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# THE DIMENSIONS OF THE MINIMUM AND MAXIMUM CYLINDRICAL LOCAL DIMENSION SETS

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Abstract. We compute the Hausdorff and packing dimensions of the cylindrical lower or upper local dimension set for a self-similar measure having different minimum and maximum of the local dimension on a self-similar set satisfying the open set condition.

## 1. Introduction

Recently using the parameter distribution, we got the parallel results ([3]) for the self-similar set(attractor of the IFS(iterated function system) consisting of $n(\geq 2)$ similitudes satisfying the OSC(open set condition)) instead of the self-similar Cantor set([1]). We used some parameter axes to get the results. We also gave an example of the different distribution sets by the differently chosen parameter axes giving the same cylindrical local dimension set. In this paper, we give the Hausdorff dimensions and packing dimensions for the minimum local dimension set and the maximum local dimension set. We([3]) had some results about the dimensions for the minimum local dimension set and the maximum local dimension set when the self-similar measure has all different values of $\{\frac{\log p_k}{\log a_k}\}_{k=1}^N$. In this paper, we just assume that the minimum and the maximum of $\{\frac{\log p_k}{\log a_k}\}_{k=1}^N$ are different. Finally we give a concrete example such that the dimensions of the minimum and maximum cylindrical local dimension sets are not zero.

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## 2. Preliminaries

Let $\mathbb{N}$ and $\mathbb{R}$ be the set of positive integers and the set of real numbers respectively. An attractor $K$ in the $d$-dimensional Euclidean space $\mathbb{R}^d$ of the IFS $(f_1, \cdots, f_N)$ of contractions where $N \geq 2$ makes each point $v \in K$ have an infinite sequence $\omega = (m_1, m_2, \cdots) \in \Sigma = \{1, \cdots, N\}^{\mathbb{N}}$ where

$$\{v\} = \bigcap_{n=1}^{\infty} K_{\omega|n}$$

for $K_{\omega|n} = K_{m_1,\cdots,m_n} = f_{m_1} \circ \cdots \circ f_{m_n}(K)$([4]) where $\omega|n$ denotes the truncation of $\omega$ to the $n$th place. In such case, we sometimes write $\pi(\omega)$ for such $v$ using the natural projection $\pi : \Sigma \to K$ and call $K_{\omega|n}$ the cylinder of $v$. We note that $K_{\omega|n}$ may be different for the same $v \in K$ since $v$ may have different codes $\omega$. Therefore we write $K_{\omega|n}$ for such distinction for the cylinder of $v$. We call such $K_{\omega|n}$ the cylinders of $K$ and call $K$ a self-similar set if the IFS $(f_1, \cdots, f_N)$ are similitudes.

Each infinite sequence $\omega = (m_1, m_2, \cdots)$ in the coding space $\Sigma$ has the unique subset $A(x_n(\omega))$ of its accumulation points in the simplex of probability vectors in $\mathbb{R}^N$ of the vector-valued sequence $\{x_n(\omega)\} = \{(u_1, \cdots, u_N)_n\}$ of the probability vectors where $u_k$ for $1 \leq k \leq N$ in the probability vector $(u_1, \cdots, u_N)_n$ for each $n \in \mathbb{N}$ is defined by

$$u_k = \frac{|\{1 \leq l \leq n : m_l = k\}|}{n}.$$

The $u_k$ for the $n$th place gives the frequency of the digit $k$ in $\omega|n = (m_1, \cdots, m_n)$. Sometimes we write $n_k(\omega|n)$ for such $u_k$. It is well-known([5]) that a set $A(x_n(\omega))$ of the accumulation points of the vector-valued sequence $\{x_n(\omega)\}$ is a continuum in $\mathbb{R}^N$.

For the self-similar measure $\gamma_{\mathbf{p}}$ on $K$ associated with the probability vector $\mathbf{p} = (p_1, \cdots, p_N) \in (0,1)^N$, we recall the cylindrical lower and upper local dimension sets([3]):

$$\underline{E}_\alpha^{(\mathbf{p})} = \pi\{\omega \in \Sigma : \liminf_{n\to\infty} \frac{\log \gamma_{\mathbf{p}}(K_{\omega|n})}{\log |K_{\omega|n}|} = \alpha\},$$

$$\overline{E}_\alpha^{(\mathbf{p})} = \pi\{\omega \in \Sigma : \limsup_{n\to\infty} \frac{\log \gamma_{\mathbf{p}}(K_{\omega|n})}{\log |K_{\omega|n}|} = \alpha\}$$

where $|K_{\omega|n}|$ denotes the diameter of the cylinder $K_{\omega|n}$. In this paper, we assume that the IFS satisfies the open set condition(OSC)([2, 4, 5]). In this paper, we assume that $0 \log 0 = 0$ for convenience.

From now on, $\dim(E)$ denotes the Hausdorff dimension of $E$ and $\mathrm{Dim}(E)$ denotes the packing dimension of $E$([4]). We note that $\dim(E) \leq \mathrm{Dim}(E)$ for every set $E$([4]).

## 3. Relation between frequency and density

From now on, we assume that the similarity ratios of the similarities $(f_1, \cdots, f_N)$ are $a_1, \cdots, a_N$ and $K$ is the self-similar set for the IFS $(f_1, \cdots, f_N)$ satisfying the OSC and $\gamma_{\mathbf{p}}$ on $K$ is the self-similar measure associated with the probability vector $\mathbf{p}$ and $(a_1, \cdots, a_N) \in (0,1)^N$ satisfying $\sum_{k=1}^N a_k^s = 1$. To avoid the degeneration case, we also assume that $\mathbf{p} = (p_1, \cdots, p_N) \neq (a_1^s, \cdots, a_N^s)$ with $\sum_{k=1}^N a_k^s = 1$( $\Longleftrightarrow$ $\frac{\log p_k}{\log a_k}$ is not the same for all $k = 1, \cdots, N$). We call the set of the elements $\mathbf{y} = (y_1, \cdots, y_N)$ satisfying $\mathbf{y} \in [0,1]^N$ satisfying $\sum_{k=1}^N y_k = 1$ the simplex in this paper.

Remark 3.1. Let $\mathbf{r} = (r_1, \cdots, r_N) \in [0,1]^N$ with $\sum_{k=1}^N r_k = 1$ and let

$$g(\mathbf{r}, \mathbf{p}) = \frac{\sum_{k=1}^N r_k \log p_k}{\sum_{k=1}^N r_k \log a_k}.$$

Then

$$\alpha_{\min} \equiv \min_{1 \leq k \leq N} \frac{\log p_k}{\log a_k} \leq g(\mathbf{r}, \mathbf{p}) \leq \max_{1 \leq k \leq N} \frac{\log p_k}{\log a_k} \equiv \alpha_{\max}.$$

We also define a function $f : (\alpha_{\min}, \alpha_{\max}) \to \mathbb{R}$ by

$$f(\alpha) = \alpha q + \beta(q)$$

for $\alpha = -\beta'(q)$ where $\sum_{k=1}^N p_k^q a_k^{\beta(q)} = 1$. The following Lemma 3.2 (2) is the key idea to explain our multifractal results.

Lemma 3.2. *Let $\mathbf{p} = (p_1, \cdots, p_N) \in (0,1)^N$ with $\sum_{k=1}^N p_k = 1$ and consider a function $\beta(q)$ satisfying $\sum_{k=1}^N p_k^q a_k^{\beta(q)} = 1$. Given $\alpha_{\min} \leq \alpha \leq \alpha_{\max}$,*

(1) *when $\alpha \in (\alpha_{\min}, \alpha_{\max})$, there exists $q_0 \in \mathbb{R}$ such that $g(\mathbf{r}, \mathbf{p}) = \alpha$ for $\mathbf{r} = (r_1, \cdots, r_N)$ where $r_k = p_k^{q_0} a_k^{\beta(q_0)}$ such that $\beta'(q_0) = -\alpha$, and when $\alpha \in \{\alpha_{\min}, \alpha_{\max}\}$, there exists a real sequence $\{q_n\}$ such that $g(\mathbf{r}, \mathbf{p}) = \alpha$ for $\mathbf{r} = (r_1, \cdots, r_N)$ where $r_k = \lim_{n\to\infty} p_k^{q_n} a_k^{\beta(q_n)}$ and $\lim_{n\to\infty} \beta'(q_n) = -\alpha$,*

(2) *when* $\alpha \in (\alpha_{\min}, \alpha_{\max})$, *if* $g(\mathbf{y}, \mathbf{p}) = \alpha$ *with* $\mathbf{y}$ *in the simplex, then* $g(\mathbf{y}, \mathbf{r}) = g(\mathbf{r}, \mathbf{r})$, *conversely if* $g(\mathbf{y}, \mathbf{r}) = g(\mathbf{r}, \mathbf{r})$ *with* $q_0 \neq 0$ *then* $g(\mathbf{y}, \mathbf{p}) = \alpha$,
*and when* $\alpha \in \{\alpha_{\min}, \alpha_{\max}\}$, *if* $g(\mathbf{y}, \mathbf{p}) = \alpha$ *with* $\mathbf{y}$ *in the simplex, then* $g(\mathbf{y}, \mathbf{r}) = g(\mathbf{r}, \mathbf{r})$.

*Proof.* It follows essentially from [3]. We only need to show the followings. Note that $\mathbf{r} = \lim_{n\to\infty} \mathbf{r}_n$ and $\mathbf{r}_n = (r_{n,1}, \cdots, r_{n,N})$ where $r_{n,k} = p_k^{q_n} a_k^{\beta(q_n)}$ for each $k = 1, \cdots, N$. Then since $\alpha_n = \frac{\sum_{k=1}^N r_{n,k} \log p_k}{\sum_{k=1}^N r_{n,k} \log a_k}$,

$$g(\mathbf{r}_n, \mathbf{r}_n) = \frac{\sum_{k=1}^N r_{n,k} \log r_{n,k}}{\sum_{k=1}^N r_{n,k} \log a_k} = \alpha_n q_n + \beta(q_n) = f(\alpha_n).$$

When $\alpha \in \{\alpha_{\min}, \alpha_{\max}\}$, since $\lim_{n\to\infty} \alpha_n = \alpha$ where $\alpha_n = -\beta'(q_n)$, noting $f(\alpha_n) = \alpha_n q_n + \beta(q_n)$ for $\alpha_n = -\beta'(q_n)$ and we have

$$\begin{aligned}
g(\mathbf{y}, \mathbf{r}) &= \frac{\sum_{k=1}^N y_k \log \lim_{n\to\infty} [p_k^{q_n} a_k^{\beta(q_n)}]}{\sum_{k=1}^N y_k \log a_k} \\
&= \lim_{n\to\infty} [q_n g(\mathbf{y}, \mathbf{p}) + \beta(q_n)] \\
&= \lim_{n\to\infty} [\alpha q_n + \beta(q_n)] \\
&= \lim_{n\to\infty} [\alpha_n q_n + \beta(q_n)] \\
&= \lim_{n\to\infty} f(\alpha_n) \\
&= \lim_{n\to\infty} g(\mathbf{r}_n, \mathbf{r}_n) \\
&= g(\mathbf{r}, \mathbf{r}).
\end{aligned}$$

□

From now on, without specific mention, we fix distinct $i, j$ respectively satisfying

$$\frac{\log p_i}{\log a_i} = \min_{1\le k\le N} \frac{\log p_k}{\log a_k} < \max_{1\le k\le N} \frac{\log p_k}{\log a_k} = \frac{\log p_j}{\log a_j}.$$

It is obvious that there is a unique $\mathbf{z} \in (0,1)^N$ for $\mathbf{y}$ in the simplex such that $g(\mathbf{y}, \mathbf{p}) = g(\mathbf{z}, \mathbf{p})$ where $\mathbf{z} = (z_1, \cdots, z_N)$ with $z_j = 1 - z_i$ and $z_k = 0$ if $k \neq i, j$. We put $z_{\mathbf{y}} = z_i$ from now on. More precisely, $z_{\mathbf{y}}$ is the projection of $\mathbf{y}$ in the simplex into the unit interval $[0,1]$ satisfying $z_{\mathbf{y}} = z_i$ where $g(\mathbf{y}, \mathbf{p}) = g(\mathbf{z}, \mathbf{p})$.

THEOREM 3.3. ([3], Theorem 3.4) *For* $\log p_i/\log a_i \le \alpha \le \log p_j/\log a_j$, *we have*

$$\overline{E}_{\alpha}^{(\mathbf{p})} = \pi(\underline{F}(t))$$

*where*

$$\underline{F}(t) = \{\omega : \min_{\mathbf{y}\in A(x_n(\omega))} z_{\mathbf{y}} = t\},$$

*and*

$$\frac{t\log p_i + (1-t)\log p_j}{t\log a_i + (1-t)\log a_j} = \alpha.$$

THEOREM 3.4. ([3], Theorem 3.5) *For* $\log p_i/\log a_i \le \alpha \le \log p_j/\log a_j$, *we have*

$$\underline{E}_{\alpha}^{(\mathbf{p})} = \pi(\overline{F}(t))$$

*where*

$$\overline{F}(t) = \{\omega : \max_{\mathbf{y}\in A(x_n(\omega))} z_{\mathbf{y}} = t\},$$

*and*

$$\frac{t\log p_i + (1-t)\log p_j}{t\log a_i + (1-t)\log a_j} = \alpha.$$

## 4. Subset relation and multifratal spectrum

In the following Theorems, let $t_0$ be the real number satisfying

$$\frac{t_0\log p_i + (1-t_0)\log p_j}{t_0\log a_i + (1-t_0)\log a_j} = g(\mathbf{r}_0, \mathbf{p})$$

for $\mathbf{r}_0 = (a_1^s, \cdots, a_N^s)$ with $\sum_{k=1}^N a_k^s = 1$. We note that when $\alpha \in (\alpha_{\min}, \alpha_{\max})$, there exists $q_0 \in \mathbb{R}$ such that $g(\mathbf{r}, \mathbf{p}) = \alpha$ for $\mathbf{r} = (r_1, \cdots, r_N)$ where $r_k = p_k^{q_0} a_k^{\beta(q_0)}$ such that $\beta'(q_0) = -\alpha$ by Lemma 3.2 (1). Therefore in the following Theorems, given $0 < t < 1$, we put $\mathbf{r} = \mathbf{r}(t) = (r_1, \cdots, r_N)$ satisfying $r_k = p_k^{q_0} a_k^{\beta(q_0)}$ for $\beta'(q_0) = -\alpha$ where

$$\alpha = \alpha(t) = \frac{t\log p_i + (1-t)\log p_j}{t\log a_i + (1-t)\log a_j}.$$

REMARK 4.1. For $t = 1$, we put $\mathbf{r}(1) = (r_1, \cdots, r_N)$ satisfying $r_k = \lim_{n\to\infty} p_k^{q_n} a_k^{\beta(q_n)}$ and $\lim_{n\to\infty} \beta'(q_n) = -\alpha(1) = -\alpha_{\min}$. In this case, we can put $q_n = n$. For $t = 0$, we put $\mathbf{r}(0) = (r_1, \cdots, r_N)$ satisfying $r_k = \lim_{n\to\infty} p_k^{q_n} a_k^{\beta(q_n)}$ and $\lim_{n\to\infty} \beta'(q_n) = -\alpha(0) = -\alpha_{\max}$. In this case, we can put $q_n = -n$.

Remark 4.2. We remark that

$$\lim_{n\to\infty} -\beta'(n) = \alpha(1) = \alpha_{\min}$$

and

$$\lim_{n\to\infty} -\beta'(-n) = \alpha(0) = \alpha_{\max}$$

since $\alpha(1)$ and $\alpha(0)$ are the slopes of the asymptotes of the convex function([4]).

We have the followings.

Theorem 4.3. *For every $n \in \mathbb{N}$, $\mathbf{r}_n = (r_1, \cdots, r_N)$ where $r_k = p_k^n a_k^{\beta(n)}$ for each $k = 1, \cdots, N$ and $\mathbf{r}_{-n} = (r_1, \cdots, r_N)$ where $r_k = p_k^{-n} a_k^{\beta(-n)}$ for each $k = 1, \cdots, N$. Then*

(1) *if $0 < t_0 < t \leq 1$, then*

$$\overline{E}^{(\mathbf{p})}_{\alpha(t)} = \pi(\underline{F}(t)) = \overline{E}^{(\mathbf{r}_n)}_{\alpha(t)n+\beta(n)}$$

(2) *if $0 \leq t < t_0 < 1$, then*

$$\overline{E}^{(\mathbf{p})}_{\alpha(t)} = \pi(\underline{F}(t)) = \underline{E}^{(\mathbf{r}_{-n})}_{\alpha(t)(-n)+\beta(-n)}$$

(3) *if $0 < t_0 < t \leq 1$, then*

$$\underline{E}^{(\mathbf{p})}_{\alpha(t)} = \pi(\overline{F}(t)) = \underline{E}^{(\mathbf{r}_n)}_{\alpha(t)n+\beta(n)}$$

(4) *if $0 \leq t < t_0 < 1$, then*

$$\underline{E}^{(\mathbf{p})}_{\alpha(t)} = \pi(\overline{F}(t)) = \overline{E}^{(\mathbf{r}_{-n})}_{\alpha(t)(-n)+\beta(-n)}.$$

*Proof.* For (1), from [3], we only need to show

$$\overline{E}^{(\mathbf{p})}_{\alpha(t)} = \overline{E}^{(\mathbf{r}_n)}_{\alpha(t)n+\beta(n)}.$$

It follows from $\gamma_{\mathbf{r}_n}(K_{\omega|n}) = \gamma_{\mathbf{p}}(K_{\omega|n})^n |K_{\omega|n}|^{\beta(n)}$ since

$$\overline{E}^{(\mathbf{p})}_{\alpha(t)} = \pi\{\omega \in \Sigma : \limsup_{n\to\infty} \frac{\log \gamma_{\mathbf{p}}(K_{\omega|n})}{\log |K_{\omega|n}|} = \alpha(t)\},$$

and

$$\overline{E}^{(\mathbf{r}_n)}_{\alpha(t)n+\beta(n)} = \pi\{\omega \in \Sigma : \limsup_{n\to\infty} \frac{\log \gamma_{\mathbf{r}_n}(K_{\omega|n})}{\log |K_{\omega|n}|} = \alpha(t)n + \beta(n)\}.$$

For (2), from [3], we only need to show

$$\overline{E}^{(\mathbf{p})}_{\alpha(t)} = \underline{E}^{(\mathbf{r}_{-n})}_{\alpha(t)(-n)+\beta(-n)}.$$

It follows from $\gamma_{\mathbf{r}_{-n}}(K_{\omega|n}) = \gamma_{\mathbf{p}}(K_{\omega|n})^{-n}|K_{\omega|n}|^{\beta(-n)}$ since

$$\overline{E}^{(\mathbf{p})}_{\alpha(t)} = \pi\{\omega \in \Sigma : \limsup_{n\to\infty} \frac{\log \gamma_{\mathbf{p}}(K_{\omega|n})}{\log |K_{\omega|n}|} = \alpha(t)\},$$

and

$$\underline{E}^{(\mathbf{r}_{-n})}_{\alpha(t)(-n)+\beta(-n)} = \pi\{\omega \in \Sigma : \liminf_{n\to\infty} \frac{\log \gamma_{\mathbf{r}_{-n}}(K_{\omega|n})}{\log |K_{\omega|n}|} = \alpha(t)(-n) + \beta(-n)\}.$$

(3), (4) also follow from the similar arguments above. □

REMARK 4.4. We note that $\mathbf{r}(1) = \lim_{n\to\infty} \mathbf{r}_n$ and $\mathbf{r}(0) = \lim_{n\to\infty} \mathbf{r}_{-n}$.

We have the followings.

THEOREM 4.5.

(1) $\dim(\pi(\underline{F}(1))) = \mathrm{Dim}(\pi(\underline{F}(1))) = \dim(\overline{E}^{(\mathbf{p})}_{\alpha(1)}) = \mathrm{Dim}(\overline{E}^{(\mathbf{p})}_{\alpha(1)}) = g(\mathbf{r}(1), \mathbf{r}(1))$

(2) $\dim(\pi(\underline{F}(0))) = \dim(\overline{E}^{(\mathbf{p})}_{\alpha(0)}) = g(\mathbf{r}(0), \mathbf{r}(0))$ *and* $\mathrm{Dim}(\pi(\underline{F}(0))) = \mathrm{Dim}(\overline{E}^{(\mathbf{p})}_{\alpha(0)}) = s$

(3) $\dim(\pi(\overline{F}(1))) = \dim(\underline{E}^{(\mathbf{p})}_{\alpha(1)}) = g(\mathbf{r}(1), \mathbf{r}(1))$ *and* $\mathrm{Dim}(\pi(\overline{F}(1))) = \mathrm{Dim}(\underline{E}^{(\mathbf{p})}_{\alpha(1)}) = s$

(4) $\dim(\pi(\overline{F}(0))) = \mathrm{Dim}(\pi(\overline{F}(0))) = \dim(\underline{E}^{(\mathbf{p})}_{\alpha(0)}) = \mathrm{Dim}(\underline{E}^{(\mathbf{p})}_{\alpha(0)}) = g(\mathbf{r}(0), \mathbf{r}(0))$.

*Proof.* From the above Theorem (1) and the proposition 2.1([3]), we have $\mathrm{Dim}(\pi(\underline{F}(1))) \le \lim_{n\to\infty}(\alpha(1)n + \beta(n))$. Noting the proof of Lemma 3.2 (2), we have $\lim_{n\to\infty}(\alpha(1)n + \beta(n)) = g(\mathbf{r}(1), \mathbf{r}(1))$. Further we have $\dim(\pi(\underline{F}(1))) \ge g(\mathbf{r}(1), \mathbf{r}(1))$ since $\{\mathbf{r}(1)\} \subset \underline{F}(1)$ and $\dim(\pi(\{\mathbf{r}(1)\})) = g(\mathbf{r}(1), \mathbf{r}(1))$([2]). Similar arguments give (4). For (2), by the above Theorem (2), $\dim(\pi(\underline{F}(0))) \le \lim_{n\to\infty}(\alpha(0)(-n) + \beta(-n)) = g(\mathbf{r}(0), \mathbf{r}(0))$ from the above similar arguments. Since $\{\mathbf{r}(0)\} \subset \underline{F}(0)$ and $\dim(\pi(\{\mathbf{r}(0)\})) = g(\mathbf{r}(0), \mathbf{r}(0))$([2]), we have $\dim(\pi(\underline{F}(0))) \ge g(\mathbf{r}(0), \mathbf{r}(0))$. We note that there is $\{\omega : A(x_n(\omega)) = \mathbf{C}\} \subset \underline{F}(0)$ satisfying $\mathbf{r}_0 = (a_1^s, ..., a_N^s) \in \mathbf{C}$ where $\sum_{k=1}^{N} a_k^s = 1$ and $\mathbf{C}$ is a continuum. Clearly $\mathrm{Dim}(\pi(\{\omega : A(x_n(\omega)) = \mathbf{C}\})) = s$([2]). (2) follows since $\mathrm{Dim}(\pi(\underline{F}(0))) \le \mathrm{Dim}(K) = s$. (3) follows from the similar arguments with the proof of (2). □

REMARK 4.6. If $\frac{\log p_k}{\log a_k} = \frac{\log p_i}{\log a_i} = \alpha(1)$, then

$$\lim_{n\to\infty} p_k^n a_k^{\beta(n)} = a_k^{\lim_{n\to\infty} f(-\beta'(n))}.$$

Putting $\lim_{n\to\infty} f(-\beta'(n)) = f(\alpha(1))$, we note that

$$1 = \lim_{n\to\infty} \sum_{k=1}^{N} p_k^n a_k^{\beta(n)} = \sum_{p_k = a_k^{\alpha(1)}} a_k^{f(\alpha(1))}.$$

This also gives the information of $\mathbf{r}(1)$, that is, its $k$th component is $a_k^{f(\alpha(1))}$ and its $k'$th component is 0 if $p_{k'} \neq a_{k'}^{\alpha(1)}$. We note that $f(\alpha_n) = \alpha_n n + \beta(n)$ where $\alpha_n = -\beta'(n) = \frac{\sum_{k=1}^N p_k^n a_k^{\beta(n)} \log p_k}{\sum_{k=1}^N p_k^n a_k^{\beta(n)} \log a_k}$. Hence we see that $f(\alpha(1)) = g(\mathbf{r}(1), \mathbf{r}(1))$. For, $g(\mathbf{r}(1), \mathbf{r}(1)) = \lim_{n\to\infty} \alpha_n n + \beta(n) = \lim_{n\to\infty} f(\alpha_n) = f(\alpha(1))$.

It also holds for the case: $\frac{\log p_k}{\log a_k} = \frac{\log p_j}{\log a_j} = \alpha(0)$. That is, $f(\alpha(0)) = g(\mathbf{r}(0), \mathbf{r}(0))$ where $f(\alpha(0)) = \lim_{n\to\infty} f(-\beta'(-n))$. We also have the information of $\mathbf{r}(0)$, that is, its $k$-th component is $a_k^{f(\alpha(0))}$ and its $k'$-th component is 0 if $p_{k'} \neq a_{k'}^{\alpha(0)}$.

REMARK 4.7. We have

$$\lim_{t\uparrow 1} g(\mathbf{r}(t), \mathbf{r}(t)) = g(\mathbf{r}(1), \mathbf{r}(1))$$

and

$$\lim_{t\downarrow 0} g(\mathbf{r}(t), \mathbf{r}(t)) = g(\mathbf{r}(0), \mathbf{r}(0)).$$

Further for $g(\mathbf{r}_n, \mathbf{r}_n) = \alpha_n n + \beta(n)$ where $\mathbf{r}_n = (r_{n,1}, \cdots, r_{n,N})$ with $r_{n,k} = p_k^n a_k^{\beta(n)}$ for each $k = 1, \cdots, N$, $\lim_{n\to\infty} f(\alpha_n) = f(\alpha(1))$ where $f(\alpha_n) = \alpha_n n + \beta(n)$ with

$$\alpha_n = -\beta'(n) = \frac{\sum_{k=1}^N p_k^n a_k^{\beta(n)} \log p_k}{\sum_{k=1}^N p_k^n a_k^{\beta(n)} \log a_k} \to \alpha(1).$$

Similarly for $g(\mathbf{r}_n, \mathbf{r}_n) = \alpha_n(-n) + \beta(-n)$ where $\mathbf{r}_n = (r_{n,1}, \cdots, r_{n,N})$ with $r_{n,k} = p_k^{-n} a_k^{\beta(-n)}$ for each $k = 1, \cdots, N$, $\lim_{n\to\infty} f(\alpha_n) = f(\alpha(0))$ where $f(\alpha_n) = \alpha_n(-n) + \beta(-n)$ with

$$\alpha_n = -\beta'(-n) = \frac{\sum_{k=1}^N p_k^{-n} a_k^{\beta(-n)} \log p_k}{\sum_{k=1}^N p_k^{-n} a_k^{\beta(-n)} \log a_k} \to \alpha(0).$$

The above facts also assure that the continuous function

$$f : (\alpha_{\min}, \alpha_{\max}) \to \mathbb{R}$$

can be extended to the continuous function

$$f : [\alpha_{\min}, \alpha_{\max}] \to \mathbb{R}.$$

Noting that $t$ determines the distribution set $F(t)$ and $\pi(F(t))$ and its Hausdorff dimension $g(\mathbf{r}(t), \mathbf{r}(t)) = \alpha q + \beta(q) = f(\alpha)$(see [3]), we have that the dimension function $\dim(\pi(F(t)))$ is continuous on $[0, 1]$. More precisely, $t \in (0, 1)$ determines $\alpha = \alpha(t)$ and $\alpha$ determines $q \in \mathbb{R}$ such that $\beta'(q) = -\alpha$ and they determine $f(\alpha) = \alpha q + \beta(q)$. Further $t \in [0, 1]$ also determines $f(\alpha)$ since $t = 1$ gives $f(\alpha_{\min})$ and $t = 0$ gives $f(\alpha_{\max})$.

We have the followings.

Corollary 4.8. *If there are unique $i, j$ such that*

$$\frac{\log p_i}{\log a_i} = \min_{1 \le k \le N} \frac{\log p_k}{\log a_k} < \max_{1 \le k \le N} \frac{\log p_k}{\log a_k} = \frac{\log p_j}{\log a_j} :$$

(1) $\dim(\pi(\underline{F}(1))) = \mathrm{Dim}(\pi(\underline{F}(1))) = \dim(\overline{E}^{(\mathbf{p})}_{\alpha(1)}) = \mathrm{Dim}(\overline{E}^{(\mathbf{p})}_{\alpha(1)}) = 0$

(2) $\dim(\pi(\underline{F}(0))) = \dim(\overline{E}^{(\mathbf{p})}_{\alpha(0)}) = 0, \mathrm{Dim}(\pi(\underline{F}(0))) = \mathrm{Dim}(\overline{E}^{(\mathbf{p})}_{\alpha(0)}) = s$

(3) $\dim(\pi(\overline{F}(1))) = \dim(\underline{E}^{(\mathbf{p})}_{\alpha(1)}) = 0, \mathrm{Dim}(\pi(\overline{F}(1))) = \mathrm{Dim}(\underline{E}^{(\mathbf{p})}_{\alpha(1)}) = s$

(4) $\dim(\pi(\overline{F}(0))) = \mathrm{Dim}(\pi(\overline{F}(0))) = \dim(\underline{E}^{(\mathbf{p})}_{\alpha(0)}) = \mathrm{Dim}(\underline{E}^{(\mathbf{p})}_{\alpha(0)}) = 0.$

*Proof.* It is not difficult to show that $g(\mathbf{r}(1), \mathbf{r}(1)) = 0 = g(\mathbf{r}(0), \mathbf{r}(0))$, if there are unique $i, j$ such that

$$\frac{\log p_i}{\log a_i} = \min_{1 \le k \le N} \frac{\log p_k}{\log a_k} < \max_{1 \le k \le N} \frac{\log p_k}{\log a_k} = \frac{\log p_j}{\log a_j}.$$

This is a variation of the exercise 11.2([4]). It easily follows from the above Theorem. $\square$

Remark 4.9. The above Corollary is a more generalized form of the theorem 4.4([3]) since the assumption of the above Corollary is weaker than the condition of all different values of $\{\frac{\log p_k}{\log a_k}\}_{k=1}^N$.

Example 4.10. Let $K = [0, 1]$ be the self-similar set for the IFS $(f_1, f_2, f_3)$ satisfying the OSC whose similarity ratios are $(a_1, a_2, a_3) = (1/9, 5/9, 1/3)$ and $\gamma_{\mathbf{p}}$ on $K$ be the self-similar measure associated with $\mathbf{p} = (p_1, p_2, p_3) = (1/4, 1/4, 1/2)$. We([3]) had two different distribution structures fixing distinct $i, j$, that is, for $(i, j) = (1, 2)$ and $(i, j) = (3, 2)$ respectively

$$\frac{\log p_i}{\log a_i} = \min_{1\le k\le 3} \frac{\log p_k}{\log a_k} = \frac{\log 2}{\log 3} < \frac{\log 4}{\log 9/5} = \max_{1\le k\le 3} \frac{\log p_k}{\log a_k} = \frac{\log p_j}{\log a_j}.$$

In this case, we have $\dim(\overline{E}^{(\mathbf{p})}_{\alpha(1)}) = \mathrm{Dim}(\overline{E}^{(\mathbf{p})}_{\alpha(1)}) = g(\mathbf{r}(1), \mathbf{r}(1))$ and $\dim(\underline{E}^{(\mathbf{p})}_{\alpha(1)}) = g(\mathbf{r}(1), \mathbf{r}(1))$, and $\mathrm{Dim}(\underline{E}^{(\mathbf{p})}_{\alpha(1)}) = 1$ where $\alpha(1) = \frac{\log 2}{\log 3}$, $\mathbf{r}(1) = (\frac{3-\sqrt{5}}{2}, 0, \frac{\sqrt{5}-1}{2})$, and $g(\mathbf{r}(1), \mathbf{r}(1)) = \frac{\log \frac{\sqrt{5}-1}{2}}{-\log 3} > 0$. However since there is unique $j(= 2)$ such that $\max_{1\le k\le 3} \frac{\log p_k}{\log a_k} = \frac{\log p_j}{\log a_j}$, we have $\dim(\overline{E}^{(\mathbf{p})}_{\alpha(0)}) = 0$ and $\mathrm{Dim}(\overline{E}^{(\mathbf{p})}_{\alpha(0)}) = 1$, and $\dim(\underline{E}^{(\mathbf{p})}_{\alpha(0)}) = \mathrm{Dim}(\underline{E}^{(\mathbf{p})}_{\alpha(0)}) = 0$. In this case, $\alpha(0) = \frac{\log 4}{\log 9/5}$, $\mathbf{r}(0) = (0, 1, 0)$, and $g(\mathbf{r}(0), \mathbf{r}(0)) = 0$.

## References


[1] I.-S. Baek, *Relation between spectral classes of a self-similar Cantor set*, J. Math. Anal. Appl. **292** (2004), no. 1, 294-302.

[2] I.-S. Baek, L. Olsen, and N. Snigireva, *Divergence points of self-similar measures and packing dimension*, Adv. Math. **214** (2007), no. 1, 267-287.

[3] I. -S. Baek, *The parameter distribution set for a self-similar measure*, Bull. Kor. Math. Soc. **49** (2012), no. 5, 1041-1055.

[4] K. J. Falconer, Techniques in fractal geometry, John Wiley and Sons, 1997.

[5] L. Olsen and S. Winter, *Normal and non-normal points of self-similar sets and divergence points of a self-similar measures*, J. London Math. Soc. **67** (2003), no. 3, 103-122.



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