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IDENTITIES INVOLVING TRIBONACCI NUMBERS

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Abstract. The $kt + r$ subscripted tribonacci numbers will be expressed by three k step apart tribonacci numbers for any $0 < r < k \in \mathbb{Z}$.

1. Introduction

A tribonacci sequence $\{T_n\}_{n \geq 0}$ is a kind of generalization of fibonacci sequence that begins with three initials $T_0 = 0$ and $T_1 = T_2 = 1$ and each next is the sum of the previous three terms. So, $T_{n+3} = T_{n+2} + T_{n+1} + T_n$, and the first few terms are $0, 1, 1, 2, 4, 7, \cdots$. The sequence can be extended to negative n that $T_{-1} = 0,\ T_{-2} = 1,\ T_{-3} = -1,\ T_{-4} = 0,\ T_{-5} = 2,\ T_{-6} = -3, \cdots$.

A number of properties of the sequence were studied by many researchers ([2], [3], [4] and [5]). In particular, by means of generating matrices, it was proved in [4] that the $4t$ subscripted tribonacci numbers satisfy $T_{4(t+1)} = 11T_{4t} + 5T_{4(t-1)} + T_{4(t-2)}$, and the sum is $\sum_{t=0}^{n} T_{4t} = (T_{4n+4} + 6T_{4n} + T_{4n-4} - T_4)/T_4^2$.

In this work we shall generalize the identities about $4t$ subscripted tribonacci numbers T_{4t} to any T_{4t+r} $(1 \leq r \leq 4)$, so that the properties in [4] are regarded as a case of $r = 4$. Moreover the identities will be extended to any $kt + r$ subscripted tribonacci T_{kt+r} for any $1 \leq r \leq k \in \mathbb{Z}$. One of our main theorem is to express T_{kt+r} by T_{2k+r}, T_{k+r} and T_r, which are k step apart terms.

2. Tribonacci table

For $k \in \mathbb{N}$, when we say k columns tribonacci table we mean a rectangle shape having k columns that consists of the all tribonacci numbers

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from T_1 in order.

$$\begin{array}{cccc} T_1 & T_2 & \cdots & T_k \\ T_{k+1} & T_{k+2} & \cdots & T_{2k} \\ T_{2k+1} & T_{2k+2} & \cdots & T_{3k} \\ \vdots & & \cdots & \vdots \end{array}$$

We shall investigate a third order linear recurrence $T_n = a_k T_{n-k} + b_k T_{n-2k} + T_{n-3k}$ for tribonacci numbers with some $a_k, b_k \in \mathbb{Z}$.

LEMMA 2.1. *$T_n = 3T_{n-2} + T_{n-4} + T_{n-6}$, $T_n = 7T_{n-3} - 5T_{n-6} + T_{n-9}$, and $T_n = 11T_{n-4} + 5T_{n-8} + T_{n-12}$ for any $n \in \mathbb{Z}$.*

Proof. Observe that $T_6 = 13 = 3 \cdot 4 + 1 = 3T_4 + T_2 + T_0$. If we assume $T_s = 3T_{s-2} + T_{s-4} + T_{s-6}$ for all $s < n$ then

$$\begin{aligned} T_n &= T_{n-1} + T_{n-2} + T_{n-3} \\ &= 3(T_{n-3} + T_{n-4} + T_{n-5}) + (T_{n-5} + T_{n-6} + T_{n-7}) \\ &\quad + (T_{n-7} + T_{n-8} + T_{n-9}) \\ &= 3T_{n-2} + T_{n-4} + T_{n-6}. \end{aligned}$$

Similar to this, we notice $T_9 = 81 = 7 \cdot 13 - 5 \cdot 2 + 0 = 7T_6 - 5T_3 + T_0$. If we assume $T_s = 7T_{s-3} - 5T_{s-6} + T_{s-9}$ for all $s < n$, then the induction hypothesis proves $T_n = 7T_{n-3} - 5T_{n-6} + T_{n-9}$.

Analogously, since $T_{12} = 504 = 11 \cdot 44 + 5 \cdot 4 + 0 = 11T_8 + 5T_4 + T_0$, the identity $T_n = 11T_{n-4} + 5T_{n-8} + T_{n-12}$ can be proved immediately by induction. □

Note that the identity $T_{4(t+1)} = 11T_{4t} + 5T_{4(t-1)} + T_{4(t-2)}$ in [4] is a special case of $T_n = 11T_{n-4} + 5T_{n-8} + T_{n-12}$ in Lemma 2.1 when $4 \mid n$. We extend Lemma 2.1 to any integer $1 \le k \le 10$.

THEOREM 2.2. *Let $1 \le k \le 10$. Then the third order recurrence $T_n = a_k T_{n-k} + b_k T_{n-2k} + T_{n-3k}$ of T_n holds with the following (a_k, b_k).*

k	(a_k, b_k)	$T_n =$	$a_k\ T_{n-k}$	$+b_k T_{n-2k}$	$+T_{n-3k}$
1	$(1,1)$	$T_n =$	T_{n-1}	$+\ T_{n-2}$	$+T_{n-3}$
2	$(3,1)$	$T_n =$	$3\ T_{n-2}$	$+\ T_{n-4}$	$+T_{n-6}$
3	$(7,-5)$	$T_n =$	$7\ T_{n-3}$	$-5\ T_{n-6}$	$+T_{n-9}$
4	$(11,5)$	$T_n =$	$11\ T_{n-4}$	$+5\ T_{n-8}$	$+T_{n-12}$
5	$(21,1)$	$T_n =$	$21\ T_{n-5}$	$+\ T_{n-10}$	$+T_{n-15}$
6	$(39,-11)$	$T_n =$	$39\ T_{n-6}$	$-11\ T_{n-12}$	$+T_{n-18}$
7	$(71,15)$	$T_n =$	$71\ T_{n-7}$	$+15\ T_{n-14}$	$+T_{n-21}$
8	$(131,-3)$	$T_n =$	$131\ T_{n-8}$	$-3\ T_{n-16}$	$+T_{n-24}$
9	$(241,-23)$	$T_n =$	$241\ T_{n-9}$	$-23\ T_{n-18}$	$+T_{n-27}$
10	$(443,41)$	$T_n =$	$443\ T_{n-10}$	$+41\ T_{n-20}$	$+T_{n-30}$

Proof. Clearly $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ shows $(a_1, b_1) = (1, 1)$. And Lemma 2.1 shows $(a_k, b_k) = (3, 1)$, $(7, -5)$ and $(11, 5)$ for $k = 2, 3, 4$ respectively.

Let $n = kt + r$ $(1 \le r \le k)$ and $5 \le k \le 10$. In order to express $T_{kt+r} = a_k T_{k(t-1)+r} + b_k T_{k(t-2)+r} + T_{k(t-3)+r}$, we shall consider k columns tribonacci tables. Let us begin with $k = 5$.

1	1	2	4	7
13	24	44	81	149
274	504	927	1705	3136
5768	10609	19513	35890	···

Then it can be observed that, for instance

$$\begin{cases} T_{16} = 5768 = (21)274 + 13 + 1 = (21)T_{11} + T_6 + T_1 \\ T_{17} = 10609 = (21)504 + 24 + 1 = (21)T_{12} + T_7 + T_2 \\ T_{18} = 19513 = (21)927 + 44 + 2 = (21)T_{13} + T_8 + T_3 \end{cases}$$

Thus, by assuming $T_s = (21)T_{s-5} + T_{s-10} + T_{s-15}$ for all $s < n$, the induction hypothesis gives rise to

$$\begin{aligned} T_n &= T_{n-1} + T_{n-2} + T_{n-3} \\ &= (21)(T_{n-6} + T_{n-7} + T_{n-8}) + (T_{n-11} + T_{n-12} + T_{n-13}) \\ &\quad + (T_{n-16} + T_{n-17} + T_{n-18}) \\ &= (21)T_{n-5} + T_{n-10} + T_{n-15}, \end{aligned}$$

so $(a_5, b_5) = (21, 1)$. Moreover from the 6 columns tribonacci table

1	1	2	4	7	13
24	44	81	149	274	504
927	1705	3136	5768	10609	19513
35890	66012	121415	223317	410744	···

we can observe that, for instance

$$\begin{cases} T_{19} = 35890 = (39)927 - (11)24 + 1 = (39)T_{13} - 11T_7 + T_1 \\ T_{20} = 66012 = (39)1705 - (11)44 + 1 = (39)T_{14} - 11T_8 + T_2 \\ T_{21} = 121415 = (39)3136 - (11)81 + 2 = (39)T_{15} - 11T_9 + T_3 \end{cases}$$

By assuming $T_s = (39)T_{s-6} - 11T_{s-12} + T_{s-18}$ for all $s < n$, we have

$$\begin{aligned} T_n &= T_{n-1} + T_{n-2} + T_{n-3} \\ &= (39)(T_{n-7} + T_{n-8} + T_{n-9}) - 11(T_{n-13} + T_{n-14} + T_{n-15}) \\ &\quad + (T_{n-19} + T_{n-20} + T_{n-21}) \\ &= (39)T_{n-6} - 11T_{n-12} + T_{n-18}, \end{aligned}$$

so $(a_6, b_6) = (39, -11)$. Now from the 7 columns tribonacci table

1	1	2	4	7	13	24
44	81	149	274	504	927	1705
3136	5768	10609	19513	35890	66012	121415
223317	410744	755476	1389537	2555757	4700770	$\cdots$

we can find the identities that, for instance

$$\begin{cases} T_{22} = 223317 = (71)3136 + (15)44 + 1 = (71)T_{15} + 15T_8 + T_1 \\ T_{23} = 410744 = (71)5768 + (15)81 + 1 = (71)T_{16} + 15T_9 + T_2 \\ T_{24} = 755476 = (71)10609 + (15)149 + 2 = (71)T_{17} + 15T_{10} + T_3, \end{cases}$$

so induction proves that $T_n = a_7T_{n-7} + b_7T_{n-14} + T_{n-21}$ with $(a_7, b_7) = (71, 15)$.

Continuing this way, the 8 columns tribonacci table, as well as 9 columns and 10 columns tribonacci table provides

$$\begin{cases} T_{25} = 1389537 = (131)10609 - (3)81 + 1 = (131)T_{17} - 3T_9 + T_1 \\ T_{26} = 2555757 = (131)19513 - (3)149 + 1 = (131)T_{18} - 3T_{10} + T_2 \\ T_{27} = 4700770 = (131)35890 - (3)274 + 2 = (131)T_{19} - 3T_{11} + T_3, \end{cases}$$

$$\begin{cases} T_{28} = 8646064 = (241)35890 - (23)149 + 1 = (241)T_{19} - (23)T_{10} + T_1 \\ T_{29} = 15902591 = (241)66012 - (23)274 + 1 = (241)T_{20} - (23)T_{11} + T_2 \\ T_{30} = 29249425 = (241)121415 - (23)504 + 2 = (241)T_{21} - (23)T_{12} + T_3, \end{cases}$$

and

$$\begin{cases} T_{31} = 53798080 = (443)121415 + (41)274 + 1 = (443)T_{21} + (41)T_{11} + T_1 \\ T_{32} = 98950096 = (443)223317 + (41)504 + 1 = (443)T_{22} + (41)T_{12} + T_2 \\ T_{33} = 181997601 = (443)410744 + (41)927 + 2 = (443)T_{23} + (41)T_{13} + T_3, \end{cases}$$

respectively. Therefore the observations and mathematical induction prove that the coefficients (a_k, b_k) for $8 \le k \le 10$ satisfying $T_n = a_kT_{n-k} + b_kT_{n-2k} + T_{n-3k}$ are equal to $(131, -3)$, $(241, -23)$, $(443, 41)$, respectively. □

We note that the subscript n of T_n could be negative, for example, in 6 columns tribonacci table, $T_{15} = (39)T_9 - 11T_3 + T_{-3} = (39)81 - 11(2) - 1 = 3136$.

A sequence $\{t_n\}$ is called a tribonacci type if it satisfies $t_n + t_{n+1} + t_{n+2} = t_{n+3}$ with any initials t_1, t_2 and t_3.

THEOREM 2.3. *For $1 \le k \le 10$, let (a_k, b_k) be the coefficient of the third order recurrence $T_n = a_kT_{n-k} + b_kT_{n-2k} + T_{n-3k}$.*

(1) *$\{a_k\}$ is a tribonacci type sequence $a_i + a_{i+1} + a_{i+2} = a_{i+3}$ with initials $a_1 = 1$, $a_2 = 3$ and $a_3 = 7$, while $\{b_k\}$ satisfies $b_i - b_{i+1} - b_{i+2} = b_{i+3}$ with $b_1 = b_2 = 1$ and $b_3 = -5$, for $1 \le i \le 7$.*
(2) *Moreover $a_k = 3T_k - T_{k-6}$ and $b_k = -a_{-k}$ for $1 \le k \le 10$.*

Proof. By Theorem 2.2, $\{a_k\}_{k=1}^{10} = \{1, 3, 7, 11, 21, 39, 71, 131, 241, 443\}$ and $\{b_k\}_{k=1}^{10} = \{1, 1, -5, 5, 1, -11, 15, -3, -23, 41\}$. So it is easy to see that $a_i + a_{i+1} + a_{i+2} = a_{i+3}$ and $b_i - b_{i+1} - b_{i+2} = b_{i+3}$ for $1 \le i \le 7$.

Moreover by means of tribonacci numbers T_i, we notice

$$a_1 = 1 = 3T_1 - T_{-5},\ a_2 = 3 = 3T_2 - T_{-4},\ a_3 = 7 = 3T_3 - T_{-3},$$

and $a_4 = a_3 + a_2 + a_1 = 3T_4 - T_{-2}$, etc. So $a_k = 3T_k - T_{k-6}$ for $1 \le k \le 10$. By considering T_k with negative k, the tribonacci type sequence $\{a_k\}$ can be extended to any $k \in \mathbb{Z}$, as follows.

k	$\cdots -10$	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	$3\cdots$
a_k	$\cdots -41$	23	3	-15	11	-1	-5	5	-1	-1	3	1	3	$7\cdots$

Then by comparing $\{a_k\}_{k=-1}^{-10} = \{-1, -1, 5, -5, -1, 11, -15, 3, 23, -41\}$ with $\{b_k\}_{k=1}^{10}$, we find that $b_k = -a_{-k}$ for $1 \le k \le 10$. □

3. The third order linear recurrence of T_n

We shall generalize the findings in Section 2 for $1 \le k \le 10$ to any integer k.

THEOREM 3.1. *Let $a_k = 3T_k - T_{k-6}$ and $b_k = -a_{-k}$ for any $k > 0$. Then any nth tribonacci number satisfies $T_n = a_k T_{n-k} + b_k T_{n-2k} + T_{n-3k}$ for every $k < n$.*

Proof. It is due to Theorem 2.3 if $1 \le k \le 10$. Since $a_k = 3T_k - T_{k-6}$ for all k, $\{a_k\}$ is a tribonacci type sequence because

$$\begin{aligned} a_k + a_{k+1} + a_{k+2} &= (3T_k - T_{k-6}) + (3T_{k+1} - T_{k-5}) + (3T_{k+2} - T_{k-4}) \\ &= 3(T_k + T_{k+1} + T_{k+2}) - (T_{k-6} + T_{k-5} + T_{k-4}) \\ &= 3T_{k+3} - T_{k-3} = a_{k+3}. \end{aligned}$$

Similarly, since $b_k = -a_{-k}$ for all k, $\{b_k\}$ satisfies

$$\begin{aligned} b_k - b_{k+1} - b_{k+2} &= -a_{-k} + a_{-(k+1)} + a_{-(k+2)} \\ &= -(a_{-k-3} + a_{-k-2} + a_{-k-1}) + a_{-k-1} + a_{-k-2} \\ &= -a_{-k-3} = b_{k+3}. \end{aligned}$$

We now suppose that the three order recurrence $a_i T_{n-i} + b_i T_{n-2i} + T_{n-3i} = T_n$ hold for all $i \le k$. Since

$$\begin{aligned} T_{n-(k-2)} &= T_{n-(k-1)} + T_{n-k} + T_{n-(k+1)}, \\ T_{n-2(k-2)} &= 3T_{n-2(k-1)} + T_{n-2k} + T_{n-2(k+1)}, \\ T_{n-3(k-2)} &= 7T_{n-3(k-1)} - 5T_{n-3k} + T_{n-3(k+1)} \end{aligned}$$

by Lemma 2.1, the mathematical induction with long calculations proves that

$$\begin{aligned} & a_{k+1}T_{n-(k+1)} + b_{k+1}T_{n-2(k+1)} + T_{n-3(k+1)} \\ & = (a_k + a_{k-1} + a_{k-2})(-T_{n-k} - T_{n-(k-1)} + T_{n-(k-2)}) \\ & \quad + (-b_k - b_{k-1} + b_{k-2})(-3T_{n-2(k-1)} - T_{n-2k} + T_{n-2(k-2)}) \\ & \quad + (-7T_{n-3(k-1)} + 5T_{n-3k} + T_{n-3(k-2)}) = T_n. \end{aligned}$$

□

Theorem 3.1 provides a good way to find huge tribonacci numbers. For instance, for 50th tribonacci number T_{50}, we may choose any k, say $k = 11$, then $a_{11} = 3T_{11} - T_5 = 3(274) - 7 = 815$ and $b_{11} = -b_{10} - b_9 + b_8 = -21$, thus

$$T_{50} = a_{11}T_{50-11} + b_{11}T_{50-22} + T_{50-33} = 5,742,568,741,225,$$

a 13 digit integer. On the other hand, if we take $k = 12$ then $a_{12} = 3T_{12} - T_6 = 3 \cdot 504 - 13 = 1499$ and $b_{12} = -b_{11} - b_{10} + b_9 = -43$, so T_{50} can be obtained by $T_{50} = a_{12}T_{38} + b_{12}T_{26} + T_{14} = 5,742,568,741,225$.

Besides the expression of a_k by six step apart tribonacci numbers, more identities for a_k can be developed in terms of three successive tribonacci numbers.

Theorem 3.2. *Let k be any integer. Then*
$a_k = T_k + 2T_{k-1} + 3T_{k-2} = T_{k+1} + T_{k-1} + 2T_{k-2} = 3T_{k+1} - 2T_k - T_{k-1}$.
So $a_k = [3 \ \ -2 \ \ -1]\begin{bmatrix} T_{k+1} \\ T_k \\ T_{k-1} \end{bmatrix}$ *and* $a_{-k} = [3 \ \ -2 \ \ -1]\begin{bmatrix} T_{-(k-1)} \\ T_{-k} \\ T_{-(k+1)} \end{bmatrix}$.

Proof. Since

$$\begin{aligned} T_{k-6} &= T_{k-3} - T_{k-4} - (T_{k-2} - T_{k-3} - T_{k-4}) = 2T_{k-3} - T_{k-2} \\ &= 2(T_k - T_{k-1} - T_{k-2}) - T_{k-2} = 2T_k - 2T_{k-1} - 3T_{k-2}, \end{aligned}$$

it follows that

$$a_k = 3T_k - T_{k-6} = 3T_k - 2T_k + 2T_{k-1} + 3T_{k-2} = T_k + 2T_{k-1} + 3T_{k-2}.$$

Hence we have

$$\begin{aligned} a_k &= (T_{k+1} - T_{k-1} - T_{k-2}) + 2T_{k-1} + 3T_{k-2} = T_{k+1} + T_{k-1} + 2T_{k-2} \\ &= T_{k+1} + T_{k-1} + 2(T_{k+1} - T_k - T_{k-1}) = [3 \ \ -2 \ \ -1]\begin{bmatrix} T_{k+1} \\ T_k \\ T_{k-1} \end{bmatrix}. \end{aligned}$$

Therefore it follows immediately from

$$a_{-k} = T_{-k} + 2T_{-(k+1)} + 3T_{-(k+2)} = T_{-(k-1)} + T_{-(k+1)} + 2T_{-(k+2)}$$

$$= \quad 3T_{-(k-1)} - 2T_{-k} - T_{-(k+1)} = [3 \;\; -2 \;\; -1]\begin{bmatrix} T_{-(k-1)} \\ T_{-k} \\ T_{-(k+1)} \end{bmatrix}.$$

□

To find T_{-k}, we may refer to [1] that $T_{-k} = \begin{vmatrix} T_{k-1} & T_k \\ T_{k-2} & T_{k-1} \end{vmatrix}$. Thus for instance, $T_{-8} = \begin{vmatrix} T_7 & T_8 \\ T_6 & T_7 \end{vmatrix} = 4$ and $T_{-9} = \begin{vmatrix} T_8 & T_9 \\ T_7 & T_8 \end{vmatrix} = -8$. Similarly $T_{-10} = 5$. So we have $a_{-9} = [3 \;\; -2 \;\; -1]\begin{bmatrix} T_{-8} \\ T_{-9} \\ T_{-10} \end{bmatrix} = 23$, while $a_9 = [3 \;\; -2 \;\; -1]\begin{bmatrix} T_{10} \\ T_9 \\ T_8 \end{bmatrix} = 241$. Therefore, it follows, for example, $T_{30} = a_9T_{21} - a_{-9}T_{12} + T_3 = 29,249,425$.

For each $n \in \mathbb{Z}$, we define two sequences

$$P_n = T_n + T_{-n} \quad \text{and} \quad Q_n = T_n - T_{-n}.$$

Then it is easy to have the table

$n\|\cdots$	1	2	3	4	5	6	7	8	9	10	11	$12\cdots$
$T_n\|\cdots$	1	1	2	4	7	13	24	44	81	149	274	$504\cdots$
$T_{-n}\|\cdots$	0	1	−1	0	2	−3	1	4	−8	5	7	$-20\cdots$
$P_n\|\cdots$	1	2	1	*4*	*9*	10	25	48	**73**	154	281	**484**$\cdots$
$Q_n\|\cdots$	1	0	*3*	4	5	*16*	23	40	89	**144**	**267**	$524\cdots$

From the table, we notice $3+4+9=16$ and $73+144+267=484$, etc.

THEOREM 3.3. *Let $k \in \mathbb{Z}$. Then the sequences $\{P_k\}$ and $\{Q_k\}$ satisfy inter-recurrence relations $P_{k-3}+Q_{k-2}+Q_{k-1} = P_k$ and $Q_{k-3}+P_{k-2}+P_{k-1} = Q_k$ with initials $\{1,2,1\}$ for P_k and $\{1,0,3\}$ for Q_k. Furthermore*

(1) $\frac{1}{2}(P_k+Q_k) = T_k$ *and* $\frac{1}{2}(P_k-Q_k) = T_{-k}$

(2) $a_k + a_{-k} = [3 \;\; -2 \;\; -1]\begin{bmatrix} P_{k-1} \\ P_k \\ P_{k+1} \end{bmatrix} + 4(T_k + T_{k-2})$

Proof. It is easy to see that

$$\begin{aligned} P_k &= T_k + T_{-k} = (T_{k-3}+T_{k-2}+T_{k-1}) + (T_{-k+3} - T_{-k+2} - T_{-k+1}) \\ &= (T_{k-3}+T_{-(k-3)}) + (T_{k-2} - T_{-(k-2)}) + (T_{k-1} - T_{-(k-1)}) \\ &= P_{k-3} + Q_{k-2} + Q_{k-1}, \end{aligned}$$

and analogously $Q_{k-3}+P_{k-2}+P_{k-1} = Q_k$. Hence (1) is easy to see.

Now we note that

$$T_{k+1}+T_{-(k-1)} = (T_{k-2}+T_{k-1}+T_k)+T_{-(k-1)}$$
$$= (T_{k-2}+T_k)+(T_{k-1}+T_{-(k-1)}) = T_{k-2}+T_k+P_{k-1},$$

and similarly

$T_{k-1}+T_{-(k+1)} = (T_{k+1}-T_{k-2}-T_k)+T_{-(k+1)} = -(T_{k-2}+T_k)+P_{k+1}$.

Therefore together with Theorem 3.2, it follows that

$$a_k+a_{-k} = [3-2-1]\begin{bmatrix} T_{k+1}+T_{-(k-1)} \\ T_k+T_{-k} \\ T_{k-1}+T_{-(k+1)} \end{bmatrix}$$
$$= [3-2-1]\begin{bmatrix} T_{k-2}+T_k+P_{k-1} \\ P_k \\ -(T_{k-2}+T_k)+P_{k+1} \end{bmatrix}$$
$$= [3-2-1]\left(\begin{bmatrix} P_{k-1} \\ P_k \\ P_{k+1} \end{bmatrix}+\begin{bmatrix} T_{k-2}+T_k \\ 0 \\ -(T_{k-2}+T_k) \end{bmatrix}\right)$$
$$= [3-2-1]\left(\begin{bmatrix} P_{k-1} \\ P_k \\ P_{k+1} \end{bmatrix}+(T_k+T_{k-2})\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right)$$
$$= [3\ \ -2\ \ -1]\begin{bmatrix} P_{k-1} \\ P_k \\ P_{k+1} \end{bmatrix}+4(T_k+T_{k-2}).$$

□

Owing to Theorem 3.3 we first construct the table of P_n and Q_n recursively, and then obtain tribonacci numbers from the table. Indeed,

$k\vert\cdots$	10	11	12	13	14	15	16	17	$18\cdots$
$P_k\vert\cdots$	154	281	484	945	1714	3089	5824	10609	19410
$Q_k\vert\cdots$	7144	267	524	909	1696	3183	5712	10609	
$\frac{1}{2}(P_k+Q_k)\vert\cdots$					1705	3136	5768	10609	
$\frac{1}{2}(P_k-Q_k)\vert\cdots$					9	-47	56	0	

shows $T_{17}=\frac{1}{2}(P_{17}+Q_{17})=10609$ and $T_{-17}=\frac{1}{2}(P_{17}-Q_{17})=0$. Hence

$$a_{17}+a_{-17} = [3-2-1]\begin{bmatrix} P_{16} \\ P_{17} \\ P_{18} \end{bmatrix}+4(T_{15}+T_{17})$$
$$= [3-2-1]\begin{bmatrix} 5824 \\ 10609 \\ 19410 \end{bmatrix}+4(3136+10609) = 31824$$

where, it can be compared with $a_{17}=31553$ and $a_{-17}=-b_{17}=271$.

Theorem 3.4. *Let $n = kt + r$ with $1 \leq r \leq k < n$. Let (a_k, b_k) be the coefficients satisfying $T_n = a_k T_{n-k} + b_k T_{n-2k} + T_{n-3k}$.*

(1) *T_n is a linear combination of any three consecutive entries of rth column in the k columns tribonacci table.*
(2) *T_n is expressed by the first three terms T_{2k+r}, T_{k+r} and T_r of rth column.*
(3) *For $0 < i < t$, if $T_n = uT_{k(i+2)+r} + vT_{k(i+1)+r} + wT_{ki+r}$ with $u, v, w \in \mathbb{Z}$ then $T_n = (a_k u + v)T_{k(i+1)+r} + (b_k u + w)T_{ki+r} + uT_{k(i-1)+r}$.*

Proof. $T_n = a_k T_{n-k} + b_k T_{n-2k} + T_{n-3k}$ in Theorem 2.3 yields

$$\begin{aligned} T_{kt+r} &= a_k T_{k(t-1)+r} + b_k T_{k(t-2)+r} + T_{k(t-3)+r} \\ &= a_k(a_k T_{k(t-2)+r} + b_k T_{k(t-3)+r} + T_{k(t-4)+r}) + b_k T_{k(t-2)+r} + T_{k(t-3)+r} \\ &= (a_k^2 + b_k)T_{k(t-2)+r} + (a_k b_k + 1)T_{k(t-3)+r} + a_k T_{k(t-4)+r} \\ &= (a_k(a_k^2 + b_k) + (a_k b_k + 1))T_{k(t-3)+r} + (b_k(a_k^2 + b_k) + a_k)T_{k(t-4)+r} \\ &\quad + (a_k^2 + b_k)T_{k(t-5)+r}. \end{aligned}$$

Hence after i step $(0 < i < t)$, if we write

$$T_{kt+r} = uT_{k(i+2)+r} + vT_{k(i+1)+r} + wT_{ki+r}$$

with $u, v, w \in \mathbb{Z}$ then in the next step we have

$$\begin{aligned} T_{kt+r} &= u(a_k T_{k(i+1)+r} + b_k T_{ki+r} + T_{k(i-1)+r}) + vT_{k(i+1)+r} + wT_{ki+r} \\ &= (a_k u + v)T_{k(i+1)+r} + (b_k u + w)T_{ki+r} + uT_{k(i-1)+r}. \end{aligned}$$

Continue this process to reach $i = 1$, then it follows that T_n is a linear combination of T_{2k+r}, T_{k+r} and T_r. □

For example, for T_{56} we may take any $k < 56$, say $k = 10$. Since $(a_{10}, b_{10}) = (443, 41)$ by Theorem 2.2, T_{56} can be obtained easily by Theorem 3.4 that

$$\begin{aligned} T_{56} &= 443T_{46} + 41T_{36} + T_{26} \\ &= (443^2 + 41)T_{36} + ((41)443 + 1)T_{26} + 443T_{16} \\ &= ((443)196290 + 18164)T_{26} + ((41)196290 + 443)T_{16} + 196290T_6 \\ &= (86974634)2555757 + (8048333)5768 + (196290)13 \\ &= 222,332,455,004,452, \end{aligned}$$

which is a 15 digit number. However, since T_n is composed of T_{n-k}, T_{n-2k} and T_{n-3k}, it may be better to choose $k \approx \frac{n}{3}$. Indeed if we take $\frac{56}{3} \approx 18 = k$, then

$$T_{56} = a_{18}T_{38} - a_{-18}T_{20} + T_2$$

and the last term $T_2 = 1$ is known easily.

LEMMA 3.5. *Let* $x \in \mathbb{R}$ *and* $n \in \mathbb{N}$. *Then* $\lfloor x+n \rfloor = \lfloor x \rfloor + n$, $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$, *and* $\lfloor -x \rfloor = \begin{cases} -\lfloor x \rfloor & \text{if } x \in \mathbb{N} \\ -\lfloor x \rfloor - 1 & \text{if } x \notin \mathbb{N}. \end{cases}$

THEOREM 3.6. *Assume the same context* (a_k, b_k) *as before. Then*

$$T_n = \begin{cases} a_{\frac{n}{3}} T_{\frac{2n}{3}} + b_{\frac{n}{3}} T_{\frac{n}{3}}, & \text{if } 3|n \\ a_{\lfloor \frac{n}{3} \rfloor} T_{\lfloor \frac{2n}{3} \rfloor + 1} + b_{\lfloor \frac{n}{3} \rfloor} T_{\lfloor \frac{n}{3} + \frac{1}{2} \rfloor + 1} + 1, & \text{if } 3 \not| n. \end{cases}$$

Proof. From Theorem 3.1, $T_n = a_k T_{n-k} + b_k T_{n-2k} + T_{n-3k}$ with $a_k = 3T_k - T_{k-6}$ and $b_k = -a_{-k}$. Now let $k = \lfloor \frac{n}{3} \rfloor$. If $3|n$ then $k = \frac{n}{3}$. Since $T_0 = 0$, we have

$$T_n = a_{\frac{n}{3}} T_{\frac{2n}{3}} + b_{\frac{n}{3}} T_{\frac{n}{3}}.$$

If $3 \not| n$, then due to Lemma 3.5 we have

$$n - k = n + (-\lfloor \frac{n}{3} \rfloor) = n + (\lfloor -\frac{n}{3} \rfloor + 1) = \lfloor n - \frac{n}{3} \rfloor + 1 = \lfloor \frac{2n}{3} \rfloor + 1,$$
$$n - 2k = \lfloor \frac{2n}{3} \rfloor + 1 - \lfloor \frac{n}{3} \rfloor = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{n}{3} + \frac{1}{2} \rfloor + 1 - \lfloor \frac{n}{3} \rfloor = \lfloor \frac{n}{3} + \frac{1}{2} \rfloor + 1.$$

Moreover since $n - 3k = 1$ or 2 and $T_1 = T_2 = 1$, it follows that

$$T_n = a_{\lfloor \frac{n}{3} \rfloor} T_{\lfloor \frac{2n}{3} \rfloor + 1} + b_{\lfloor \frac{n}{3} \rfloor} T_{\lfloor \frac{n}{3} + \frac{1}{2} \rfloor + 1} + 1.$$

□

For example, $T_{67} = a_{22} T_{45} + b_{22} T_{23} + 1 = (664183)272809183135 + (1189)410744 + 1 = 181,195,222,170,528,322$, an 18 digit number.

4. Sum of Tribonacci numbers in a row

THEOREM 4.1. $\sum_{i=0}^{t} T_{4i} = \frac{1}{T_4^2} \left(T_{4t+4} + 6T_{4t} + T_{4t-4} - T_4 \right)$.

Proof. Let $t = 3$. Lemma 2.1 shows $T_{4(4)} = 11T_{4(3)} + 5T_{4(2)} + T_4$, so we have

$$\begin{aligned} T_4^2 \sum_{i=0}^{3} T_{4i} &= 16T_{4(3)} + 16T_{4(2)} + 16T_4 + 16T_0 \\ &= (11T_{4(3)} + 5T_{4(2)} + T_4) + 5T_{4(3)} + 11T_{4(2)} + 15T_4 \\ &= T_{4(4)} + 5T_{4(3)} + 11T_{4(2)} + 15T_4 \\ &= T_{4(4)} + 5T_{4(3)} + (T_{4(3)} - 5T_4) + 15T_4 = T_{4(4)} + 6T_{4(3)} + 10T_4 \\ &= T_{4(4)} + 6T_{4(3)} + T_{4(2)} - T_4 = T_{4(3)+4} + 6T_{4(3)} + T_{4(3)-4} - T_4, \end{aligned}$$

since $10T_4 = 40 = T_8 - T_4$. Assume $T_4^2 \sum_{i=0}^{t} T_{4i} = T_{4t+4} + 6T_{4t} + T_{4t-4} - T_4$ is true. Then it follows that

$$\begin{aligned}
&T_{4(t+1)+4} + 6T_{4(t+1)} + T_{4(t+1)-4} - T_4 \\
&= [11T_{4t+4} + 5T_{4(t-1)+4} + T_{4(t-2)+4}] + 6T_{4(t+1)} + T_{4(t+1)-4} - T_4 \\
&= 16T_{4t+4} + (T_{4t+4} + 6T_{4t} + T_{4t-4} - T_4) \\
&= T_4^2 T_{4t+4} + T_4^2 \sum_{i=0}^{t} T_{4i} = T_4^2 \sum_{i=0}^{t+1} T_{4i}.
\end{aligned}$$

□

Theorem 4.1 was discussed in [4] as a sum of $4t$ subscripted tribonacci numbers. But in our context, it can be explained as a sum of entries of 4th column in the 4 columns tribonacci table. We now shall study the sum of entries of any rth column in the 4 columns tribonacci table. Consider $T_n = T_{4t+r}$ ($t \geq 0$ and $1 \leq r \leq 4$) as an entry placed at the $(t+1)$th row and rth column in the table, and let

$$s_t^{(4,r)} = \sum_{i=0}^{t} T_{4i+r} = T_r + T_{4+r} + \cdots + T_{4t+r}$$

be the partial sum of $t+1$ entries of rth column.

THEOREM 4.2. *Consider $s_t^{(4,r)}$ with $1 \leq r \leq 4$. Then for $t \geq 3$,*

(1) $s_t^{(4,r)} = \begin{cases} 11s_{t-1}^{(4,r)} + 5s_{t-2}^{(4,r)} + s_{t-3}^{(4,r)} - T_4 & \text{if } r = 1 \\ 11s_{t-1}^{(4,r)} + 5s_{t-2}^{(4,r)} + s_{t-3}^{(4,r)} + T_4 & \text{if } r \neq 1. \end{cases}$

(2) $s_t^{(4,r)} = \begin{cases} \frac{1}{T_4^2}(T_{4(t+1)+r} + 6T_{4t+r} + T_{4(t-1)+r} + T_4 & \text{if } r = 1 \\ \frac{1}{T_4^2}(T_{4(t+1)+r} + 6T_{4t+r} + T_{4(t-1)+r} - T_4 & \text{if } r \neq 1. \end{cases}$

(3) $s_t^{(4,r)} = 12s_{t-1}^{(4,r)} - 6s_{t-2}^{(4,r)} - 4s_{t-3}^{(4,r)} - s_{t-4}^{(4,r)}$.

Proof. The 4 columns tribonacci table makes the table of $s_t^{(4,r)}$ as follows.

4 columns table			
1	1	2	4
7	13	24	44
81	149	274	504
927	1705	3136	5768
10609	19513	35890	···

and

t	$s_t^{(4,1)}$	$s_t^{(4,2)}$	$s_t^{(4,3)}$	$s_t^{(4,4)}$
0	1	1	2	4
1	8	14	26	48
2	89	163	300	552
3	1016	1868	3436	6320
4	11625	21381	39326	72332

When $t = 3$, we notice $1016 = (11)89 + (5)8 + 1 - 4$, and it can be written as

$$s_3^{(4,1)} = 11s_2^{(4,1)} + 5s_1^{(4,1)} + s_0^{(4,1)} - T_4.$$

Similar to this, we observe that

$s_3^{(4,2)} = 11s_2^{(4,2)} + 5s_1^{(4,2)} + s_0^{(4,2)} + T_4,$

$s_3^{(4,3)} = 11s_2^{(4,3)} + 5s_1^{(4,3)} + s_0^{(4,3)} + T_4,$

and $s_3^{(4,4)} = 11s_2^{(4,4)} + 5s_1^{(4,4)} + s_0^{(4,4)} + T_4.$

Assume that $s_{t-1}^{(4,r)} = 11s_{t-2}^{(4,r)} + 5s_{t-3}^{(4,r)} + s_{t-4}^{(4,r)} \pm T_4$ with minus sign if $r = 1$, otherwise plus. Then Theorem 2.2 together with induction hypothesis yields

$s_t^{(4,r)} = \sum_{i=0}^t T_{4i+r} = s_{t-1}^{(4,r)} + T_{4t+r}$

$= (11s_{t-2}^{(4,r)} + 5s_{t-3}^{(4,r)} + s_{t-4}^{(4,r)} \pm T_4) + (11T_{4(t-1)+r} + 5T_{4(t-2)+r} + T_{4(t-3)+r})$

$= 11s_{t-1}^{(4,r)} + 5s_{t-2}^{(4,r)} + s_{t-3}^{(4,r)} \pm T_4,$

this proves (1). Moreover we can check the next identities that

$T_{17} + 6T_{13} + T_9 + T_4 = 10609 + 6(927) + 81 + 4 = 4^2(1016) = T_4^2\ s_3^{(4,1)},$

$T_{18} + 6T_{14} + T_{10} - T_4 = 19513 + 6(1705) + 149 - 4 = 4^2(1868) = T_4^2\ s_3^{(4,2)},$

$T_{19} + 6T_{15} + T_{11} - T_4 = 35890 + 6(3136) + 274 - 4 = 4^2(3436) = T_4^2\ s_3^{(4,3)},$

$T_{20} + 6T_{16} + T_{12} - T_4 = 66012 + 6(5768) + 504 - 4 = 4^2(6320) = T_4^2\ s_3^{(4,4)}.$

So if we assume $T_4^2\ s_{t-1}^{(4,r)} = T_{4t+r} + 6T_{4(t-1)+r} + T_{4(t-2)+r} \pm T_4$, then

$$\begin{aligned}
&T_{4(t+1)+r} + 6T_{4t+r} + T_{4(t-1)+r} \pm T_4 \\
&= (11T_{4t+r} + 5T_{4(t-1)+r} + T_{4(t-2)+r}) + 6T_{4t+r} + T_{4(t-1)+r} \pm T_4 \\
&= (T_{4t+r} + 6T_{4(t-1)+r} + T_{4(t-2)+r} \pm T_4) + 16T_{4t+r} \\
&= T_4^2 s_{t-1}^{(4,r)} + T_4^2 T_{4t+r} \ = T_4^2(s_{t-1}^{(4,r)} + T_{4t+r}) \ = T_4^2 s_t^{(4,r)},
\end{aligned}$$

this is (2). On the other hand, it is noticed that

$12s_3^{(4,1)} - 6s_2^{(4,1)} - 4s_1^{(4,1)} - s_0^{(4,1)} = 12(1016) - 6(89) - 4(8) - 1 = s_4^{(4,1)},$

and so on. We assume $s_{t-1}^{(4,r)} = 12s_{t-2}^{(4,r)} - 6s_{t-3}^{(4,r)} - 4s_{t-4}^{(4,r)} - s_{t-5}^{(4,r)}$. Then

$$\begin{aligned}
s_t^{(4,r)} &= s_{t-1}^{(4,r)} + T_{4t+r} \\
&= (12s_{t-2}^{(4,r)} - 6s_{t-3}^{(4,r)} - 4s_{t-4}^{(4,r)} - s_{t-5}^{(4,r)}) \\
&\quad + (11T_{4(t-1)+r} + 5T_{4(t-2)+r} + T_{4(t-3)+r}) \\
&= 12(s_{t-2}^{(4,r)} + T_{4(t-1)+r}) - T_{4(t-1)+r} - 6(s_{t-3}^{(4,r)} + T_{4(t-2)+r}) \\
&\quad + 11T_{4(t-2)+r} - 4(s_{t-4}^{(4,r)} + T_{4(t-3)+r}) + 5T_{4(t-3)+r} - s_{t-5}^{(4,r)}
\end{aligned}$$

$$
\begin{aligned}
&= 12s_{t-1}^{(4,r)} - 6s_{t-2}^{(4,r)} - 4s_{t-3}^{(4,r)} - s_{t-5}^{(4,r)} \\
&\quad - (T_{4(t-1)+r} - 11T_{4(t-2)+r} - 5T_{4(t-3)+r}) \\
&= 12s_{t-1}^{(4,r)} - 6s_{t-2}^{(4,r)} - 4s_{t-3}^{(4,r)} - s_{t-5}^{(4,r)} - T_{4(t-4)+r} \\
&= 12s_{t-1}^{(4,r)} - 6s_{t-2}^{(4,r)} - 4s_{t-3}^{(4,r)} - s_{t-4}^{(4,r)}.
\end{aligned}
$$

□

Theorem 4.1 is a special case of (2) in Theorem 4.2.

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