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ON f-DERIVATIONS OF BE-ALGEBRAS

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Abstract. In this paper, we introduce the notion of f-derivation in a BE- algebra, and consider the properties of f-derivations. Also, we characterize the fixed set $Fix_d(X)$ and $Kerd$ by f-derivations. Moreover, we prove that if d is a f-derivation of a BE-algebra, every f-filter F is a a d-invariant.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras([4, 5]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [2, 3], Q. P. Hu and X. Li introduced a wide class of abstracts: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. The notion of a BE-algebra is a dualization of a generalization of a BCK-algebra. In this paper, we introduce the notion of f-derivation in a BE- algebra, and consider the properties of f-derivations. Also, we characterize the fixed set $Fix_d(X)$ and $Kerd$ by f-derivations. Moreover, we prove that if d is a f-derivation of a BE-algebra, every f-filter F is a a d-invariant.

2. Preliminaries

In what follows, let X denote an BE-algebra unless otherwise specified.

By a *BE-algebra* we mean an algebra $(X; *, 1)$ of type (2, 0) with a single binary operation "$*$" that satisfies the following identities: for any $x, y, z \in X$,

(BE1) $x * x = 1$ for all $x \in X$,

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(BE2) $x * 1 = 1$ for all $x \in X$,
(BE3) $1 * x = x$ for all $x \in X$,
(BE4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.

A BE-algebra $(X, *, 1)$ is said to be *self-distributive* if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$. A non-empty subset S of a BE-algebra X is called a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. For any x, y in a BE-algebra X, we define $x \vee y = (y * x) * x$.

In a BE-algebra, the following identities are true: for any $x, y, z \in X$,

(p1) $x * (y * x) = 1$.
(p2) $x * ((x * y) * y)) = 1$.
(p3) Let X be a self-distributive BE-algebra. If $x \le y$, then $z * x \le z * y$ and $y * z \le x * z$.

Definition 2.1. A non-empty subset F of X is called a *filter* of X if

(F1) $1 \in F$,
(F2) If $x \in F$ and $x * y \in F$, then $y \in F$.

Definition 2.2. Let X be a BE-algebra. We say that X is *commutative* if

$$(x * y) * y = (y * x) * x$$

for all $x, y \in X$.

Definition 2.3. A self-map d on a BE-algebra X is called a *derivation* if

$$d(x * y) = (x * d(y)) \vee (d(x) * y)$$

for every $x, y \in X$.

Example 2.4. Let $X = \{1, a, b\}$ be a set in which "$*$" is defined by

$*$	1	a	b
1	1	a	b
a	1	1	b
b	1	a	1

Then X is a BE-algebra. Define a map $d : X \to X$ by

$$d(x) = \begin{cases} 1 & \text{if } x = 1, a \\ b & \text{if } x = b \end{cases}$$

Then it is easy to check that d is a derivation of a BE-algebra X.

Definition 2.5. A self-map d on a BE-algebra X is called to be *regular* if $d(1) = 1$.

DEFINITION 2.6. Let X be a BE-algebra. We define the binary operation "$\leq$" as the following,

$$x \leq y \Leftrightarrow x * y = 1$$

for all $x, y \in X$.

3. f-derivations of BE-algebras

DEFINITION 3.1. Let X be a BE-algebra. A function $d : X \to X$ is called an *f-derivation* on X if there exists an endomorphism $f : X \to X$ such that

$$d(x * y) = (f(x) * d(y)) \vee (d(x) * f(y))$$

for every $x, y \in X$.

EXAMPLE 3.2. Let $X = \{1, a, b\}$ be a set in which "$*$" is defined by

$*$	1	a	b
1	1	a	b
a	1	1	1
b	1	1	1

Then X is a BE-algebra. Define a map $d : X \to X$ by

$$d(x) = \begin{cases} 1 & \text{if } x = 1, b \\ b & \text{if } x = a \end{cases}$$

and define an endomorphism $f : X \to X$ by

$$f(x) = \begin{cases} 1 & \text{if } x = 1 \\ b & \text{if } x = a, b \end{cases}$$

Then it is easy to check that d is a f-derivation of a BE-algebra X.

EXAMPLE 3.3. Let $X = \{1, a, b, c\}$ be a set in which "$*$" is defined by

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	1	1	1

Then X is a BE-algebra. Define a map $d : X \to X$ by

$$d(x) = \begin{cases} 1 & \text{if } x = 1, b, c \\ a & \text{if } x = a \end{cases}$$

and define an endomorphism $f : X \to X$ by

$$f(x) = \begin{cases} 1 & \text{if } x = 1, b \\ a & \text{if } x = a \\ b & \text{if } x = c \end{cases}$$

Then it is easy to check that d is a f-derivation of a BE-algebra X.

EXAMPLE 3.4. Let $X = \{1, a, b, c\}$ be a set in which "$*$" is defined by

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	1
b	1	c	1	c
c	1	1	b	1

Then X is a BE-algebra. Define a map $d : X \to X$ by

$$d(x) = \begin{cases} 1 & \text{if } x = 1, b \\ c & \text{if } x = a, c \end{cases}$$

and define an endomorphism $f : X \to X$ by

$$f(x) = \begin{cases} 1 & \text{if } x = 1, b \\ a & \text{if } x = a, c \end{cases}$$

Then it is easy to check that d is a f-derivation of a BE-algebra X.

EXAMPLE 3.5. Let $X = \{1, a, b\}$ be a set in which "$*$" is defined by

$*$	1	a	b
1	1	a	b
a	1	1	b
b	1	1	1

Then X is a BE-algebra. Define a map $d : X \to X$ by

$$d(x) = \begin{cases} 1 & \text{if } x = 1, a \\ a & \text{if } x = b \end{cases}$$

and define an endomorphism $f : X \to X$ by

$$f(x) = \begin{cases} 1 & \text{if } x = 1, a \\ b & \text{if } x = b \end{cases}$$

Then it is easy to check that d is a f-derivation of a BE-algebra X. But d is not a derivation of X since

$$\begin{aligned} a = d(b) = d(a * b) &\neq (a * d(b)) \vee (d(a) * b) \\ &= (a * a) \vee (1 * b) = 1 \vee b = (b * 1) * 1 = 1 * 1 = 1. \end{aligned}$$

PROPOSITION 3.6. *Every endomorphism f of a BE-algebra X is its f-derivation.*

Proof. Let X be a BE-algebra and let f be an endomorphism on X. Then

$$f(x) * f(y) \vee f(x) * f(y) = f(x) * f(y) = f(x * y)$$

for all $x, y \in X$. This completes the proof. □

PROPOSITION 3.7. *Let X be a BE-algebra. Then every f-derivation of X is regular.*

Proof. Since f is an endomorphism on X, we have $f(1) = 1$. Hence we have

$$\begin{aligned} d(1) = d(x * 1) &= (f(x) * d(1)) \vee (d(x) * f(1)) \\ &= (f(x) * d(1)) \vee (d(x) * 1) \\ &= (f(x) * d(1)) \vee 1 \\ &= 1. \end{aligned}$$

This completes the proof. □

PROPOSITION 3.8. *Let X be a BE-algebra and let d be a f-derivation on X. Then $d(x) = d(x) \vee f(x)$ for all $x \in X$.*

Proof. For all $x \in X$, we have

$$\begin{aligned} d(x) = d(1 * x) &= (f(1) * d(x)) \vee (d(1) * f(x)) \\ &= (f(1) * d(x)) \vee (1 * f(x)) = (1 * d(x)) \vee f(x) \\ &= d(x) \vee f(x). \end{aligned}$$

□

PROPOSITION 3.9. *Let X be a BE-algebra. If d is a f-derivation of X, then the following identities hold:*

(1) *$f(x) \leq d(x)$ for all $x \in X$,*
(2) *$d(x) * f(y) \leq f(x) * d(y)$ for all $x, y \in X$.*

Proof. (1) By Proposition 3.8, we have

$$\begin{aligned} f(x) * d(x) &= f(x) * (d(x) \vee f(x)) = f(x) * ((f(x) * d(x)) * d(x)) \\ &= (f(x) * d(x)) * (f(x) * d(x)) \\ &= 1 \end{aligned}$$

which implies $f(x) \leq d(x)$.

(2) From (1) and (p3), we have $d(x) * f(y) \leq f(x) * f(y) \leq f(x) * d(y)$. □

THEOREM 3.10. *Let X be a BE-algebra and let d be a f-derivation of X. Then we have $d(x * y) = f(x) * d(y)$ for all $x, y \in X$.*

Proof. Let d be a f-derivation on X and $x, y \in X$. Then we have $d(x) * f(y) \leq f(x) * d(y)$ from Proposition 3.9 (2). Hence we get

$$\begin{aligned} d(x * y) &= (f(x) * d(y)) \vee (d(x) * f(y)) \\ &= ((d(x) * f(y)) * (f(x) * d(y))) * (f(x) * d(y)) \\ &= 1 * (f(x) * d(y)) = f(x) * d(y). \end{aligned}$$

□

PROPOSITION 3.11. *Let X be a BE-algebra and let d be a f-derivation of X. If it satisfies $d(x * y) = d(x) * f(y)$ for all $x, y \in X$, we have $d(x) = f(x)$.*

Proof. Let d be a f-derivation of X. If it satisfies $d(x*y) = d(x)*f(y)$ for all $x, y \in X$, we have

$$\begin{aligned} d(x) &= d(1 * x) = d(1) * f(x) \\ &= 1 * f(x) = f(x). \end{aligned}$$

This completes the proof. □

PROPOSITION 3.12. *Let X be a BE-algebra and let d be a f-derivation of X. If it satisfies $f(x) * d(y) = d(x) * f(y)$ for all $x, y \in X$, then $d(x) = f(x)$ for all $x \in X$.*

Proof. Let d be a f-derivation of X. If it satisfies $f(x) * d(y) = d(x) * f(y)$ for all $x, y \in X$, we have

$$\begin{aligned} d(x) &= d(1 * x) = f(1) * d(x) \\ &= d(1) * f(x) = 1 * f(x) \\ &= f(x) \end{aligned}$$

from Theorem 3.10. This completes the proof. □

THEOREM 3.13. *Let d be a f-derivation on X. If $d \circ f = f \circ d$, then we have $d(f(x) * d(x)) = 1$ for all $x \in X$.*

Proof. Let d be a f-derivation on X and $d \circ f = f \circ d$. For all $x \in X$, we have

$$\begin{aligned} d(f(x) * d(x)) &= (f(f(x)) * d(d(x))) \vee (d(f(x)) * f(d(x))) \\ &= (f(f(x)) * d(d(x))) \vee (f(d(x)) * f(d(x))) \\ &= (f(f(x)) * d(d(x))) \vee 1 = 1. \end{aligned}$$

□

DEFINITION 3.14. Let X be a BE-algebra and let d be a f-derivation on X. If $x \leq y$ implies $d(x) \leq d(y)$ for all $x, y \in X$, then d is called an *isotone f-derivation* of X.

PROPOSITION 3.15. *Let d be a f-derivation of a BE-algebra X. If $d(x) \vee d(y) \leq d(x \vee y)$ for all $x, y \in X$, then d is an isotone f-derivation of X.*

Proof. Suppose that $d(x) \vee d(y) \leq d(x \vee y)$ and $x \leq y$. Then we have $d(x) \leq d(x) \vee d(y) \leq d(x \vee y) = d(y)$. □

Let d be a f-derivation of X. Define a set $Fix_d(X)$ by

$$Fix_d(X) := \{x \in X \mid d(x) = f(x)\}$$

for all $x \in X$.

PROPOSITION 3.16. *Let d be a f-derivation of a BE-algebra X. Then $Fix_d(X)$ is a subalgebra of X.*

Proof. Clearly, $1 \in Fix_d(X)$ and so $Fix_d(X)$ is non-empty. Let $x, y \in Fix_d(X)$. Then we have $d(x) = f(x)$ and $d(y) = f(y)$, and so

$$d(x*y) = (f(x)*d(y)) \vee (d(x)*f(y)) = (f(x)*f(y)) \vee (f(x)*f(y)) = f(x*y).$$

This implies $x * y \in Fix_d(X)$. □

PROPOSITION 3.17. *Let X be a BE-algebra and let d be a f-derivation of X. If $x, y \in Fix_d(X)$, then we have $x \vee y \in Fix_d(X)$.*

Proof. Let $x, y \in Fix_d(X)$. Then we have $d(x) = f(x)$ and $d(y) = f(y)$, and so

$$\begin{aligned} d(x \vee y) &= d((y * x) * x) = (f(y * x) * d(x)) \vee (d(y * x) * f(x)) \\ &= ((f(y) * f(x)) * f(x)) \vee (((f(y) * d(x)) \vee (d(y) * f(x))) * f(x)) \\ &= (f(y) * f(x)) * f(x) \vee ((f(y) * f(x)) \vee (f(y) * f(x))) * f(x) \\ &= (f(y) * f(x)) * f(x)) \vee ((f(y) * f(x)) * f(x)) \\ &= ((f(y) * f(x)) * f(x)) = f((y * x) * x) = f(x \vee y). \end{aligned}$$

This completes the proof. □

Let d be a f-derivation of X. Define a $Kerd$ by

$$Kerd = \{x \in X \mid d(x) = 1\}$$

for all $x \in X$.

PROPOSITION 3.18. *Let d be a f-derivation of X. Then $Kerd$ is a subalgebra of X.*

Proof. Clearly, $1 \in Kerd$, and so $Kerd$ is non-empty. Let $x, y \in Kerd$. Then $d(x) = 1$ and $d(y) = 1$. Hence we have

$$\begin{aligned} d(x * y) &= (f(x) * d(y)) \vee (d(x) * f(y)) \\ &= (f(x) * 1) \vee (1 * f(y)) = 1 \vee f(y) \\ &= (f(y) * 1) * 1 = 1 * 1 \\ &= 1, \end{aligned}$$

and so $x * y \in Kerd$. Thus $Kerd$ is a subalgebra of X. □

PROPOSITION 3.19. *Let X be a commutative BE-algebra and let d be a f-derivation of X. If $x \in Kerd$ and $x \leq y$, then we have $y \in Kerd$.*

Proof. Let $x \in Kerd$ and $x \leq y$. Then $d(x) = 1$ and $x * y = 1$.

$$\begin{aligned} d(y) &= d(1 * y) = d((x * y) * y) \\ &= d((y * x) * x) \\ &= (f(y * x) * d(x)) \vee (d(y * x) * f(x)) \\ &= (f(y * x) * 1) \vee (d(y * x) * f(x)) \\ &= 1 \vee (d(y * x) * f(x)) \\ &= 1, \end{aligned}$$

and so $y \in Kerd$. This completes the proof. □

PROPOSITION 3.20. *Let X be a BE-algebra and let d be a f-derivation of X. If $y \in Kerd$, then we have $x * y \in Kerd$ for all $x \in X$.*

Proof. Let $y \in Kerd$. Then $d(y) = 1$. Thus we have

$$\begin{aligned} d(x * y) &= (f(x) * d(y)) \vee (d(x) * f(y)) \\ &= (f(x) * 1) \vee (d(x) * f(y)) \\ &= 1 \vee (d(x) * f(y)) \\ &= 1, \end{aligned}$$

which implies $x * y \in Kerd$. □

PROPOSITION 3.21. *Let X be a BE-algebra and let d be a f-derivation of X. If $x \in Kerd$, then we have $x \vee y \in Kerd$ for all $y \in X$.*

Proof. Let $x \in Kerd$. Then $d(x) = 1$. Then we have

$$\begin{aligned} d(x \vee y) = d((y * x) * x) &= (f(y * x) * d(x)) \vee (d(y * x) * f(x)) \\ &= (f(y * x) * 1) \vee (d(y * x) * f(x)) \\ &= 1 \vee (d(y * x) * f(x)) \\ &= 1, \end{aligned}$$

which implies $x \vee y \in Kerd$. □

PROPOSITION 3.22. *Let X be a BE-algebra and let d be a f-derivation. If d is an endomorphism on X, then $Kerd$ is a filter of X.*

Proof. Clearly, $1 \in Kerd$. Let $x, x * y \in Kerd$, respectively. Then $d(x) = 1$ and $d(x * y) = 1$. Thus, we have $1 = d(x * y) = d(x) * d(y) = 1 * d(y) = d(y)$, which implies $y \in Kerd$. This completes the proof. □

Let X be a BE-algebra. We define the binary operation " $+$ " as the following

$$x + y = (x * y) * y$$

for all $x, y \in X$. Clearly, X is a commutative BE-algebra if and only if $x + y = y + x$ for all $x, y \in X$.

PROPOSITION 3.23. *Let X be a commutative BE-algebra. Then the followings hold for all $x, y, z \in X$,*

(1) $x + 1 = 1 + x$.
(2) $x + y = y + x$.
(3) $x + (y + z) = (x + y) + z$.
(4) $x + x = x$.

Proof. The proof is clear. □

PROPOSITION 3.24. *Let X be a BE-algebra and let d be a f-derivation of X. Then the followings hold for all $x, y \in X$,*

(1) $d(x+1)=1$.
(2) $d(x+x)=f(x)+d(x)$.
(3) $d(x)=f(x)+d(x)$.

Proof. (1) Let $x\in X$. Then we get

$$\begin{aligned} d(x+1) &= d((x*1)*1) = (f(x*1)*d(1)) \vee (d(x*1)*f(1)) \\ &= (f(1)*d(1)) \vee (1*1) = 1 \vee 1 = (1*1)*1 \\ &= 1. \end{aligned}$$

(2) Let $x\in X$. Then we have

$$\begin{aligned} d(x+x) &= d((x*x)*x) = (f(x*x)*d(x)) \vee (d(x*x)*f(x)) \\ &= (f(1)*d(x)) \vee (d(1)*f(x)) = (1*d(x)) \vee (1*f(x)) \\ &= d(x) \vee f(x) \\ &= (f(x)*d(x))*d(x) = f(x)+d(x). \end{aligned}$$

(3) Let $x\in X$. Then we have

$$\begin{aligned} d(x) &= d(1*x) = (f(1)*d(x)) \vee (d(1)*f(x)) \\ &= (1*d(x)) \vee (1*f(x) = d(x)) \vee f(x) \\ &= (f(x)*d(x))*d(x) \\ &= f(x)+d(x) \end{aligned}$$

□

Definition 3.25. Let X be a BE-algebra. A non-empty set F of X is called a *normal filter* of X if it satisfies the following conditions:
(NF1) $1\in F$,
(NF2) $x\in X$ and $y\in F$ imply $x*y\in F$.

Example 3.26. Let $X=\{1,a,b,c\}$ be a set in which "$*$" is defined by

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	1	1	c
c	1	1	1	1

Then X is a BE-algebra. Let $F=\{1,a\}$. Then F is a normal filter of X.

Proposition 3.27. *Let X be a BE-algebra and let d be a f-derivation of X. Then $Fix_d(X)$ is a normal filter of X.*

Proof. Clearly, $1 \in Fix_d(X)$. Let $x \in X$ and $y \in Fix_d(X)$. Then we have $d(y) = f(y)$, and so

$$\begin{aligned} d(x * y) &= f(x) * d(y) \\ &= f(x) * f(y) \\ &= f(x * y), \end{aligned}$$

which implies $x * y \in Fix_d(X)$ from Theorem 3.10. This completes the proof. □

Proposition 3.28. *Let X be a BE-algebra and let d be a f-derivation of X. Then $Kerd$ is a normal filter of X.*

Proof. Clearly, $1 \in Kerd$. Let $x \in X$ and $y \in Kerd$. Then we have $d(y) = 1$, and so

$$\begin{aligned} d(x * y) &= (f(x) * d(y)) \vee (d(x) * f(y)) \\ &= (f(x) * 1) \vee (d(x) * f(y)) \\ &= 1 \vee (d(x) * f(y)) = 1, \end{aligned}$$

which implies $x * y \in Kerd$. Hence $Kerd$ is a normal filter of X. □

Proposition 3.29. *Let X be a self-distributive BE-algebra and let d be a f-derivation of X. Then $F_a = \{x \in X \mid a \leq d(x)\}$ is a normal filter of X.*

Proof. Clearly, $a \leq d(1) = 1$ for any $a \in X$, and so $1 \in F_a$. Let $x \in X$ and $y \in F_a$. Then we have $a * d(y) = 1$, and so from Theorem 3.10,

$$\begin{aligned} a * d(x * y) &= a * (f(x) * d(y)) = (a * f(x)) * (a * d(y)) \\ &= (a * f(x)) * 1 \\ &= 1, \end{aligned}$$

which implies $x * y \in F_a$. Hence F_a is a normal filter of X. □

Definition 3.30. Let f be a map on X. . A filter F of a BE-algebra X is said to be *f-filter* if $f(F) \subseteq F$.

Definition 3.31. Let d be a self-map of a BE-algebra X. A f-filter F of X is said to be a d-invariant if $d(F) \subseteq F$.

Proposition 3.32. *Let X be a self-distributive BE-algebra X. If d is a f-derivation of X, then every f-filter F is d-invariant.*

Proof. Let F be a f-filter of X. Let $y \in d(F)$. Then $y = d(x)$ for some $x \in F$. It follows from Proposition 3.9(1) that $f(x) * y = f(x) * d(x) = 1 \in F$. Since F is a f-filter of X, we have $y \in F$. Thus $d(F) \subseteq F$. Hence F is d-invariant. □

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