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A DETERMINANT FORMULA FOR THE RELATIVE CONGRUENT ZETA FUNCTION IN CYCLOTOMIC FUNCTION FIELDS

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Abstract. In this paper we give a determinant formula for the relative congruent zeta function in cyclotomic function fields.

1. Introduction

Let $\mathrm{k} = \mathbb{F}_q(T)$ be the rational function field over the finite field $\mathbb{F}_q$ and $\mathbb{A} = \mathbb{F}_q[T]$. Let $\infty = (1/T)$ be the prime of k associated to $1/T$, which is called the infinite prime of k. Write $\mathbb{A}^+ = \{1 \neq M \in \mathbb{A} : M \text{ is monic}\}$ and $\mathbb{A}^+_{\mathrm{irr}} = \{P \in \mathbb{A}^+ : P \text{ is irreducible}\}$ for simplicity. For any $M \in \mathbb{A}^+$, write K_M for the Mth cyclotomic function field and K_M^+ for the maximal real subfield of K_M.

It is known that there exists a polynomial $P_{K_M}(X) \in \mathbb{Z}[X]$ such that

$$\zeta(s, K_M) = \frac{P_{K_M}(q^{-s})}{(1-q^{-s})(1-q^{1-s})},$$

where $\zeta(s, K_M)$ is the congruence zeta function of K_M, and $P_{K_M}(1)$ is equal to the divisor class number h_{K_M} of K_M. Let $\zeta^{(-)}(s, K_M) = \zeta(s, K_M)/\zeta(s, K_M^+)$, called the relative congruence zeta function of K_M and $P^{(-)}_{K_M}(X) = P_{K_M}(X)/P_{K_M^+}(X)$, where $P_{K_M^+}(X)$ is the polynomial corresponding to the congruence zeta function $\zeta(s, K_M^+)$ for K_M^+. Then we have $\zeta^{(-)}(s, K_M) = P^{(-)}_{K_M}(q^{-s})$ and $P^{(-)}_{K_M}(1) = h^-_{K_M}$, where $h^-_{K_M} = h_{K_M}/h_{K_M^+}$ is the relative divisor class number of K_M.

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In recent paper [5], Shiomi has expressed the polynomial $P^{(-)}_{K_M}(X)$ as determinant of matrix $D^{(-)}_M(X)$ up to some polynomial $J^{(-)}_{K_M}(X)$. Since $h^-_{K_M} = P^{(-)}_{K_M}(1)$, this determinant formula for $P^{(-)}_{K_M}(X)$ can be regarded as generalization of that for relative class number $h^{(-)}_{K_M}$ ([4], [2], [1]).

The aim of this paper is to give another determinant formula for the polynomial $P^{(-)}_{K_M}(X)$. In contrast to the fact that in the determinant in [5], the matrix $D^{(-)}_M(X)$ is a product of matrices with entries containing polynomials with complex coefficients, the entries of matrices in our determinant formula are polynomials with integer coefficients.

2. Zeta and L-functions

Let F be a finite extension of k which is contained in some cyclotomic extension K_M. Let $N \in \mathbb{A}^+$ be the conductor of F, that is, K_N is the smallest cyclotomic function field containing F. Let $\zeta(s, F)$ be the congruence zeta function of F given by

$$\zeta(s,F) = \prod_{\mathfrak{p}} \Big(1 - \frac{1}{N\mathfrak{p}^s}\Big)^{-1},$$

where $\mathfrak{p}$ runs over all primes of F. It is well known that there exists a polynomial $P_F(X) \in \mathbb{Z}[X]$ of degree $2g$, where g is the genus of F, such that

$$\zeta(s,F) = \frac{P_F(q^{-s})}{(1-q^{-s})(1-q^{1-s})}. \tag{2.1}$$

Moreover, the polynomial $P_F(X)$ satisfies $P_F(0) = 1$ and $P_F(1) = h_F$, where h_F is the divisor class number of F. Let $\mathcal{O}_F$ be the integral closure of $\mathbb{A}$ in F and $\zeta(s, \mathcal{O}_F)$ be the zeta function of $\mathcal{O}_F$ given by

$$\zeta(s,\mathcal{O}_F) = \prod_{\mathfrak{p}} \Big(1 - \frac{1}{N\mathfrak{p}^s}\Big)^{-1},$$

where $\mathfrak{p}$ runs over all prime ideals of $\mathcal{O}_F$. Then the functions $\zeta(s, F)$ and $\zeta(s, \mathcal{O}_F)$ satisfy the following equality

$$\zeta(s,F) = \zeta(s,\mathcal{O}_F)(1-q^{-s})^{-[F^+:\mathrm{k}]}. \tag{2.2}$$

Let X_F be the group of primitive Dirichlet characters of $\mathbb{A}$ associated to F. For $\chi \in X_F$, let $L(s,\chi)$ be the L-function associated to χ given by

$$L(s,\chi) = \prod_{P\in\mathbb{A}^+_{\mathrm{irr}}} \left(1-\chi(P)q^{-s\deg P}\right)^{-1}.$$

Then we have

$$\zeta(s,\mathcal{O}_F) = \prod_{\chi\in X_F} L(s,\chi). \tag{2.3}$$

Let $\chi_0 \in X_F$ denote the trivial character. Since $L(s,\chi_0) = (1-q^{1-s})^{-1}$, from (2.1), (2.2) and (2.3), we get

$$\prod_{\chi_0\neq\chi\in X_F} L(s,\chi) = (1-q^{-s})^{[F^+:\mathrm{k}]-1} P_F(q^{-s}). \tag{2.4}$$

For any $\chi \in X_F$, let $F_\chi \in \mathbb{A}^+$ be the conductor of χ and $\tilde{\chi} = \chi\circ\pi_\chi$, where $\pi_\chi : (\mathbb{A}/N\mathbb{A})^* \to (\mathbb{A}/F_\chi\mathbb{A})^*$ is the canonical homomorphism. Then we have

$$L(s,\tilde{\chi}) = L(s,\chi) \prod_{Q\in\mathbb{A}^+_{\mathrm{irr}},\, Q|N} \left(1-\chi(Q)q^{-s\deg Q}\right). \tag{2.5}$$

Thus, by (2.4) and (2.5), we have

$$\prod_{\chi_0\neq\chi\in X_F} L(s,\tilde{\chi}) = (1-q^{-s})^{[F^+:\mathrm{k}]-1} P_F(q^{-s}) J_F(q^{-s}), \tag{2.6}$$

where $J_F(X)$ is the polynomial given by

$$J_F(X) = \prod_{\chi_0\neq\chi\in X_F,} \prod_{Q\in\mathbb{A}^+_{\mathrm{irr}},\, Q|N} \left(1-\chi(Q)X^{\deg Q}\right).$$

Let $\zeta^{(-)}(s,F) = \zeta(s,F)/\zeta(s,F^+)$ be the relative congruence zeta function of F and $P_F^{(-)}(X) = P_F(X)/P_{F^+}(X)$. Then by (2.1) and (2.4), we have

$$\zeta^{(-)}(s,F) = P_F^{(-)}(q^{-s}) = \prod_{\chi\in X_F^-} L(s,\chi)$$

where $X_F^- = X_F \backslash X_{F^+}$. Finally, by (2.5), we have

$$\prod_{\chi\in X_F^-} L(s,\tilde{\chi}) = P_F^{(-)}(q^{-s}) J_F^{(-)}(q^{-s}), \tag{2.7}$$

where

$$J_F^{(-)}(X) = \prod_{\chi\in X_F^-,} \prod_{Q\in\mathbb{A}^+_{\mathrm{irr}},\, Q|N} \left(1-\chi(Q)X^{\deg Q}\right).$$

Finally we give some remarks on the polynomial $J_F^{(-)}(X)$. They satisfy the following equality ([5, Proposition 3.1])

$$J_F^{(-)}(X) = \prod_{Q \in \mathbb{A}^+_{\mathrm{irr}},\, Q|N} \frac{(1 - X^{f_Q \deg Q})^{g_Q}}{(1 - X^{f_Q^+ \deg Q})^{g_Q^+}},$$

where f_Q, f_Q^+ are the residue class degrees of Q in $F/\mathrm{k}, F^+/\mathrm{k}$, respectively and g_Q, g_Q^+ are the number of primes over Q in F, F^+, respectively. Hence we see that $J_F^{(-)}(X) \in \mathbb{Z}[X]$ and in particular, $J_F^{(-)}(X) = 1$ if N is a power of some $Q \in \mathbb{A}^+_{\mathrm{irr}}$.

3. A determinants formula for $P_{K_M}^{(-)}(X)$

From now on, we consider the case $F = K_M$, the Mth cyclotomic function field for $M \in \mathbb{A}^+$ Let $\mathcal{R}_M = (\mathbb{A}/M\mathbb{A})^*/\mathbb{F}_q^*$. For $\alpha \in (\mathbb{A}/M\mathbb{A})^*$, there exists a unique polynomial $A_\alpha \in \mathbb{A}$ such that $\deg A_\alpha < \deg M$ and $A_\alpha + M\mathbb{A} = \alpha$. Write $\mathrm{sgn}_M(\alpha) \in \mathbb{F}_q^*$ for the leading coefficient of A_α, $\deg_M(\alpha) := \deg A_\alpha$. For $\sigma \in \mathrm{Gal}(K_M/\mathrm{k})$, let α_σ be the element of $(\mathbb{A}/M\mathbb{A})^*$, which corresponds to σ under the canonical isomorphism $\mathrm{Gal}(K_M/\mathrm{k}) \cong (\mathbb{A}/M\mathbb{A})^*$. We define a function $f : \mathrm{Gal}(K_M/k) \to \mathbb{Z}[X]$ by

$$f(\sigma) := \begin{cases} X^{\deg_M(\alpha_\sigma)}, & \text{if } \mathrm{sgn}_M(\alpha_\sigma) = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Now we define the matrix

$$E_{K_M}^{(-)}(X) = \left(f(\sigma\tau^{-1}) - f(\sigma(\tau/\mathrm{sgn}_M(\alpha_\tau))^{-1})\right)_{\sigma,\tau},$$

where $\sigma, \tau \in \mathrm{Gal}(K_M/\mathrm{k})$ with $\mathrm{sgn}_M(\alpha_\sigma) \neq 1, \mathrm{sgn}_M(\alpha_\tau) \neq 1$. Then we have the following main theorem, which gives a determinant formula for the relative congruence zeta functions in cyclotomic function fields.

THEOREM 3.1. *With notations as above, we have*

$$\det E_{K_M}^{(-)}(X) = P_{K_M}^{(-)}(X) J_{K_M}^{(-)}(X).$$

Proof. For $\chi \in X^{-}_{K_M}$, as in the proof of [5, Theorem 3.1] or [3, Lemma 3], we have

$$\begin{aligned} L(s,\tilde{\chi}) &= \sum_{\alpha \in (\mathbb{A}/M\mathbb{A})^*, \operatorname{sgn}_M(\alpha)=1} \tilde{\chi}(\alpha) q^{-\deg_M(\alpha)s} \\ &= \sum_{\sigma \in \mathrm{Gal}(K_M/\mathrm{k})} \tilde{\chi}(\alpha_\sigma) f(\sigma)(q^{-s}). \end{aligned}$$

Noting that under the isomorphism $\mathrm{Gal}(K_M/\mathrm{k}) \cong (\mathbb{A}/M\mathbb{A})^*$, $\{\sigma \in \mathrm{Gal}(K_M/\mathrm{k}) | \operatorname{sgn}_M(\alpha_\sigma) = 1\}$ represents $\mathrm{Gal}(K_M/\mathrm{k})/\mathrm{Gal}(K^{+}_M/\mathrm{k})$, we have, from [1, Corollary 2.2(ii)],

$$\prod_{\chi \in X^{-}_{K_M}} L(s,\tilde{\chi}) = \det \left(f(\sigma\tau^{-1})(q^{-s}) - f(\sigma(\tau/\operatorname{sgn}_M(\alpha_\tau))^{-1})(q^{-s}) \right)_{\sigma,\tau}.$$

Now from (2.7), we get the result. □

Example 3.2. *For $q = 3$, $M = T^2+1$, since M is irreducible, we have $\det E^{(-)}_{K_M}(X) = P^{(-)}_{K_M}(X)$. Then we have $\{A_{\alpha_\sigma} | \sigma \in \mathrm{Gal}(K_M/\mathrm{k}), \operatorname{sgn}_M(\alpha_\sigma) \neq 1\} = \{-1, -T, -T+1, -T+2\}$ and so*

$$\begin{aligned} P^{(-)}_{K_M}(X) &= \det E^{(-)}_{K_M}(X) \\ &= \begin{vmatrix} 1 & -X & X & X \\ X & 1 & X & -X \\ X & X & 1 & X \\ X & -X & -X & 1 \end{vmatrix} \\ &= 1 - 2X^2 + 9X^4. \end{aligned}$$

Thus we have the same result as in [5, Example 5.1].

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