THE RELATIVE ENTROPY UNDER THE R-CGMY PROCESSES

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ABSTRACT. We consider the relative entropy for two R-CGMY processes, which are CGMY processes with Y equal to 1, to choose an equivalent martingale measure (EMM) when the underlying asset of a derivative follows a R-CGMY process in the financial market. Since the R-CGMY process leads to an incomplete market, we have to use a proper technique to choose an EMM among a variety of EMMs. In this paper, we derive the closed form expression of the relative entropy for R-CGMY processes.

1. Introduction

Many alterative stochastic processes have been developed in the financial market because of some drawbacks of the classical Black-Scholes model [1] such as volatility smile effects. We consider the CGMY processes proposed by Carr, Geman, Madan, and Yor [2]. Since the CGMY process leads to an incomplete market, we need an adequate technique to choose an EMM with respect to the market measure. We are interested in the relative entropy to select an EMM. Kim and Lee [5] introduced the closed form of the relative entropy under the CGMY processes, except for Y equal to 0 and 1. Küchler and Tappe [6] obtained the relative entropy for the CGMY processes with Y equal to 0. The purpose of this paper is to represent the relative entropy in the closed form under the CGMY processes with Y equal to 1.

This paper is organized as follows. In section 2 we start by introducing briefly the concept about the R-CGMY process and its properties. As our main results, in section 3 we represent the relative entropy in the closed form under the R-CGMY processes and study the existence of the model preserving minimal entropy martingale measure. Finally, this paper ends with conclusions in section 4.

2. The R-CGMY process and its properties

In order to overcome the drawbacks of the Black-Scholes model which cannot capture the stylized facts such as fat tails and volatility smiles in the financial market, we are interested in Lévy processes. In this section we consider a Lévy process X_t defined by the following Lévy measure

$$\nu(dx) = C\left(\frac{e^{-G|x|}}{|x|^{1+Y}}1_{x<0} + \frac{e^{-Mx}}{x^{1+Y}}1_{x>0}\right)dx,$$

where $C > 0, G > 0, M > 0, 0 < Y < 2$, and 1_A is the indicator function of a set A. The condition $0 < Y < 2$ ensures that the Lévy measure integrates x^2 in the neighborhood of the origin. This process X_t is called the CGMY process introduced by Carr, Geman, Madan, and Yor [2]. When the parameter Y is zero, it is well-known for the variance gamma (VG) process shown by Madan and Seneta [7]. The VG process is obtained by the time changed Brownian motion with drift where time is given by a gamma process. In this paper, we focus on the CGMY process with Y equal to 1. Let us now consider a Lévy process X_t with the Lévy triplet

$$(2.1) \qquad\qquad \sigma = 0,$$

$$(2.2) \qquad \nu(dx) = C\left(\frac{e^{-G|x|}}{x^2}1_{x<0} + \frac{e^{-Mx}}{x^2}1_{x>0}\right)dx,$$

$$(2.3) \qquad\qquad \gamma = \mu + C(\ln G - \ln M) - \int_{|x|>1} x\nu(dx).$$

The Lévy process X_t above is called the restricted CGMY (R-CGMY) process with parameters (C, G, M, μ). The characteristic function $\Phi(\cdot)$ of the R-CGMY process X_t with parameters (C, G, M, μ) is given by

$$\Phi_{X_t}(z; C, G, M, \mu) = \mathbb{E}[\exp izX_t] = \exp[t\Psi_{X_t}(z; C, G, M, \mu)],$$

where the characteristic exponent $\Psi(\cdot)$ is

$$\Psi_{X_t}(z; C, G, M, \mu) = iz(\mu + C(\ln G - \ln M)) + C(M - iz) \ln \left(1 - \frac{iz}{M} \right)$$

$$+ C(G + iz) \ln \left(1 + \frac{iz}{G} \right).$$

It allows us to compute the cumulants of the R-CGMY process X_t with parameters (C, G, M, μ) by differentiating the logarithm of the characteristic function successively. The *cumulants* or *semi-invariants* of the R-CGMY process X_t are defined by

$$c_n(X_t) = \frac{1}{i^n} \frac{\partial^n \ln \Phi_{X_t}}{\partial z^n}(0).$$

Then the cumulants of the R-CGMY process X_t can be obtained as follows.

THEOREM 2.1. *Suppose that X_t is a R-CGMY process with parameters (C, G, M, μ). Then its cumulants c_n are given by*

$$c_1 = t \left(\mu - C \left(\ln \frac{1}{G} - \ln \frac{1}{M} \right) \right), \qquad c_2 = tC \left(\frac{1}{G} + \frac{1}{M} \right),$$

$$c_3 = -tC \left(\frac{1}{G^2} - \frac{1}{M^2} \right), \qquad\qquad c_4 = 2tC \left(\frac{1}{G^3} + \frac{1}{M^3} \right).$$

Proof. It can be easily proven by differentiating the logarithm of the characteristic function. $\qquad\square$

The *skewness* $s(X_t)$ and *kurtosis* $\kappa(X_t)$ are defined by normalizing the cumulants c_n by the n-th power of the standard deviation of X_t as below.

$$s(X_t) = \frac{c_3(X_t)}{c_2(X_t)^{3/2}} \quad \text{and} \quad \kappa(X_t) = \frac{c_4(X_t)}{c_2(X_t)^2}.$$

It is known that the financial market is negatively skewed and fat tailed. The R-CGMY process X_t has a negative skewness if G is smaller than M and is fat tailed because C, G, and M are positive.

2.1. Transformation of measures for Lévy processes

For a given market measure, we have to find an equivalent martingale measure (EMM) with respect to the market measure in order to evaluate derivatives in the financial market. In this section we describe a general result to find equivalent measures for Lévy processes. For more details, see [8].

LEMMA 2.2. *Let $(X_t, \mathbf{P})$ and $(X_t, \mathbf{Q})$ be two Lévy processes on $\mathbb{R}$ with Lévy triplets (σ^2, ν, γ) and $(\bar{\sigma}^2, \bar{\nu}, \bar{\gamma})$, respectively. Then $\mathbf{P}|_{\mathcal{F}_t}$ and $\mathbf{Q}|_{\mathcal{F}_t}$ are mutually absolutely continuous for all t if and only if the following conditions are satisfied:*

$$(2.4) \qquad\qquad\qquad \sigma^2 = \bar{\sigma}^2,$$

$$(2.5) \qquad\qquad \int_{-\infty}^{\infty} (e^{\psi(x)/2} - 1)^2 \nu(dx) < \infty,$$

where $\psi(x)$ is the logarithm of the Radon-Nikodym density of $\bar{\nu}$ with respect to ν, i.e., $e^{\psi(x)} = \frac{d\bar{\nu}}{d\nu}$. If $\sigma = 0$, then

$$(2.6) \qquad\qquad \bar{\gamma} - \gamma = \int_{|x| \leq 1} x(\bar{\nu} - \nu)(dx).$$

When $\mathbf{P}$ and $\mathbf{Q}$ are equivalent, the Radon-Nikodym derivative is

$$(2.7) \qquad\qquad\qquad \frac{d\mathbf{Q}|_{\mathcal{F}_t}}{d\mathbf{P}|_{\mathcal{F}_t}} = e^{U_t},$$

where $(U_t)_{t \geq 0}$ is a Lévy process with Lévy triplet $(\sigma_U{}^2, \nu_U, \gamma_U)$ given by

$$(2.8) \qquad\qquad\qquad \sigma_U{}^2 = \sigma^2 \eta^2,$$

$$(2.9) \qquad\qquad\qquad \nu_U = \nu \psi^{-1},$$

$$(2.10) \qquad \gamma_U = -\frac{1}{2}\sigma^2\eta^2 - \int_{-\infty}^{\infty} (e^x - 1 - x1_{|x| \leq 1}(x))(\nu\psi^{-1})(dx),$$

and η is chosen so that

$$(2.11) \qquad\qquad \bar{\gamma} - \gamma - \int_{-1}^{1} x(\bar{\nu} - \nu)(dx) = \sigma^2\eta,$$

if $\sigma > 0$ and η is zero if $\sigma = 0$.

We apply Lemma 2.2 to R-CGMY processes.

THEOREM 2.3. *Let us consider two R-CGMY processes $(X_t, \mathbf{P})$ and $(X_t, \mathbf{Q})$ on $\mathbb{R}$ with parameters (C, G, M, μ) and $(\bar{C}, \bar{G}, \bar{M}, \bar{\mu})$, respectively. Then $\mathbf{P}|_{\mathcal{F}_t}$ and $\mathbf{Q}|_{\mathcal{F}_t}$ are equivalent for all $t > 0$ if and only if $C = \bar{C}$ and $\mu = \bar{\mu}$. Moreover, when $\mathbf{P}$ and $\mathbf{Q}$ are equivalent, the Radon-Nikodym derivative of $\mathbf{Q}$ with respect to $\mathbf{P}$ is $\frac{d\mathbf{Q}|_{\mathcal{F}_t}}{d\mathbf{P}|_{\mathcal{F}_t}} = e^{U_t}$, where $(U_t, \mathbf{P})$ is a Lévy process with Lévy triplet $(\sigma_U{}^2, \nu_U, \gamma_U)$ given by*

$$\sigma_U = 0,$$

$$\nu_U = \nu \circ \psi^{-1},$$

$$\gamma_U = -\int_{-\infty}^{\infty} (e^x - 1 - x1_{|x|\leq 1}(x))(\nu \circ \psi^{-1})(dx),$$

where $\psi(x) = (\bar{G} - G)x1_{x<0} - (\bar{M} - M)x1_{x>0}$.

Proof. According to Cont and Tankov [3], we can see the fact that the condition (2.5) holds if and only if $C = \bar{C}$. Since σ is zero, the condition (2.6) has to be satisfied under the condition $C = \bar{C}$. We can easily obtain that $\int_{-\infty}^{\infty} x(\bar{\nu} - \nu)(dx) = C(\ln \bar{G} - \ln \bar{M} - \ln G + \ln M)$. Hence the condition (2.6) holds if $\mu = \bar{\mu}$. This proof is complete. □

3. The relative entropy for R-CGMY processes

There are a variety of EMMs with respect to the market measure in the incomplete market while there is the unique EMM in the complete market. Since the CGMY process leads to the incomplete market, we have to choose an EMM associated with the market measure. To do this, we consider the *relative entropy* or the *Kullback-Leibler distance* between two measures. Let $\mathbf{P}$ and $\mathbf{Q}$ be two equivalent probability measures on the measurable space $(\Omega, \mathcal{F})$. The relative entropy of $\mathbf{Q}$ with respect to $\mathbf{P}$ is defined by

$$\mathcal{E}(\mathbf{Q}|\mathbf{P}) = \mathbb{E}_{\mathbf{P}}\left[\frac{d\mathbf{Q}}{d\mathbf{P}} \ln \frac{d\mathbf{Q}}{d\mathbf{P}}\right],$$

where $\mathbb{E}_{\mathbf{P}}[\cdot]$ is the expectation with repect to the measure $\mathbf{P}$. It can be easily observed by the Jensen's inequality that the relative entropy $\mathcal{E}(\mathbf{Q}|\mathbf{P})$ above is nonnegative with $\mathcal{E}(\mathbf{Q}|\mathbf{P}) = 0$ if and only if $\frac{d\mathbf{Q}}{d\mathbf{P}} = 1$ almost surely.

If two measures correspond to exponential Lévy models, the relative entropy can be obtained in terms of the corresponding Lévy triplets as follows. For more details, see [4].

LEMMA 3.1. *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be equivalent measures on* $(\Omega, \mathcal{F})$ *generated by exponential Lévy models with Lévy triplets* (σ^2, ν, γ) *and* $(\bar{\sigma}^2, \bar{\nu}, \bar{\gamma})$, *respectively. Suppose that* σ^2 *is positive. Then the relative entropy* $\mathcal{E}(\mathbf{Q}|\mathbf{P})$ *is given by*

$$\mathcal{E}(\mathbf{Q}|\mathbf{P}) = \frac{T}{2\sigma^2}\left\{\bar{\gamma} - \gamma - \int_{-1}^{1} x(\bar{\nu} - \nu)(dx)\right\}^2$$

(3.1)
$$+ T\int_{-\infty}^{\infty}\left(\frac{d\bar{\nu}}{d\nu}\ln\frac{d\bar{\nu}}{d\nu} + 1 - \frac{d\bar{\nu}}{d\nu}\right)\nu(dx).$$

When $\sigma^2 = 0$, then

$$(3.2) \qquad \mathcal{E}(\mathbf{Q}|\mathbf{P}) = T \int_{-\infty}^{\infty} \left(\frac{d\bar{\nu}}{d\nu} \ln \frac{d\bar{\nu}}{d\nu} + 1 - \frac{d\bar{\nu}}{d\nu} \right) \nu(dx).$$

As our main result, we show that the relative entropy can be expressed by the following closed form applying Lemma 3.1 to the Lévy triplets of two R-CGMY processes.

THEOREM 3.2. *Let us consider that $(X_t, \mathbf{P})$ and $(X_t, \mathbf{Q})$ follow R-CGMY processes on $\mathbb{R}$ with parameters (C, G, M, μ) and $(\bar{C}, \bar{G}, \bar{M}, \bar{\mu})$, respectively. We also assume that $\mathbf{P}$ and $\mathbf{Q}$ are equivalent measures, that is, $C = \bar{C}$ and $\mu = \bar{\mu}$. If $\bar{G} < 2G$ and $\bar{M} < 2M$, then we have the following relative entropy of $\mathbf{Q}$ with respect to $\mathbf{P}$*

$$(3.3) \quad \mathcal{E}(\mathbf{Q}|\mathbf{P}) = TC \left\{ M \left(\frac{\bar{M}}{M} - 1 - \ln \frac{\bar{M}}{M} \right) + G \left(\frac{\bar{G}}{G} - 1 - \ln \frac{\bar{G}}{G} \right) \right\}.$$

Proof. Since $\sigma = 0$, the relative entropy is given by

$$\mathcal{E}(\mathbf{Q}|\mathbf{P}) = T \int_{-\infty}^{\infty} (\psi(x)e^{\psi(x)} - e^{\psi(x)} + 1)\nu(dx),$$

where $\psi(x) = (\bar{G} - G)x 1_{x<0} - (\bar{M} - M)x 1_{x>0}$. Using the Taylor expansion, we have

$$\psi(x)e^{\psi(x)} - e^{\psi(x)} + 1$$

$$= \left(\sum_{n=1}^{\infty} \frac{1}{n!} \{(M - \bar{M})x\}^{n+1} - \sum_{n=2}^{\infty} \frac{1}{n!} \{(M - \bar{M})x\}^n \right) 1_{x>0}$$

$$+ \left(\sum_{n=1}^{\infty} \frac{1}{n!} \{(\bar{G} - G)x\}^{n+1} - \sum_{n=2}^{\infty} \frac{1}{n!} \{(\bar{G} - G)x\}^n \right) 1_{x<0}.$$

Hence the relative entropy can be rewritten as

$$\mathcal{E}(\mathbf{Q}|\mathbf{P}) = T \int_0^{\infty} \sum_{n=1}^{\infty} \frac{1}{n!} \{(M - \bar{M})x\}^{n+1} \nu(dx)$$

$$(3.4) \qquad\qquad - T \int_0^{\infty} \sum_{n=2}^{\infty} \frac{1}{n!} \{(M - \bar{M})x\}^n \nu(dx)$$

$$+ T \int_{-\infty}^0 \sum_{n=1}^{\infty} \frac{1}{n!} \{(\bar{G} - G)x\}^{n+1} \nu(dx)$$

$$- T \int_{-\infty}^0 \sum_{n=2}^{\infty} \frac{1}{n!} \{(\bar{G} - G)x\}^n \nu(dx).$$

Computing the first integral on the right-hand side of the equation (3.4), we have

$$T \int_0^\infty \sum_{n=1}^\infty \frac{1}{n!} \{(M - \bar{M})x\}^{n+1} \nu(dx)$$

$$= T \sum_{n=1}^\infty \frac{1}{n!} C(M - \bar{M})^{n+1} \int_0^\infty e^{-Mx} x^{n-1} dx$$

$$= T \sum_{n=1}^\infty \frac{1}{n!} C(M - \bar{M})^{n+1} \frac{1}{M^n} \Gamma(n)$$

$$= TCM(1 - \frac{\bar{M}}{M}) \sum_{n=1}^\infty \frac{1}{n} (1 - \frac{\bar{M}}{M})^n$$

$$= TCM \left(\frac{\bar{M}}{M} - 1 \right) \ln \frac{\bar{M}}{M}.$$

We can interchange integration and summation above by using Lebegue dominated convergence theorem and the infinite summation is replaced by the logarithm due to $\bar{M} < 2M$. Next, let us compute the second integral on the right-hand side of the equation (3.4). Then we have

$$T \int_0^\infty \sum_{n=2}^\infty \frac{1}{n!} \{(M - \bar{M})x\}^n \nu(dx)$$

$$= T \sum_{n=2}^\infty \frac{1}{n!} C(M - \bar{M})^n \int_0^\infty e^{-Mx} x^{n-2} dx$$

$$= T \sum_{n=2}^\infty \frac{1}{n!} C(M - \bar{M})^n \frac{1}{M^{n-1}} \Gamma(n-1)$$

$$= TCM \sum_{n=2}^\infty \frac{1}{n(n-1)} \left(1 - \frac{\bar{M}}{M} \right)^n$$

$$= TCM \left\{ (1 - \frac{\bar{M}}{M}) \sum_{n=1}^\infty \frac{1}{n} \left(1 - \frac{\bar{M}}{M} \right)^n - \sum_{n=2}^\infty \frac{1}{n} \left(1 - \frac{\bar{M}}{M} \right)^n \right\}$$

$$= TCM \left(\frac{\bar{M}}{M} \ln \frac{\bar{M}}{M} + 1 - \frac{\bar{M}}{M} \right).$$

Using similar arguments in the third and the fourth integrals of the equation (3.4), the relative entropy can be expressed by (3.3).   □

THEOREM 3.3. *For given $C, G, M > 0$, and $T > 0$, let f be a real-valued function on $(0, 2G) \times (0, 2M)$ such that*

$$(3.5) \qquad\qquad f(\bar{G}, \bar{M}) = \mathcal{E}(\mathbf{Q}|\mathbf{P}),$$

where $\mathcal{E}(\mathbf{Q}|\mathbf{P})$ is given by (3.3). Then f is convex on the domain $(0, 2G) \times (0, 2M)$.

Proof. For each point $(\bar{G}_0, \bar{M}_0) \in (0, 2G) \times (0, 2M)$,

$$\frac{\partial f}{\partial G}(\bar{G}_0, \bar{M}_0) = TC\left(1 - \frac{G}{G_0}\right), \quad \frac{\partial f}{\partial M}(\bar{G}_0, \bar{M}_0) = TC\left(1 - \frac{M}{M_0}\right),$$

$$\frac{\partial^2 f}{\partial \bar{G}^2}(\bar{G}_0, \bar{M}_0) = TC\frac{G}{G_0^2}, \quad \frac{\partial^2 f}{\partial \bar{M}^2}(\bar{G}_0, \bar{M}_0) = TC\frac{M}{M_0^2}, \quad \frac{\partial^2 f}{\partial \bar{G} \partial \bar{M}}(\bar{G}_0, \bar{M}_0) = 0.$$

Therefore, since $\frac{\partial^2 f}{\partial \bar{G}^2}(\bar{G}_0, \bar{M}_0) > 0$ and $\left(\left(\frac{\partial^2 f}{\partial \bar{G} \partial \bar{M}}\right)^2 - \frac{\partial^2 f}{\partial \bar{G}^2}\frac{\partial^2 f}{\partial \bar{M}^2} \right)(\bar{G}_0, \bar{M}_0) <$ 0 for each point $(\bar{G}_0, \bar{M}_0) \in (0, 2G) \times (0, 2M)$, f is convex on the domain $(0, 2G) \times (0, 2M)$. $\qquad\square$

For a given market measure $\mathbf{P}$ under which the log return of a stock price process X_t follows a R-CGMY process, we can choose the unique corresponding EMM $\mathbf{Q}$ that has the minimal entropy among R-CGMY processes by using Theorem 3.3. This unique EMM is called the model preserving minimal entropy martingale measure.

4. Conclusion

In this paper we show the existence of the closed form of the relative entropy when two processes follow the R-CGMY processes. Since the relative entropy with respect to the market measure is convex, there exists the model preserving minimal entropy martingale measure. Thus, we can apply the relative entropy to selecting an EMM among a variety of EMMs in the financial market.

References

[1] F. Black and M. S. Scholes, *The pricing of options and corporate liabilities*, J. Polit. Econ. **81** (1973), 637-654.

[2] P. Carr, H. Geman, D. B. Madan, and M. Yor, *The fine structure of asset returns: An empirical investigation*, J. Bus. **75** (2002), 305-332.

[3] R. Cont and P. Tankov, *Financial Modelling with Jump Processes*, Chapman & Hall/CRC, Boca Raton, FL, 2004.

[4] R. Cont and P. Tankov, *Non-parametric calibration of jump-diffusion option pricing models*, J. Comput. Financ. **7** (2004), 1-49.

[5] Y. S. Kim and J. H. Lee, *The relative entropy in CGMY processes and its applications to finance*, Math. Methods Oper. Res. **66** (2007), 327-338.

[6] U. Küchler and S. Tappe, *Bilateral gamma distributions and processes in financial mathematics*, Stochastic Process. Appl. **118** (2008), 261-283.

[7] D. B. Madan and E. Seneta, *The variance gamma (V.G.) model for share market returns*, J. Bus. **63** (1990), 511-524.

[8] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, Cambridge, UK, 1999.

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