of (9) allows the determination of the PGFs according to their definitions.

$$\lim_{n \rightarrow \infty} \{E[z^{W_{n+1}}] - E[z^{W_n + X_n}]\} = W_Q^*(z) - W_Q^*(z) X^*(z).$$

Let us define the discrete random variable I_n as the length of idle period preceding the $(n+1)$th arriving customer and the PGF $I^*(z) = \lim_{n \rightarrow \infty} E[z^{I_n}]$. Here, the meaning of I_n is identical to that of a continuous case, and it is also expressed as $I_n = -\min(0, W_n + v_n - u_n)$. Then, the right-hand side of (9) can be written as follows :

$$\Pr\{W_n + X_n < 0\}\{1 - E[z^{W_n + X_n} | W_n + X_n < 0]\}$$
$$= a_0 \{1 - I^*(-z)\}.$$

Its derivation is identical to that of a continuous case (refer to [Figure 1]). Finally, the PGF of the waiting time distribution of a discrete-time GI/G/1 queue is given by

$$W_Q^*(z) = \frac{a_0 \{1 - I^*(-z)\}}{1 - X^*(z)}. \quad (10)$$

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