DYNAMICAL SYSTEMS WITH SPECIFICATION

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Abstract. In this paper we prove that C^1-generically, if a diffeomorphism f on a closed C^∞ manifold M satisfies weak specification on a locally maximal set $\Lambda \subset M$ then Λ is hyperbolic for f. As a corollary we obtain that C^1-generically, every diffeomorphism with weak specification is Anosov.

1. Introduction

The notion of the specification was introduced by Bowen [3] to construct the equilibrium state and make a Markov partition of Axiom A diffeomorphisms. After that the notion of weak specification was introduced by Ruelle [10] to study variational principle for Z^μ-action. The notions have turned out to be important in the study of ergodic theory of dynamical systems (for more details, see [2, 3, 5, 7, 8, 10, 12, 13]).

The definition of specification (or weak specification) seems to be complicated and strong, but it is satisfied by many examples. For example, Lind [8] showed that hyperbolic toral automorphisms satisfy specification and ergodic central spin toral automorphisms satisfy weak specification. Moreover those properties for solenoidal automorphisms are well discussed by Aoki *et al* [2] .

Recently some interesting properties of specification and weak specification from viewpoints of topological dynamics and ergodic theory are obtained by Lampart and Oprocha [6] and Yamamoto [13], respectively. In particular, Sakai *et al* [11] have studied weak specification from a viewpoint of geometric theory of dynamical systems, and characterized the diffeomorphisms with weak specification under the C^1-stable

assumption. Note that weak specification in here is called specification in [11].

In this paper we prove that C^1-generically, if a diffeomorphism f on a closed C^∞ manifold M satisfies weak specification on a locally maximal set $\Lambda \subset M$ then Λ is hyperbolic for f. As a corollary we obtain that C^1-generically, every diffeomorphism with weak specification is Anosov. More precisely, we prove the following theorem.

THEOREM 1.1. *There exists a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ such that if $f \in \mathcal{R}$ satisfies weak specification on a locally maximal set $\Lambda \subset M$ then Λ is hyperbolic for f.*

COROLLARY 1.2. *C^1-generically, every diffeomorphism with weak specification is Anosov.*

Let M be a closed C^∞ manifold, and let $\mathrm{Diff}(M)$ be the space of diffeomorphisms of M endowed with the C^1-topology. Denote by d the distance on M induced from a Riemannian metric $\| \cdot \|$ on the tangent bundle TM.

Recall that a closed invariant set Λ is called *hyperbolic* for f if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E \oplus F$ and there exist constants $C > 0, 0 < \lambda < 1$ such that $\|Df^n|_{E(x)}\| \leq C\lambda^n$ and $\|Df^{-n}|_{F(f^n(x))}\| \leq C\lambda^n$ for all $x \in \Lambda$ and $n \geq 0$. We say that f is *Anosov* if M is hyperbolic for f.

It is well known that if p is a hyperbolic periodic point of f with period k then the sets $W^s(p) = \{x \in M : f^{kn}(x) \to p$ as $n \to \infty\}$ and $W^u(p) = \{x \in M : f^{-kn}(x) \to p$ as $n \to \infty\}$ are C^1-injectively immersed submanifolds of M. Let p, q be hyperbolic periodic points of f. We say that p and q are *homoniclly related*, and write $p \sim q$ if $W^s(p)\overline{\pitchfork}W^u(q) \neq \phi$ and $W^u(p)\overline{\pitchfork}W^s(q) \neq \phi$. It is clear that if $p \sim q$ then index$(p) = $ index(q), that is, dim$W^s(p) = $ dim$W^u(q)$.

We say that f satisfies *Axiom A* if the periodic points are dense in the set $\Omega(f)$ of nonwandering points of f and $\Omega(f)$ is hyperbolic; and Axiom A diffeomorphism f satisfies *strong transversality condition* if $W^s(p)\overline{\pitchfork}W^u(q)$ for any $p, q \in \Omega(f)$.

For $\delta > 0$, a sequence of points $\{x_i\}_{i=a}^b$ in $M(-\infty \leq a < b \leq \infty)$ is called a *δ-pseudo-orbit* (or *δ-chain*) of f if $d(f(x_i), x_{i+1}) < \delta$ for all $a \leq i \leq b-1$. For a closed invariant set $\Lambda \subset M$, we say that f satisfies *shadowing* on Λ (or Λ is *shadowable* for f) if for every $\epsilon > 0$, there is $\delta > 0$ such that for any δ-pseudo-orbit $\{x_i\}_{i=a}^b \subset \Lambda$ of f $(-\infty \leq a < b \leq \infty)$, there is $y \in M$ satisfying $d(f^i(y), x_i) < \epsilon$ for all $a \leq i \leq b-1$. In the case, we say that the δ-pseudo orbit $\{x_i\}_{i=a}^b$ is *ϵ-shadowed* by the point y.

Notice that only δ-pseudo-orbit of f contained in Λ can be ϵ-shadowed, but the shadowing point $y \in M$ is not necessary contained in Λ.

We say that $f \in \text{Diff(M)}$ satisfies *weak specification* (or Λ admits a *weak specification* for f) on a closed invariant set $\Lambda \subset M$ if for every $\epsilon > 0$ there is a positive integer $N = N(\epsilon) > 0$ such that for any $k \geq 2$, for any k points $x_1, x_2, \cdots, x_k$ in Λ and any integers

$$a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k \quad \text{with} \quad a_i - b_{i-1} \geq N$$

there is $y \in M$ satisfying $d(f^j(y), f^j(x_i)) < \epsilon$ for all $a_i \leq j \leq b_i$, $1 \leq i \leq k - 1$. If such a point y can be chosen as a periodic point, then we say that f satisfies *specification* on Λ (or Λ admits *specification* for f). We say that f satisfies *weak specification* (or *specification*) if f satisfies weak specification (or specification) on M, respectively.

This means that given specified pieces $\{f^j(x_i)\}$ of orbits at different times, if there is enough time delay, then these pieces can be well approximated by the same pieces of the orbit of a point.

If f satisfies weak specification, then we can see that f is *topologically mixing*; that is, for any nonempty open sets $U, V \subset M$ there is an integer $N > 0$ such that for any $n \geq N$, $U \cap f^{-n}(V) \neq \emptyset$ (for more details, see [5]). By definition, it is clear that if f is topologically mixing, then it is *transitive*, that is, there is a dense orbit. A set Λ is said to be *locally maximal* if there is a compact neighborhood U of Λ such that $\Lambda = \cap_{n \in \mathbb{N}} f^n(U)$.

A toral automorphism $A : T^n \longrightarrow T^n$ is said to be *central spin automorphism* if A has unitary eigenvalues, and on the eigenspace of unitary eigenvalues, the associated linear map is an isometry with an appropriate metric. It is proved by Lind [8] that ergodic central spin toral automorphisms satisfy weak specification. However we see that ergodic central spin toral automorphisms do not satisfy shadowing. In fact, we can show that the ergodic central spin toral automorphims given by Lind in [8, page 99] does not satisfy shadowing.

Of course, shadowing does not imply weak specification. For example, let us consider a diffeomorphism on S^1 given by $f(\theta) = \theta + \varepsilon \sin(\theta)$ for $0 < \varepsilon < \frac{1}{2}$. Then $(0,0) \in S^1$ is a repelling fixed point of f and $(0,-1) \in S^1$ is an attracting fixed point of f. We can see that f does not satisfy weak specification since S^1 is not transitive for f, but it satisfy shadowing.

We say that a subset $\mathcal{R} \subset \text{Diff(M)}$ is *residual* if $\mathcal{R}$ contains the intersection of a countable family of open and dense subsets of Diff(M); in this case $\mathcal{R}$ is dense in Diff(M). A dynamic property P is said to be C^1-*generic* if P holds for all diffeomorphisms which belong to some residual subset of Diff(M).

The notion of shadowing is closely related to the notion of hyperbolicity. For example, it is well known that the C^1 interior of the set of diffeomorphisms satisfying shadowing is exactly the same as the set of diffeomorphisms satisfying both Axiom A and the strong transversality condition (i.e., structurally stable).

However the following conjecture posed by Abdenur and Díaz in [1] is still open.

CONJECTURE 1.3. C^1 generically, shadowing implies structural stability. That is, there is a residual set $\mathcal{R} \subset \mathrm{Diff}(M)$ such that if $f \in \mathcal{R}$ satisfies shadowing then it is structurally stable.

In [1], the structurally stable diffeomorphism is called the hyperbolic diffeomorphism. It is known that if f is Anosov then it is structurally stable; and if f is structurally stable then it satisfies shadowing. In general, a diffeomorphism with weak specification is not Anosov as we see in the case of ergodic central spin automorphisms on torus. However our theorem shows that every C^1 generic diffeomorphism with weak specification is Anosov.

2. Proof of Theorem 1.1

Combining the Kupka-Smale theorem with Pugh's closing lemma, we obtain a residual set $\mathcal{R}_1$ of $\mathrm{Diff}^1(\mathrm{M})$ such that for any $f \in \mathcal{R}_1$ and any closed invariant set $\Lambda \subset M$ which admits weak specification, all periodic points of f are hyperbolic and all their invariant manifolds are transverse, and the periodic points of f are dense in Λ.

First we see that if $f \in \mathcal{R}_1$ satisfies weak specification on a closed invariant set $\Lambda \subset M$ then every periodic point in Λ has the same index; that is, $W^s(p) \cap W^u(q) \neq \emptyset$ for any periodic points $p, q \in \Lambda$. This is a direct consequence of [11, Lemma 2.1].

Next, take a countable basis $\beta = \{U_n\}_{n \in \mathbb{N}}$ of M such that the union of two elements in β belongs to β. For each $U_n \in \beta$, we define by $\mathcal{H}_n$ the set of all diffeomorphisms f such that f has a C^1 neighborhood $\mathcal{U}(f)$ of f with the following properties: for any $g \in \mathcal{U}(f)$, if the maximal g-invariant set in U_n, $\Lambda_g(U_n) = \bigcap_{i \in \mathbb{Z}} g^i(U_n)$, is transitive for g and $\Lambda_g(U_n)$ is non-hyperbolic for g, then g has two hyperbolic periodic points in $\Lambda_g(U_n)$ with different indices. Then it is clear that $\mathcal{H}_n$ is open in $\mathrm{Diff}(\mathrm{M})$ for each $n \in \mathbb{N}$.

To show that $\mathcal{H}_n$ is dense in $\mathrm{Diff}(\mathrm{M})$ for each $n \in \mathbb{N}$, we suppose that there is $f \in \mathrm{Diff}(\mathrm{M}) - \overline{\mathcal{H}_n}$. Then we can find a C^1 neighborhood $\mathcal{U}(f)$

of f with $\mathcal{U}(f) \cap \mathcal{H}_n = \emptyset$. For any $g \in \mathcal{U}(f)$ and any C^1 neighborhood $\mathcal{V}(g) \subset \mathcal{U}(f)$ of g, we can take $h \in \mathcal{V}(g)$ such that $\Lambda_h(U_n) = \bigcap_{i \in \mathbb{Z}} h^i(U_n)$ is transitive for h, $\Lambda_h(U_n)$ is non-hyperbolic for h and every hyperbolic periodic point in $\Lambda_h(U_n)$ has the same index. This means that for any $g \in \mathcal{U}(f)$, $\Lambda_g(U_n)$ is transitive and non-hyperbolic, and every hyperbolic periodic point in $\Lambda_g(U_n)$ has the same index. Consequently we have that $\Lambda_f(U_n)$ is robustly transitive (i.e., there is a C^1 neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, $\Lambda_g(U_n)$ is transitive for g), every periodic point in $\Lambda_g(U_n)$ is hyperbolic and has the same index. In the process of the proof of Theorem 1.1 in [9], we can see that such $\Lambda_f(U_n)$ must be hyperbolic (see also Theorem 1.3 in [11]). This contradicts to the above fact that $\Lambda_f(U_n)$ should be non-hyperbolic. Consequently we have proved that $\mathcal{H}_n$ is dense in Diff(M) for each $n \in \mathbb{N}$.

Put

$$\mathcal{R} = \mathcal{R}_1 \bigcap (\bigcap_{n=1}^{\infty} \mathcal{H}_n).$$

Then $\mathcal{R}$ is a desired residual subset of Diff(M).

Let $f \in \mathcal{R}$, and let $\Lambda \subset M$ be a locally maximal set which admits a weak specification. Then there exists a basic element $U_{n_0} \in \beta$ such that $\Lambda = \bigcap_{i \in \mathbb{Z}} f^i(U_{n_0})$.

Suppose that Λ is not hyperbolic for f. Since $f \in \mathcal{H}_{n_0}$ and $\Lambda = \Lambda_f(U_{n_0})$ is transitive for f, we can see that f has two hyperbolic periodic points in Λ with different indices. This is a contradiction to the fact that $f \in \mathcal{R}_1$. The contradiction shows that Λ is hyperbolic for f, and so completes the proof of Theorem 1.1.

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