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TEMPORAL REGULARITY OF THE EULER EQUATIONS

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Abstract. This paper investigates temporal regularity of solutions for the incompressible Euler equations in a critical Besov space $\mathbf{B}^{\frac{d}{p}+1}_{p,1}(\mathbb{R}^d)$ for $1 \le p \le d$.

1. Main theorem

We are interested in the non-stationary Euler equations of an ideal incompressible fluid

$$\begin{aligned} \frac{\partial}{\partial t}\, v + (v, \nabla) v &= -\nabla p, \\ \operatorname{div}\, v &= 0. \end{aligned} \tag{1.1}$$

Here $v(x,t) = (v^1, v^2, \cdots, v^d)$ is the Eulerian velocity of a fluid flow and $(v, \nabla) v^k = \sum_{i=1}^{d} v^i\, \partial_i\, v^k$, $k = 1, 2, \cdots, d$ with $\partial_i \equiv \frac{\partial}{\partial x_i}$.

The best local existence and uniqueness results known for the Euler equations (1.1) in Besov spaces are a series of theorems in the space $\mathbf{B}^{\frac{d}{p}+1}_{p,1}(\mathbb{R}^d)$ with $1 \le p \le \infty$ (see the introductions in [2, 5, 6] for details and the references therein). J. Bourgain and D. Li have very recently posted in [1] strong local ill-posedness results in the Sobolev spaces $\mathbf{W}^{\frac{d}{p}+1,p}(\mathbb{R}^d)$ for $1 < p < \infty$ and in the Besov spaces $\mathbf{B}^{\frac{d}{p}+1}_{p,q}(\mathbb{R}^d)$ with $1 < p < \infty$, $1 < q \le \infty$ and $d = 2$ or 3.

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While a lot of observations have been made on the spatial regularity, this paper will cover the temporal regularity of the Euler equations. More precisely, all the results of the existence theory for the Euler equations insist only on the spatial continuity of the solutions, whereas this paper investigates the temporal regularity of the solutions. To do this, we introduce the trajectory flows $X(x,t)$ along v satisfying a system of ordinary differential equations

$$\begin{cases} \dfrac{\partial}{\partial t}X(x,t) &= v(X(x,t),t), \\ \quad X(x,0) &= x. \end{cases} \tag{1.2}$$

Here we use an abbreviation $\tilde{v}(x,t) = v(X(x,t),t)$. Then our main result can be described as follows.

THEOREM 1.1. *Let $1 \leq p \leq d$, and v be a solution for the Euler equations (1.1) staying inside of $L^\infty([0,T];\mathbf{B}_{p,1}^{\frac{d}{p}+1})$. Then v is continuous with respect to time on $[0,T]$ with values in $\mathbf{B}_{p,1}^{\frac{d}{p}+1}(\mathbb{R}^d)$ and $\tilde{v}$ is continuously differentiable on $[0,T]$ with values in $\mathbf{B}_{p,1}^{\frac{d}{p}+1}(\mathbb{R}^d)$.*

NOTATION. Throughout this paper, the notation $X \lesssim Y$ means that $X \leq CY$, where C is a fixed but unspecified constant. Unless explicitly stated otherwise, C may depend on the dimension d and various other parameters (such as exponents), but not on the functions or variables $(u, v, f, g, x_i, \cdots)$ involved.

2. Preliminary estimates

We begin with some notations. Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz class of rapidly decreasing functions. Consider a nonnegative radial function $\chi \in \mathcal{S}(\mathbb{R}^d)$ satisfying $\operatorname{supp}\chi \subset \{\xi \in \mathbb{R}^d : |\xi| \leq \frac{5}{6}\}$, and $\chi = 1$ for $|\xi| \leq \frac{3}{5}$. Set $h_j(\xi) \equiv \chi(2^{-j-1}\xi) - \chi(2^{-j}\xi)$, and it can be easily seen that

$$\chi(\xi) + \sum_{j=0}^{\infty} h_j(\xi) = 1 \quad \text{for } \xi \in \mathbb{R}^d.$$

Let φ_j and Φ be functions defined by $\varphi_j \equiv \mathcal{F}^{-1}(h_j)$, $j \geq 0$ and $\Phi \equiv \mathcal{F}^{-1}(\chi)$, where $\mathcal{F}$ represents the Fourier transform on $\mathbb{R}^d$. Note that φ_j is a mollifier of φ_0, that is, $\varphi_j(x) \equiv 2^{jd}\varphi_0(2^j x)$ (or $\hat{\varphi}_j(\xi) = \hat{\varphi}(2^{-j}\xi)$).

One can readily check that

$$\Phi(x) + \sum_{j=0}^{k-1} \varphi_j(x) = 2^{kd}\Phi(2^k x) \text{ for } k \geq 1.$$

For $f \in \mathcal{S}'(\mathbb{R}^d)$ and $j \in \mathbb{Z}$, denote $\Delta_j f \equiv h_j(D)f = \varphi_j * f$ if $j \geq 0$, $\Delta_{-1} f \equiv \Phi * f$ and $\Delta_j f = 0$ if $j \leq -2$. The partial sums are also defined as $S_k f \equiv \sum_{j=-\infty}^{k} \Delta_j f$ for $k \in \mathbb{Z}$.

Assume that $s \in \mathbb{R}$, and $1 \leq p, q \leq \infty$. Then the Besov spaces $B^s_{p,q}(\mathbb{R}^d)$ are defined by

$$f \in B^s_{p,q}(\mathbb{R}^d) \Leftrightarrow \{\|2^{js}\Delta_j f\|_{L^p}\}_{j\in\mathbb{Z}} \in l^q$$

and homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^d)$ are

$$f \in \dot{B}^s_{p,q}(\mathbb{R}^d) \Leftrightarrow \{\|2^{js}\varphi_j * f\|_{L^p}\}_{j\in\mathbb{Z}} \in l^q.$$

The corresponding spaces of vector-valued functions are denoted by the bold faced symbols. For example, the product space $B^s_{p,q}(\mathbb{R}^d)^d$ is denoted by $\mathbf{B}^s_{p,q}(\mathbb{R}^d) \equiv B^s_{p,q}(\mathbb{R}^d)^d$.

We summarize some of the estimates which will be used later. We first recall the Bony's para-product formula which decomposes the product fg of two functions f and g into three parts:

$$fg = T_f g + T_g f + R(f, g),$$

where $T_f g$ represents Bony's *para-product* of f and g defined by $T_f g \equiv \sum_j S_{j-2} f \ \Delta_j g$ and $R(f, g)$ denotes the *remainder* of the para-product $R(f, g) \equiv \sum_{|i-j|\leq 1} \Delta_i f \Delta_j g$. The estimates of para-product parts in $B^s_{p,1}(\mathbb{R}^d)$ are provided as follows.

Remark 2.1. Let $s \in \mathbb{R}$ and $1 \leq p \leq \infty$.

1. **(Para-product estimate)** For any $f, g \in B^s_{p,1}(\mathbb{R}^d)$, we have

$$\|T_f g\|_{B^s_{p,1}} \lesssim \|f\|_{L^\infty} \|g\|_{B^s_{p,1}},$$

and we also have for each $i = 1, 2, \cdots, d$,

$$\|T_{\partial_i f} g\|_{B^s_{p,1}} \lesssim \|f\|_{L^\infty} \|\nabla g\|_{\mathbf{B}^s_{p,1}}.$$

2. **(Product formula)** For $s > 0$ and any $f, g \in B^s_{p,1}(\mathbb{R}^d)$, we have

$$\|fg\|_{B^s_{p,1}} \lesssim \|f\|_{L^\infty} \|g\|_{B^s_{p,1}} + \|f\|_{B^s_{p,1}} \|g\|_{L^\infty}$$

and

$$\|f \cdot \nabla g\|_{\mathbf{B}^s_{p,1}} \lesssim \|f\|_{L^\infty} \|\nabla g\|_{\mathbf{B}^s_{p,1}} + \|\nabla f\|_{\mathbf{B}^s_{p,1}} \|g\|_{L^\infty}.$$

3. **(Commutator estimate)** For any differentiable function f and any function g, we have the following commutator estimate

$$\| [f, \Delta_j]\, \partial_i g\|_{L^p} \lesssim \|\nabla f\|_{\mathbf{L}^\infty} \|g\|_{L^p}, \quad i = 1, 2, \cdots, d,$$

where the commutator $[f, \Delta_j]\, h$ is defined as $f\Delta_j h - \Delta_j(fh)$.

4. For any vector field $u = (u_1, u_2, \cdots, u_d)$ and a function g, we have

$$\sum_{j=-\infty}^{\infty} 2^{js} \left\|(S_j u, \nabla)\Delta_j g - \Delta_j (u, \nabla) g\right\|_{L^p}$$
$$\lesssim \|u\|_{\mathbf{B}^s_{p,1}} \|\nabla g\|_{\mathbf{L}^\infty} + \|g\|_{B^s_{p,1}} \|\nabla u\|_{\mathbf{L}^\infty}. \tag{2.1}$$

We also have the estimate

$$\sum_{j=-\infty}^{\infty} 2^{js} \left\|(S_{j-2} u, \nabla)\Delta_j g - \Delta_j (u, \nabla) g\right\|_{L^p}$$
$$\lesssim \|\nabla u\|_{\mathbf{B}^s_{p,1}} \|g\|_{L^\infty} + \|g\|_{B^s_{p,1}} \|\nabla u\|_{\mathbf{L}^\infty}. \tag{2.2}$$

5. **(Pressure estimate)** For $s > 0$ and any pair of divergence free vector fields u and v, we have

$$\|\pi(u, v)\|_{\mathbf{B}^{s+1}_{p,1}} \lesssim \|\nabla u\|_{\mathbf{L}^\infty} \|v\|_{\mathbf{B}^{s+1}_{p,1}} + \|u\|_{\mathbf{B}^{s+1}_{p,1}} \|\nabla v\|_{\mathbf{L}^\infty}. \tag{2.3}$$

We also have

$$\|\pi(u, v)\|_{\mathbf{B}^s_{p,1}} \lesssim \|\nabla u\|_{\mathbf{L}^\infty} \|v\|_{\mathbf{B}^s_{p,1}} + \|u\|_{\mathbf{B}^{s+1}_{p,1}} \|v\|_{\mathbf{L}^\infty}, \tag{2.4}$$

and

$$\|\pi(u, v)\|_{\mathbf{B}^s_{p,1}} \lesssim \|\nabla v\|_{\mathbf{L}^\infty} \|u\|_{\mathbf{B}^s_{p,1}} + \|v\|_{\mathbf{B}^{s+1}_{p,1}} \|u\|_{\mathbf{L}^\infty}, \tag{2.5}$$

where we set

$$\begin{aligned} \pi(u, v) &\equiv \sum_{i,j=1}^{d} \nabla \Delta^{-1} \partial_i u^j \partial_j v^i \\ &= \nabla \Delta^{-1} \mathrm{div}\ ((u, \nabla) v). \end{aligned}$$

6. **(Composition estimate)** For $0 \le s < 1$, $F \in B^s_{p,1}$, and bi-Lipschitz volume-preserving map $X : \mathbb{R}^d \to \mathbb{R}^d$, we have the following estimate:

$$\|F \circ X\|_{B^s_{p,1}} \lesssim \big(1 + \log(\|\nabla_x X\|_{L^\infty} \|\nabla_x X^{-1}\|_{L^\infty})\big) \|F\|_{B^s_{p,1}}.$$

The proofs of this properties can be found in [6]. In it, the same estimates are proved for the special case $p = 1$, however, all the estimates in [6] are valid for arbitrary p with $1 \le p \le \infty$. The original version of the Composition estimate was proved by M. Vishik in the space $B^0_{\infty,1}(\mathbb{R}^d)$ in [9], and D. Chae later generalized it to the Besov spaces $B^0_{p,q}(\mathbb{R}^d)$ and the Triebel-Lizorkin spaces $F^0_{p,q}(\mathbb{R}^d)$. The version in $B^s_{p,1}(\mathbb{R}^d)$ can be considered as a slight generalization of those.

3. The proof

We choose a solution v for the Euler equations (1.1) staying inside of $C([0,T];\mathbf{B}_{p,1}^{\frac{d}{p}+1})$, and we set $w_\ell \equiv S_\ell v$ and $\tilde{w}_\ell \equiv S_\ell \tilde{v}$ for $\ell \in \mathbb{N}$. We will demonstrate that the two sequences $\{w_\ell\}_{\ell\in\mathbb{N}}$ and $\{\dot{\tilde{w}}_\ell\}_{\ell\in\mathbb{N}}$ converge to v and $\dot{\tilde{v}}$ in $L^\infty([0,T];\mathbf{B}_{p,1}^{\frac{d}{p}+1})$, respectively and also prove that each w_ℓ and each $\dot{\tilde{w}}_\ell$ are continuous with respect to time on $[0,T]$ with values in $\mathbf{B}_{p,1}^{\frac{d}{p}+1}(\mathbb{R}^d)$. Then the argument will produce the desired result.

Step 1. Take the Δ_j operator and add the term $(S_j v, \nabla)\Delta_j v$ on both sides of (1.1) and we obtain that

$$\frac{\partial}{\partial t}\,\Delta_j v + (S_j v,\nabla)\Delta_j v = (S_j v,\nabla)\Delta_j v - \Delta_j (v,\nabla) v - \Delta_j \nabla p$$

for $j \in \mathbb{N}$. The interchangeability of the two operators $\frac{\partial}{\partial t}$ and Δ_j in the left hand side follows from the fact that $\frac{\partial}{\partial t} v \in \mathbf{L}^\infty([0,T]\times\mathbb{R}^d)$.

Consider the *trajectory flow* $\{X_j(x,t)\}$ *along* $S_j v$ defined by the solutions of the ordinary differential equations

$$\begin{cases} \frac{\partial}{\partial t} X_j(x,t) &= (S_j v)\,(X_j(x,t),t), \\ X_j(x,0) &= x \end{cases}$$

(observe $\operatorname{div} S_j v = 0$ implies that $x \mapsto X_j(x,t)$ is a volume preserving mapping). Then since $t \mapsto \|\Delta_j v(t)\|_{\mathbf{L}^p}$ is absolutely continuous on $[0,T]$, we get

$$\|\Delta_j v(t)\|_{\mathbf{L}^p} \le \|\Delta_j v_0\|_{\mathbf{L}^p} + \int_0^t \|\Delta_j \nabla p\|_{\mathbf{L}^p}\, d\tau + \int_0^t \|(S_j v,\nabla)\Delta_j v - \Delta_j((v,\nabla)v)\|_{\mathbf{L}^p}\, d\tau,$$

where $v_0 \equiv v(0)$. This implies that for $t \in [0,T]$

$$\begin{aligned}\|v(t)-w_\ell(t)\|_{\mathbf{B}^{\frac{d}{p}+1}_{p,1}} &\lesssim \sum_{j\geq \ell} 2^{j(\frac{d}{p}+1)}\|\Delta_j v(t)\|_{\mathbf{L}^p}\\ &\lesssim \sum_{j\geq \ell} 2^{j(\frac{d}{p}+1)}\|\Delta_j v_0\|_{\mathbf{L}^p} + \int_0^t \sum_{j\geq \ell} 2^{j(\frac{d}{p}+1)}\|\Delta_j \nabla p\,\|_{\mathbf{L}^p}\, d\tau\\ &\quad + \int_0^t \sum_{j\geq \ell} 2^{j(\frac{d}{p}+1)}\|(S_{j-2}v,\nabla)\Delta_j v - \Delta_j((v,\nabla)v)\|_{\mathbf{L}^p}\, d\tau.\end{aligned}$$

The first term of the right hand side converges to zero as ℓ tends to infinity because $v_0 \in \mathbf{B}^{\frac{d}{p}+1}_{p,1}(\mathbb{R}^d)$. By virtue of the properties 4 and 5 in Remark 2.1 and the fact that $v(t) \in \mathbf{B}^{\frac{d}{p}+1}_{p,1}(\mathbb{R}^d)$, the second and third terms of the right hand side also converge to zero as ℓ goes to infinity. Hence the sequence $\{w_\ell\}_{\ell\in\mathbb{N}}$ converges to v in $L^\infty([0,T];\mathbf{B}^{\frac{d}{p}+1}_{p,1})$.

Step 2. By the same argument used in Step 1, we have

$$\frac{\partial \tilde{v}}{\partial t}\,(x,t) = -(\nabla p)(X(x,t),t). \tag{3.1}$$

Then we obtain

$$\|\Delta_j \dot{\tilde{v}}(t)\|_{\mathbf{L}^p} \leq \|\Delta_j(\nabla p)(X(\cdot,t),t)\|_{\mathbf{L}^p} = \|\Delta_j(\nabla p)\|_{\mathbf{L}^p}\,, \tag{3.2}$$

where the dot symbol represents the partial derivative with respect to t, that is, $\dot{} \equiv \frac{\partial}{\partial t}$. It is important to obtain the following estimate:

Lemma 3.1. *We have*

$$\|(\nabla p)(X(\cdot,t),t)\|_{\mathbf{B}^{\frac{d}{p}+1}_{p,1}} \lesssim \|\nabla p\|_{\mathbf{B}^{\frac{d}{p}+1}_{p,1}}. \tag{3.3}$$

Proof. First, we take ∇-operator on both sides of (1.2) to get the identity

$$\frac{\partial}{\partial t}\nabla_x X(x,t) = (\nabla u)(X(x,t),t)\cdot \nabla_x X(x,t). \tag{3.4}$$

Taking $\mathbf{L}^\infty$-norm on both sides of (3.4), we have

$$\|\nabla_x X(\cdot,t)\|_{\mathbf{L}^\infty} \leq 1 + \int_0^t \|\nabla u(X(\cdot,\tau),\tau)\|_{\mathbf{L}^\infty}\|\nabla_x X(\cdot,\tau)\|_{\mathbf{L}^\infty}\, d\tau.$$

Gronwall's inequality yields that

$$\|\nabla_x X(\cdot,t)\|_{\mathbf{L}^\infty} \leq \exp\left\{\int_0^t \|\nabla u(\cdot,\tau)\|_{\mathbf{L}^\infty} d\tau\right\} \leq C.$$

Similarly, we can get $\|\nabla_x X^{-1}(\cdot,t)\|_{\mathbf{L}^\infty} \leq C$.

Taking $\dot{\mathbf{B}}^{\frac{d}{p}+1}_{p,1}$-norm on both sides of

$$X(x,t) - x = \int_0^t v(X(x,\tau),\tau)d\tau,$$

we have

$$\|\Delta_j(X(\cdot,t)-\cdot)\|_{\dot{\mathbf{B}}^{\frac{d}{p}+1}_{p,1}} \leq \int_0^t \|\Delta_j v(X(\cdot,\tau),\tau)\|_{\dot{\mathbf{B}}^{\frac{d}{p}+1}_{p,1}} d\tau. \tag{3.5}$$

Owing to the fact that $\frac{d}{p}+1 \geq 2$ and the Bersteins's inequality, we can notice that the left side of (3.5) is greater than equal to a constant times of $\|X(\cdot,t)\|_{\dot{\mathbf{B}}^{\frac{d}{p}+1}_{p,1}}$. On the other hand, we can find that the integrand of the right side of (3.5) is less than equal to a constant times of $\|X(\cdot,\tau)\|_{\dot{\mathbf{B}}^{\frac{d}{p}+1}_{p,1}} + 1$ by using the homogeneous version of the properties 2, 6 in Remark 2.1 (we may use the homogeneous version of the property 2 repetitively if necessary). Therefore Gronwall's inequality implies $\|X(\cdot,t)\|_{\dot{\mathbf{B}}^{\frac{d}{p}+1}_{p,1}}$ is finite for each $t \in [0,T]$.

In all, by the same argument in above, using the property 2 (repetitively if necessary) and property 6 in Remark 2.1 to the right side of the following estimate:

$$\begin{aligned}
&\|(\nabla p)(X(\cdot,t),t)\|_{\mathbf{B}^{\frac{d}{p}+1}_{p,1}} \\
&\leq \|\Delta_{-1}(\nabla p)(\cdot,t)\|_{\mathbf{L}^p} + \sum_{j=0}^{\infty} \|\Delta_j\left[\{D(\nabla p)(X(\cdot,t),t)\}(\nabla X)(\cdot,t)\right]\|_{\mathbf{L}^p},
\end{aligned}$$

we can get the estimate (3.3). □

Lemma 3.1 implies two facts; one is that $\|(\nabla p)(X(\cdot,t),t)\|_{\mathbf{B}^{\frac{d}{p}+1}_{p,1}}$ is finite. Also, from (3.1) (or (3.2)), we know $\|\dot{\tilde{v}}(t)\|_{\mathbf{B}^{\frac{d}{p}+1}_{p,1}}$ is finite.(Let us keep in mind this fact for a reference at Step 3.) Therefore from the estimate

$$\begin{aligned}
\|\dot{\tilde{v}}(t) - \dot{\tilde{w}}_\ell(t)\|_{\mathbf{B}^{\frac{d}{p}+1}_{p,1}} &\lesssim \sum_{j\geq \ell} 2^{j(\frac{d}{p}+1)} \|\Delta_j \dot{\tilde{v}}(t)\|_{\mathbf{L}^p} \\
&\lesssim \sum_{j\geq \ell} 2^{j(\frac{d}{p}+1)} \|\Delta_j \nabla p(X(\cdot,t),t)\|_{\mathbf{L}^p},
\end{aligned}$$

we can observe that $\{\dot{\tilde{w}}_\ell\}_{\ell\in\mathbb{N}}$ also converges to $\dot{\tilde{v}}$ in $L^\infty([0,T];\mathbf{B}_{p,1}^{\frac{d}{p}+1})$.

Step 3. Applying the properties 2 and 5 in Remark 2.1 to the Euler equations (1.1), we have

$$\|\dot{v}\|_{\mathbf{B}_{p,1}^{\frac{d}{p}}} \le \|(v,\nabla)v\|_{\mathbf{B}_{p,1}^{\frac{d}{p}}} + \|\nabla p\|_{\mathbf{B}_{p,1}^{\frac{d}{p}}} \lesssim \|v\|_{\mathbf{B}_{p,1}^{\frac{d}{p}}} \|v\|_{\mathbf{B}_{p,1}^{\frac{d}{p}+1}}.$$

Hence we obtain $\dot{v} \in L^\infty([0,T];\mathbf{B}_{p,1}^{\frac{d}{p}})$. Sobolev imbedding theorem now delivers that $v \in W^{1,\infty}([0,T];\mathbf{B}_{p,1}^{\frac{d}{p}}) \subset C([0,T];\mathbf{B}_{p,1}^{\frac{d}{p}})$.

We recall that $\dot{\tilde{v}} \in L^\infty([0,T];\mathbf{B}_{p,1}^{\frac{d}{p}+1})$ from Step 2. We can also differentiate both sides of the Euler equations (3.1)

$$\frac{\partial^2\tilde{v}}{\partial t^2}(x,t) = -(\nabla\dot{p})(X(x,t),t) - [(v,\nabla)\nabla p](X(x,t),t).$$

Then we take $\mathbf{B}_{p,1}^{\frac{d}{p}-[\frac{d}{p}]}$-norm on both sides to get

$$\|\ddot{\tilde{v}}\|_{\mathbf{B}_{p,1}^{\frac{d}{p}-[\frac{d}{p}]}} \lesssim \|\nabla\dot{p}\|_{\mathbf{B}_{p,1}^{\frac{d}{p}-[\frac{d}{p}]}} + \|[(v,\nabla)\nabla p]\|_{\mathbf{B}_{p,1}^{\frac{d}{p}-[\frac{d}{p}]}},$$

where $[a]$ represents the greatest integer less than equal to a. The first term of the right hand side is less than equal to some constant times of $\|\dot{v}\|_{\mathbf{B}_{p,1}^{\frac{d}{p}}} \|v\|_{\mathbf{B}_{p,1}^{\frac{d}{p}+1}}$ and the second term is finite. Therefore we can see that

$$v \in W^{2,\infty}([0,T];\mathbf{B}_{p,1}^{\frac{d}{p}-[\frac{d}{p}]}) \subset C^1([0,T];\mathbf{B}_{p,1}^{\frac{d}{p}-[\frac{d}{p}]}).$$

Step 4. From the estimate

$$\begin{aligned}\|w_\ell(s)-w_\ell(t)\|_{\mathbf{B}_{p,1}^{\frac{d}{p}+1}} &= \|S_\ell(v(s)-v(t))\|_{\mathbf{B}_{p,1}^{\frac{d}{p}+1}} \\ &\lesssim \sum_{j=-1}^{\ell+1} 2^{j(\frac{d}{p}+1)}\|\Delta_j(v(s)-v(t))\|_{\mathbf{L}^p} \\ &\lesssim 2^{2(\ell+1)}\|v(s)-v(t)\|_{\mathbf{B}_{p,1}^{\frac{d}{p}}}\end{aligned}$$

together with the fact that $v \in C([0,T];\mathbf{B}_{p,1}^{\frac{d}{p}})$, we can deduce that each w_ℓ is continuous on $[0,T]$ with values in $\mathbf{B}_{p,1}^{\frac{d}{p}+1}(\mathbb{R}^d)$. Therefore this fact together with the result of Step 1 implies that v is continuous with respect to time on $[0,T]$ with values in $\mathbf{B}_{p,1}^{\frac{d}{p}+1}(\mathbb{R}^d)$.

Step 5. Finally, from the similar estimate

$$\|\dot{\tilde{w}}_\ell(s) - \dot{\tilde{w}}_\ell(t)\|_{\mathbf{B}^{\frac{d}{p}+1}_{p,1}} = \|S_\ell(\dot{\tilde{v}}(s) - \dot{\tilde{v}}(t))\|_{\mathbf{B}^{\frac{d}{p}+1}_{p,1}}$$

$$\lesssim 2^{(\ell+1)([\frac{d}{p}]+1)}\|\dot{\tilde{v}}(s) - \dot{\tilde{v}}(t)\|_{\mathbf{B}^{\frac{d}{p}-[\frac{d}{p}]}_{p,1}}$$

together with the fact that $\dot{\tilde{v}} \in C([0,T];\mathbf{B}^{\frac{d}{p}-[\frac{d}{p}]}_{p,1})$, we can deduce that each $\dot{\tilde{w}}_\ell$ is continuous on $[0,T]$ with values in $\mathbf{B}^{\frac{d}{p}+1}_{p,1}(\mathbb{R}^d)$. Therefore the limit $\dot{\tilde{v}}$ is continuous on $[0,T]$ with values in $\mathbf{B}^{\frac{d}{p}+1}_{p,1}(\mathbb{R}^d)$.

The proof is now completed. □

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