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COMMON FIXED POINTS FOR TWO MAPPINGS WITH EXPANSIVE PROPERTIES ON COMPLEX VALUED METRIC SPACES

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Abstract. In this paper, we use two mappings satisfying certain expansive conditions to construct convergent sequences in complex valued metric spaces, and then we prove that the limits of the convergent sequences are the points of coincidence or common fixed points for the two mappings. The main theorems in this paper are the generalizations and improvements of the corresponding results in real metric spaces, cone metric spaces and topological vector space-valued cone metric spaces.

1. Introduction

Real metric spaces have been widely generalized and improved. For example, cone metric spaces ([7]) and topological vector space-valued cone metric spaces([3]). A number of authors discussed and obtained some fixed point and common fixed point theorems in these spaces, greatly generalized and improved some corresponding results. Recently, Azam et al. ([1]) introduced a partial order $\leq$ on the set $\mathbb{C}$ of complex numbers, used the idea in ([3, 7]) to define a complex metric d on a nonempty set X and a complex metric space (X, d), and gave coincidence point theorems and common fixed point theorems for two mappings satisfying a contractive type condition. The authors in ([11, 14, 15]) further generalized and improved the corresponding results in ([1]).

In what follows, we recall some notations and definitions that will be utilized in our subsequent discussion.

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Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\leq$ on $\mathbb{C}$ as follows:

$$z_1 \leq z_2 \Leftrightarrow \mathrm{Re}(z_1) \leq \mathrm{Re}(z_2), \mathrm{Im}(z_1) \leq \mathrm{Im}(z_2).$$

Consequently, $z_1 \leq z_2$ if and only if one of the following conditions is satisfied:

(C1) $\mathrm{Re}(z_1) = \mathrm{Re}z_2,\ \ \mathrm{Im}z_1 = \mathrm{Im}z_2$;
(C2) $\mathrm{Re}(z_1) < \mathrm{Re}z_2,\ \ \mathrm{Im}z_1 = \mathrm{Im}z_2$;
(C3) $\mathrm{Re}(z_1) = \mathrm{Re}z_2,\ \ \mathrm{Im}z_1 < \mathrm{Im}z_2$;
(C4) $\mathrm{Re}(z_1) < \mathrm{Re}z_2,\ \ \mathrm{Im}z_1 < \mathrm{Im}z_2$.

In particular, we write $z_1 < z_2$ if only (C4) is satisfied.

Obviously, the following statements hold:

(i) If $b \geq a \geq 0$, then $az \leq bz$ for any $z \in \mathbb{C}$ with $0 \leq z$;
(ii) if $0 \leq z_1 < z_2$, then $|z_1| < |z_2|$;
(iii) if $z_1 \leq z_2$ and $z_2 < z_3$, then $z_1 < z_3$;
(iv) if $z_1 \leq z_2$ and $z \in \mathbb{C}$, then $z + z_1 \leq z + z_2$.

Definition 1.1. ([1, 11, 14, 15]) Let X be a nonempty set. If a mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions:

(i) $0 \leq d(x, y)$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z, \in X$.

Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Example 1.2. ([14]) *Let $X = \mathbb{C}$. Define a mapping $d : X \times X \to \mathbb{C}$ as follows*

$$d(z_1, z_2) = e^{ik}|z_1 - z_2|,\ \forall\, z_1, z_2 \in X,$$

where $k \in \mathbb{R}$. Then (X, d) is a complex valued metric space.

Example 1.3. *Let $X = \{a, b, c\}$. Define a mapping $d : X \times X \to \mathbb{C}$ by*

$$d(a, a) = d(b, b) = d(c, c) = 0,$$

$d(a, b) = d(b, a) = 3{+}4i,\ d(a, c) = d(c, a) = 2{+}3i,\ d(b, c) = d(c, b) = 4{+}5i.$

Obviously, (X, d) is a complex valued metric space.

Definition 1.4. ([1, 11, 14, 15]) Let (X, d) be a complex valued metric space, $\{x_n\}_{n\geq 1}$ a sequence in X and $x \in X$.

(i) If for any $c \in \mathbb{C}$ with $0 < c$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < c$ for all $n > n_0$, then $\{x_n\}$ is said to converge to x and x is the limit point of $\{x_n\}$.We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii) If for any $c \in \mathbb{C}$ with $0 < c$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ and any $m \in \mathbb{N}$, $d(x_n, x_{n+m}) < c$, then $\{x_n\}$ is said to be a Cauchy sequence.

(iii) If every Cauchy sequence in X is convergent, then X is said to be complete.

Definition 1.5. Let (X, d) and (Y, ρ) be two complex valued metric spaces, $f : X \to Y$ a mapping and $x^* \in X$. f is said to be continuous at x^* if for each $\epsilon \in \mathbb{C}$ with $0 < \epsilon$, there exists $\delta \in \mathbb{C}$ with $0 < \delta$ such that $d(x, x^*) < \delta$ implies $\rho(fx, fx^*) < \epsilon$.

Definition 1.6. ([2]) Let X be a nonempty set, $f, g : X \to X$ two mappings. f and g are called weakly compatible if $x \in X$ and $fx = gx$, then $fgx = gfx$.

The following definitions of three different expansive maps can be found in ([17]): Let X be a real metric space and $T : X \to X$ a mapping.

If there exists a constant number $a > 1$ such that for each $x, y \in X$, $d(Tx, Ty) \geq ad(x, y)$, then T is said to be a I-expansive mapping.

If there exist non-negative real numbers a, b, c with $a+b+c > 1$ such that for each $x, y \in X$ with $x \neq y$, $d(Tx, Ty) \geq ad(x, Tx) + bd(y, Ty) + cd(x, y)$, then T is said to be a II-expansive mapping.

If there exists $h > 1$ such that for each $x, y \in X$, $d(Tx, Ty) \geq h \min\{d(x, Tx), d(y, Ty), d(x, y)\}$, then T is said to be a III-expansive mapping.

Wang, Li and Gao ([17]) obtained important fixed point theorems in complete real metric spaces for the above expansive type maps, and the authors in ([5, 6, 8, 9, 12, 13]) obtained coincidence point and common fixed point theorems for two maps with expansive conditions in real metric spaces, cone metric spaces and CMTS([4, 16]) respectively, widely generalized and improved the corresponding results in ([17]).

In this paper, we consider some expansive type conditions which are the generalizations of three expansive definitions([17]) defined in real metric spaces, and then discuss the existence problems of coincidence point and common fixed point for two mappings. The obtained results are the generalizations, improvements and new versions of the corresponding conclusions in real metric spaces, cone metric spaces, topological vector space valued cone metric spaces and CMTS.

2. Basic lemmas

Now, we give some lemmas that will be utilized in our subsequent discussion.

LEMMA 2.1. ([1, 11, 14, 15]) *Let $\{x_n\}$ be a sequence in a complex valued metric space (X, d). Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.*

LEMMA 2.2. ([1, 11, 14, 15]) *Let $\{x_n\}$ be a sequence in a complex valued metric space (X, d). Then $\{x_n\}$ converges to $x \in X$ if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

LEMMA 2.3. *Let (X, d) be a complex valued metric space, $\{x_n\}$ a sequence in X. If $\{x_n\}$ is convergent, then its limit point is unique.*

Proof. Suppose that $\{x_n\}$ has two limit points x and y. For any $c \in \mathbb{C}$ with $0 < c$, since $x_n \to x$ and $x_n \to y$, there exist $N_1, N_2 \in \mathbb{N}$ such that $d(x_n, x) < \frac{c}{2}$ for all $n > N_1$, and $d(x_n, y) < \frac{c}{2}$ for all $n > N_2$, hence for all $n > \max\{N_1, N_2\}$, $d(x, y) \le d(x_n, x) + d(x_n, y) < c$, so $|d(x, y)| < |c|$. Since c is arbitrary, we obtain $|d(x, y)| = 0$, i.e., $x = y$. Therefore, the limit point of $\{x_n\}$ is unique. □

LEMMA 2.4. *If (X, d) is a complex valued metric space, $\{x_n\}$ converges to $x \in X$, $\{y_n\}$ converges to $y \in X$. Then*

$$\lim_{n\to\infty} d(x_n, y_n) = d(x, y); \quad \lim_{n\to\infty} |d(x_n, y_n)| = |d(x, y)|.$$

In particular, for any fixed element $z \in X$, the following holds

$$\lim_{n\to\infty} d(x_n, z) = d(x, z); \quad \lim_{n\to\infty} |d(x_n, z)| = |d(x, z)|.$$

Proof. Obviously, the following holds

$$d(x_n, y_n) \le d(x_n, x)+d(x, y)+(y, y_n) \Rightarrow d(x_n, y_n)-d(x, y) \le d(x_n, x)+d(y_n, y);$$

$$d(x, y) \le d(x_n, x)+d(x_n, y_n)+d(y_n, y) \Rightarrow -[d(x_n, x)+d(y_n, y)] \le d(x_n, y_n)-d(x, y).$$

Hence we obtain

$$-[d(x_n, x) + d(y_n, y)] \le [d(x_n, y_n) - d(x, y)] \le [d(x_n, x) + d(y_n, y)].$$

Let $d(x_n, x)+d(y_n, y) = \alpha+\beta i$ and $d(x_n, y_n)-d(x, y) = \delta+\sigma i$. Since $\alpha, \beta \ge 0$, so $|\delta| \le |\alpha|$ and $|\sigma| \le |\beta|$, hence $|\delta + \sigma i| \le |\alpha + \beta i|$. Therefore

$$|d(x_n, y_n) - d(x, y)| \le |d(x_n, x) + d(y_n, y)|,$$

so

$$\begin{aligned}\big||d(x_n, y_n)| - |d(x, y)|\big| &\le |d(x_n, y_n) - d(x, y)| \\ &\le |d(x_n, x) + d(y_n, y)| \le |d(x_n, x)| + |d(y_n, y)|.\end{aligned}$$

Since $x_n \to x$ and $y_n \to y$, by Lemma 2.2, we have

$$\lim_{n\to\infty} \big| |d(x_n, y_n)| - |d(x, y)| \big| = \lim_{n\to\infty} |d(x_n, y_n) - d(x, y)| = 0.$$

Hence

$$\lim_{n\to\infty} d(x_n, y_n) = d(x, y); \quad \lim_{n\to\infty} |d(x_n, y_n)| = |d(x, y)|.$$

□

Lemma 2.5. *Let (X, d) and (Y, ρ) be two complex valued metric spaces and $f : X \to Y$ a mapping. Then f is continuous at $x^* \in X$ if and only if $x_n \to x^*$ in X implies $fx_n \to fx^*$ in Y.*

Proof. ($\Rightarrow$) Suppose that $x_n \to x^*$. For each $\epsilon \in \mathbb{C}$ with $0 < \epsilon$, there exists $\delta \in \mathbb{C}$ with $0 < \delta$ such that $d(x, x^*) < \delta$ implies $\rho(fx, fx^*) < \epsilon$. So for δ and $x_n \to x^*$, there exists $N \in \mathbb{N}$ such that $d(x_n, x^*) < \delta$ as $n > N$, hence $\rho(fx_n, fx^*) < \epsilon$ as $n > N$, therefore $fx_n \to fx^*$.

($\Leftarrow$) Suppose that f is not continuous at x^*, then there exists $\epsilon_0 \in \mathbb{C}$ with $0 < \epsilon_0$ such that for each $\delta \in \mathbb{C}$ with $0 < \delta$, there exists $x_\delta \in X$ satisfying $d(x_\delta, x^*) < \delta$, but $\rho(fx_\delta, fx^*) \geq \epsilon_0$. Fix δ, then for each $n \in \mathbb{N}$, $0 < \frac{\delta}{n}$. Hence for $\frac{\delta}{n}$, there exists $x_n \in X$ such that $d(x_n, x^*) < \frac{\delta}{n}$, but $\rho(fx_n, fx^*) \geq \epsilon_0$, $\forall\, n \in \mathbb{N}$. Obviously, for each $c \in \mathbb{C}$ with $0 < c$, there exists $N \in \mathbb{N}$ such that $\frac{\delta}{n} < c$ as $n > N$, hence $d(x_n, x^*) < c$ as $n > N$. This shows that $x_n \to x^*$. But $\rho(fx_n, fx^*) \geq \epsilon_0$, this implies that fx_n is not convergent to fx^*. The contradiction shows that f is continuous at x^*. □

Lemma 2.6. (Cauchy Principle) *Let $\{x_n\}$ be a sequence in a complex valued metric space (X, d). If there exists $0 \leq h < 1$ such that for all $n \in \mathbb{N}$,*

$$d(x_{n+1}, x_n) \leq h\, d(x_n, x_{n-1}).$$

Then $\{x_n\}$ is a Cauchy sequence.

Proof. By given conditions, we obtain

$$|d(x_{n+1}, x_n)| \leq h\,|d(x_n, x_{n-1})| \leq \cdots \leq h^{n-1}|d(x_1, x_2)|, \; \forall\, n = 1, 2, 3, \cdots,$$

hence, for all $n, m \in \mathbb{N}$,

$$\begin{aligned} |d(x_n, x_{n+m})| &\leq |d(x_n, x_{n+1})| + |d(x_{n+1}, x_{n+2})| + \cdots + d(x_{n+m-1}, x_{n+m})| \\ &\leq (h^{n-1} + h^n + h^{n+m-2})|d(x_1, x_2)| \\ &\leq \frac{h^{n-1}}{1-h}|d(x_1, x_2)|. \end{aligned}$$

So $\{x_n\}$ is a Cauchy sequence by Lemma 2.1. □

LEMMA 2.7. ([10]) *Let $f, g : X \to X$ be weakly compatible. If f and g have a unique point of coincidence, that is, there exist an element $x \in X$ and a unique element $w \in X$ satisfying $w = fx = gx$, then w is the unique common fixed point of f and g. In this case, x is said to be a coincidence point of f and g, w is said to be a point of coincidence of f and g.*

3. Common fixed point theorems

THEOREM 3.1. *Let (X, d) be a complex valued metric space, $S, T : X \to X$ two mappings satisfying $SX \subset TX$. Suppose that for each $x, y \in X$ with $x \neq y$,*

$$d(Tx, Ty) \geq \alpha\, d(Sx, Tx) + \beta\, d(Sy, Ty) + \gamma\, d(Sx, Sy), \tag{3.1}$$

where $\alpha, \beta, \gamma \geq 0$. If (i) TX or SX is complete; (ii) $\alpha + \beta + \gamma > 1$; (iii) $\alpha < 1$ or $\beta < 1$. Then S and T have a point of coincidence. Furthermore, if $\alpha, \beta, \gamma \geq 0$ satisfy $\gamma > 1$ and $\alpha < 1$ or $\beta < 1$, then S and T have a unique point of coincidence. And if S and T are also weakly compatible, then S and T have a unique common fixed point.

Proof. For any given $x_0 \in X$, we can use the condition $SX \subset TX$ to construct two sequences $\{x_n\}$ and $\{y_n\}$ satisfying

$$y_n = Sx_n = Tx_{n+1},\ \forall\, n = 0, 1, 2, \cdots .$$

If there exists n satisfying $x_n = x_{n+1}$, then y_n is the point of coincidence of S and T. Hence we assume that

$$x_n \neq x_{n+1},\ \forall\ n = 1, 2, \cdots .$$

Suppose that $\alpha < 1$. Take $x = x_n, y = x_{n+1}$, then by (3.1),

$$d(Tx_n, Tx_{n+1}) \geq \alpha\, d(Sx_n, Tx_n) + \beta\, d(Sx_{n+1}, Tx_{n+1}) + \gamma\, d(Sx_n, Sx_{n+1}),$$

that is,

$$d(y_{n-1}, y_n) \geq \alpha\, d(y_n, y_{n-1}) + \beta\, d(y_{n+1}, y_n) + \gamma\, d(y_n, y_{n+1}),$$

hence

$$d(y_n, y_{n+1}) \leq \frac{1-\alpha}{\beta+\gamma}\, d(y_{n-1}, y_n),\ \forall\, n = 1, 2, 3, \cdots . \tag{3.2}$$

Suppose that $\beta < 1$. Take $x = x_{n+1}, y = x_n$, then by (3.1),

$$d(Tx_{n+1}, Tx_n) \geq \alpha\, d(Sx_{n+1}, Tx_{n+1}) + \beta\, d(Sx_n, Tx_n) + \gamma\, d(Sx_{n+1}, Sx_n),$$

that is

$$d(y_n, y_{n-1}) \geq \alpha\, d(y_{n+1}, y_n) + \beta\, d(y_n, y_{n-1}) + \gamma\, d(y_{n+1}, y_n),$$

hence

$$d(y_n, y_{n+1}) \leq \frac{1-\beta}{\alpha+\gamma}\, d(y_{n-1}, y_n),\ \forall\, n = 1, 2, 3, \cdots . \tag{3.3}$$

Combining (3.2), (3.3) and (ii), we obtain

$$d(y_n, y_{n+1}) \leq h\, d(y_{n-1}, y_n),\ \forall\, n = 1, 2, 3, \cdots , \tag{3.4}$$

where $h = \max\{\frac{1-\beta}{\alpha+\gamma}, \frac{1-\alpha}{\beta+\gamma}\} < 1$. This shows that $\{y_n\}$ is a Cauchy sequence by Lemma 2.6.

Suppose that TX is complete. Since $y_n = Sx_n = Tx_{n+1} \in TX$, there exists $z \in X$ such that $y_n \to Tz$ as $n \to \infty$.

When $\alpha \neq 0$, we take $x = z, y = x_{n+1}$ and use (3.1) to obtain

$$d(Tz, Tx_{n+1}) \geq \alpha\, d(Sz, Tz) + \beta\, d(Sx_{n+1}, Tx_{n+1}) + \gamma\, d(Sz, Sx_{n+1}),$$

hence

$$d(Tz, y_n) \geq \alpha\, d(Sz, Tz) + \beta\, d(y_{n+1}, y_n) + \gamma\, d(Sz, y_{n+1}) \geq \alpha\, d(Sz, Tz),$$

so

$$|d(Tz, y_n)| \geq \alpha\, |d(Sz, Tz)|.$$

Let $n \to \infty$, then $|d(Sz, Tz)| = 0$ by Lemma 2.4, hence $Sz = Tz$.

when $\beta \neq 0$, we take $x = x_{n+1}, y = z$ and use (3.1) to obtain

$$d(y_n, Tz) \geq \alpha\, d(y_{n+1}, y_n) + \beta\, d(Sz, Tz) + \gamma\, d(y_{n+1}, Sz) \geq \beta\, d(Sz, Tz),$$

hence

$$|d(y_n, Tz)| \geq \beta\, |d(Sz, Tz)|.$$

Let $n \to \infty$, then $|d(Sz, Tz)| = 0$ by Lemma 2.4, hence $Sz = Tz$.

When $\gamma \neq 0$, we take $x = x_{n+1}, y = z$ and use (3.1) to obtain

$$d(y_n, Tz) \geq \alpha\, d(y_{n+1}, y_n) + \beta\, d(Sz, Tz) + \gamma\, d(y_{n+1}, Sz) \geq \gamma\, d(y_{n+1}, Sz),$$

hence

$$|d(y_n, Tz)| \geq \gamma\, |d(y_{n+1}, Sz)|.$$

Let $n \to \infty$, then $|d(Sz, Tz)| = 0$ by Lemma 2.4, hence $Sz = Tz$.

So in any case, $Tz = Sz$ holds. Let $u = Tz = Sz$, then u is a point of coincidence of S and T.

Suppose that SX is complete. Since $y_n = Sx_n \in SX \subset TX$, there exist $z_1, z_2 \in X$ satisfying $y_n \to Sz_1 = Tz_2$. Hence we can similarly obtain that $Sz_2 = Tz_2$, therefore S and T have a point of coincidence.

If α, β, γ satisfy $\gamma > 1$ and $\alpha < 1$ or $\beta < 1$, then they also satisfy (ii) and (iii), hence S and T have a point of coincidence $u = Sz = Tz$.

Suppose that $w = Sv = Tv$ is also a point of coincidence of S and T. Let $x = z, y = v$, then by (3.1),

$$\begin{aligned} d(u,w) = d(Tz,Tv) &\geq \alpha\, d(Sz,Tz) + \beta\, d(Sv,Tv) + \gamma\, d(Sz,Sv) \\ &\geq \gamma\, d(Sz,Sv) = \gamma\, d(u,w), \end{aligned}$$

hence $d(u,w) = 0$ since $\lambda > 1$. So $u = w$, i.e., u is the unique point of coincidence of S and T. The last result follows from Lemma 2.7. □

Remark 3.2. 1) If $S = 1_X$, then the condition of Theorem 3.1 is the II-expansive condition in ([17]), hence the condition of Theorem 3.1 generalizes the condition in ([17]). And Theorem 3.1 for case $S = 1_X$ is a generalization of a fixed point theorem for a II-expansive mapping in ([17]).

2) Theorem 3.1 is the generalization and improvement of [[13], Theorem 2.1] in complex valued metric spaces.

3) In Theorem 3.1, we not only give the existence of point of coincidence for two mappings, but also give the sufficient condition of existence of unique point of coincidence. On the other hand, in Theorem 3.1, we discuss the unique existence of common fixed point, but other authors, including the authors in [13], do not discuss the uniqueness.

Example 3.3. *Consider the complex valued metric space (X,d) in Example 1.3. Define two mappinggs $T,S : X \to X$ by*

$$Ta = a, Tb = c, Tc = b, \;\; Sa = a, Sb = a, Sc = c.$$

Obviously, $TX = X$ is complete, $SX \subset TX$ and S and T are weakly compatible. Take $\alpha = \frac{1}{16}, \beta = \frac{2}{16}, \gamma = \frac{17}{16}$.

It is easy to check that

$$\begin{aligned} d(Ta,Tb) = 2+3i &\geq \frac{1}{16}\,0 + \frac{2}{16}\,(2+3i) + \frac{17}{16}\,0 \\ &= \alpha\, d(Sa,Ta) + \beta\, d(Sb,Tb) + \gamma\, d(Sa,Sb); \\ d(Ta,Tc) = 3+4i &\geq \frac{1}{16}\,0 + \frac{2}{16}\,(4+5i) + \frac{17}{16}\,(2+3i) \\ &= \alpha\, d(Sa,Ta) + \beta\, d(Sc,Tc) + \gamma\, d(Sa,Sc); \\ d(Tb,Ta) = 2+3i &\geq \frac{1}{16}\,(2+3i) + \frac{2}{16}\,0 + \frac{17}{16}\,0 \\ &= \alpha\, d(Sb,Tb) + \beta\, d(Sa,Ta) + \gamma\, d(Sb,Sa); \end{aligned}$$

$$\begin{aligned} d(Tb,Tc) = 4+5i &\geq \frac{1}{16}\,(2+3i) + \frac{2}{16}\,(4+5i) + \frac{17}{16}\,(2+3i) \\ &= \alpha\, d(Sb,Tb) + \beta\, d(Sc,Tc) + \gamma\, d(Sb,Sc); \\ d(Tc,Ta) = 3+4i &\geq \frac{1}{16}\,(4+5i) + \frac{2}{16}\,0 + \frac{17}{16}\,(2+3i) \\ &= \alpha\, d(Sc,Tc) + \beta\, d(Sa,Ta) + \gamma\, d(Sc,Sa); \\ d(Tc,Tb) = 4+5i &\geq \frac{1}{16}\,(4+5i) + \frac{2}{16}\,(2+3i) + \frac{17}{16}\,(2+3i) \\ &= \alpha\, d(Sc,Tc) + \beta\, d(Sb,Tb) + \gamma\, d(Sc,Sb). \end{aligned}$$

Hence, T, S, α, β and γ satisfy all conditions of Theorem 3.1. So T and S have a unique common fixed point. In fact, a is the unique common fixed point of T and S.

Using Theorem 3.1, we can give following fixed point theorems.

COROLLARY 3.4. *Let (X,d) be a complex valued metric space, $S : X \to X$ a mapping. If for each $x, y \in X$ with $x \neq y$,*

$$d(x,y) \geq \alpha\, d(Sx,x) + \beta\, d(Sy,y) + \gamma\, d(Sx,Sy),$$

where $\alpha, \beta, \gamma \geq 0$. Suppose that (i) SX is complete; (ii) $\alpha+\beta+\gamma > 1$; (iii) $\alpha < 1$ or $\beta < 1$. Then S has a fixed point. In particular, if $\gamma > 1$ and $\alpha < 1$ or $\beta < 1$, then S has a unique fixed point.

Proof. Take $T = 1_X$, then all conditions of Theorem 3.1 are satisfied. Hence there exist $u, z \in X$ such that $u = Sz = Tz = z$, therefore u is a fixed point of S. The rest of the argument is similar to Theorem 3.1. □

COROLLARY 3.5. *Let (X,d) be a complex valued metric space, $T : X \to X$ a mapping. Suppose that for each $x, y \in X$ with $x \neq y$,*

$$d(Tx,Ty) \geq \alpha\, d(T^2x,Tx) + \beta\, d(T^2y,Ty) + \gamma\, d(T^2x,T^2y),$$

where $\alpha, \beta, \gamma \geq 0$. If (i) TX is complete; (ii) $\alpha+\beta+\gamma > 1$; (iii) $\alpha < 1$ or $\beta < 1$. Then T has a fixed point. In particular, if $\gamma > 1$ and $\alpha < 1$ or $\beta < 1$, then T has a unique fixed point.

Proof. Let $S = T^2$, then all conditions of Theorem 3.1 are satisfied, hence there exist $u, z \in X$ such that $u = Tz = Sz = T^2z$, so $u = Tz$ is a fixed point of T. Suppose that $\gamma > 1$ and v is a fixed point of T. Let $x = u, y = v$, then we easily obtain $d(u,v) \geq \gamma d(u,v)$ by the given conditions, hence $u = v$, i.e., u is the unique fixed point of T. □

THEOREM 3.6. *Let (X,d) be a complex valued metric space, $S, T : X \to X$ two mappings satisfying $SX \subset TX$. If for each $x, y \in X$ with*

$x \neq y$,

$$d(Tx,Ty) + \alpha\, d(Sx,Ty) + \beta\, d(Sy,Tx) \geq \gamma\, d(Sx,Sy), \tag{3.5}$$

where $\alpha, \beta, \gamma \geq 0$. Suppose that (i) TX or SX is complete; (ii) $1+2\alpha < \gamma$ or $1+2\beta < \gamma$. Then S and T have a point of coincidence. Furthermore, if $1+\alpha+\beta < \gamma$, then S and T have a unique point of coincidence. If S and T are also weakly compatible, then S and T have a unique common fixed point.

Proof. Just as Theorem 3.1, we construct $\{x_n\}$ and $\{y_n\}$ such that

$$y_n = Sx_n = Tx_{n+1},\ x_n \neq x_{n+1},\ \forall\ n = 0,1,2,\cdots.$$

Suppose that $1+2\alpha < \gamma$. Taking $x = x_{n+2}, y = x_{n+1}$ and using (3.5), we obtain

$$d(y_{n+1}, y_n) + \alpha\, d(y_{n+2}, y_n) \geq \gamma\, d(y_{n+2}, y_{n+1}),$$

hence

$$d(y_{n+1}, y_n) + \alpha\, [d(y_{n+2}, y_{n+1}) + d(y_{n+1}, y_n)] \geq \gamma\, d(y_{n+2}, y_{n+1}).$$

So

$$d(y_{n+2}, y_{n+1}) \leq \frac{1+\alpha}{\gamma-\alpha} d(y_{n+1}, y_n),\ \forall\, n = 0,1,2,\cdots. \tag{3.6}$$

Suppose that $1+2\beta < \gamma$. Taking $x = x_{n+1}, y = x_{n+2}$ and using (3.5), we obtain

$$d(y_n, y_{n+1}) + \beta\, d(y_{n+2}, y_n) \geq \gamma\, d(y_{n+1}, y_{n+2}),$$

hence

$$d(y_{n+1}, y_n) + \beta\, [d(y_{n+2}, y_{n+1}) + d(y_{n+1}, y_n)] \geq \gamma\, d(y_{n+2}, y_{n+1}),$$

and so

$$d(y_{n+2}, y_{n+1}) \leq \frac{1+\beta}{\gamma-\beta} d(y_{n+1}, y_n),\ \forall\, n = 0,1,2,\cdots. \tag{3.7}$$

Let $h = \frac{1+\alpha}{\gamma-\alpha}$ or $h = \frac{1+\beta}{\gamma-\beta}$, then $0 < h < 1$. by (3.6) and (3.7),

$$d(y_{n+2}, y_{n+1}) \leq h\, d(y_{n+1}, y_n),\ \forall\, n = 0,1,2,\cdots. \tag{3.8}$$

Hence $\{y_n\}$ is a Cauchy sequence by Lemma 2.6.

Suppose that TX is complete. Since $y_n = Sx_n = Tx_{n+1} \in TX$, there exists $z \in X$ satisfying $y_n \to Tz$.

Suppose that $1+2\beta < \gamma$. Taking $x = x_{n+1}, y = z$ and using (3.5), we obtain

$$d(y_n, Tz) + \alpha\, d(y_{n+1}, Tz) + \beta\, d(Sz, y_n) \geq \gamma\, d(y_{n+1}, Sz), \forall\, n = 1,2,\cdots,$$

hence

$|d(y_n, Tz)|+\alpha\,|d(y_{n+1}, Tz)|+\beta\,|d(Sz, y_n)| \geq \gamma\,|d(y_{n+1}, Sz)|, \forall\, n = 1, 2, \cdots.$

Let $n \to \infty$, then by Lemma 2.4, we have

$$\beta\,|d(Sz, Tz)| \geq \gamma\,|d(Tz, Sz)|.$$

Hence $Sz = Tz$ since $\beta < \gamma$. Similarly, we can also obtain that $Sz = Tz$ for the case $1 + 2\alpha < \gamma$. Denote $u = Tz = Sz$, then u is a point of coincidence of S and T.

Suppose that SX is complete. Since $y_n = Sx_n \in SX \subset TX$, there exist $z_1, z_2 \in X$ such that $y_n \to Sz_1 = Tz_2$, then we can similarly prove that $Sz_2 = Tz_2$, so S and T have a point of coincidence.

If $1 + \alpha + \beta < \gamma$, then $1 + 2\alpha < \gamma$ or $1 + 2\beta < \gamma$, hence S and T have a point of coincidence. Suppose that $w = Sv = Tv$ is also a point of coincidence of S and T. Taking $x = z, y = v$ and using (3.5), we obtain

$$d(Tz, Tv) + \alpha\, d(Sz, Tv) + \beta\, d(Sv, Tz) \geq \gamma\, d(Sz, Sv),$$

i.e.,

$$(1 + \alpha + \beta)d(u, w) \geq \gamma d(u, w).$$

Hence $d(u, w) = 0$, i.e., $u = w$. So u is the unique point of coincidence of S and T. The last result follows from Lemma 2.7. □

Example 3.7. *Consider the space (X, d) and two mappings S and T in Example 3.3. Let $\alpha = \beta = 0.1, \gamma = 1.3$, then $1+\alpha+\beta = 1.2 < 1.3 = \gamma$. By careful calculations, one find the fact all the conditions of Theorem 3.6 are fulfilled. So S and T have a unique common fixed point. In fact, a is the unique common fixed point of S and T.*

Using Theorem 3.6, we give the following fixed point results.

Corollary 3.8. *Let (X, d) be a complete complex valued metric space, $T : X \to X$ an onto mapping. Suppose that for each $x, y \in X$ with $x \neq y$,*

$$d(Tx, Ty) + \alpha\, d(x, Ty) + \beta\, d(y, Tx) \geq \gamma\, d(x, y),$$

where $\alpha, \beta, \gamma \geq 0$. If $1 + \alpha + \beta < \gamma$, then T has a unique fixed point.

Corollary 3.9. *Let (X, d) be a complex valued metric space, $S : X \to X$ a mapping. Suppose that for each $x, y \in X$ with $x \neq y$,*

$$d(x, y) + \alpha\, d(Sx, y) + \beta\, d(Sy, x) \geq \gamma\, d(Sx, Sy),$$

where $\alpha, \beta, \gamma \geq 0$. If (i) SX is complete; (ii) $1 + \alpha + \beta < \gamma$, then S has a unique fixed point.

Corollary 3.10. *Let (X, d) be a complex valued metric space, $T : X \to X$ a mapping. Suppose that for each $x, y \in X$ with $x \neq y$,*

$$d(Tx, Ty) + \alpha\, d(T^2x, Ty) + \beta\, d(T^2y, Tx) \geq \gamma\, d(T^2x, T^2y),$$

where $\alpha, \beta, \gamma \geq 0$. If (i) TX is complete; (ii) $1 + \alpha + \beta < \gamma$, then T has a unique fixed point.

Theorem 3.11. *Let (X, d) be a complete complex valued metric space, $S, T : X \to X$ two onto mappings. Suppose that for each $x, y \in X$ with $x \neq y$,*

$$d(Sx, Ty) + \alpha\, d(x, Ty) + \beta\, d(y, Sx) \geq \gamma\, d(x, y), \tag{3.9}$$

where $\alpha, \beta, \gamma \geq 0$, and $\gamma > 2\max\{\alpha, \beta\} + 1$, then S and T have a unique common fixed point.

Proof. Take any $x_0 \in X$, we construct a sequence $\{x_n\}$ as follows

$$x_{2n} = Sx_{2n+1},\ x_{2n+1} = Tx_{2n+2}, n = 0, 1, 2, \cdots.$$

If there exists n such that $x_{2n} = x_{2n+1}$, then taking $x = x_{2n+1}, y = x_{2n+2}$ and using (3.9), we obtain

$$d(x_{2n}, x_{2n+1}) + \beta d(x_{2n+2}, x_{2n}) \geq \gamma d(x_{2n+1}, x_{2n+2}),$$

which implies that

$$d(x_{2n}, x_{2n+1}) + \beta[d(x_{2n+2}, x_{2n+1}) + d(x_{2n+1}, x_{2n})] \geq \gamma d(x_{2n+1}, x_{2n+2}),$$

i.e.,

$$\beta d(x_{2n+2}, x_{2n+1}) \geq \gamma d(x_{2n+1}, x_{2n+2}).$$

Hence $d(x_{2n+1}, x_{2n+2}) = 0$ since $\beta < \gamma$, that is, $x_{2n+1} = x_{2n+2}$. Therefore $Sx_{2n+1} = x_{2n} = x_{2n+1} = Tx_{2n+2} = Tx_{2n+1}$ implies that x_{2n+1} is a common fixed point of S and T.

If there exists n such that $x_{2n+1} = x_{2n+2}$, then taking $x = x_{2n+3}, y = x_{2n+2}$ and using (3.9), we obtain

$$d(x_{2n+2}, x_{2n+1}) + \alpha d(x_{2n+3}, x_{2n+1}) \geq \gamma d(x_{2n+3}, x_{2n+2}),$$

which implies that

$$d(x_{2n+2}, x_{2n+1}) + \alpha[d(x_{2n+3}, x_{2n+2}) + d(x_{2n+2}, x_{2n+1})] \geq \gamma d(x_{2n+1}, x_{2n+2}),$$

i.e.,

$$\alpha d(x_{2n+3}, x_{2n+2}) \geq \gamma d(x_{2n+3}, x_{2n+2}).$$

Hence $d(x_{2n+3}, x_{2n+2}) = 0$, that is, $x_{2n+3} = x_{2n+2}$. Therefore $Tx_{2n+2} = x_{2n+1} = x_{2n+2} = Sx_{2n+3} = Sx_{2n+2}$ implies that x_{2n+2} is a common fixed point of S and T.

Hence from now on, we assume that $x_n \neq x_{n+1}$, $\forall\, n = 0, 1, 2, \cdots$.

Taking $x = x_{2n+1}, y = x_{2n+2}$ and using (3.9), we obtain

$$d(x_{2n}, x_{2n+1}) + \beta d(x_{2n+2}, x_{2n}) \geq \gamma d(x_{2n+1}, x_{2n+2}),$$

which implies that

$$d(x_{2n}, x_{2n+1}) + \beta[d(x_{2n+2}, x_{2n+1}) + d(x_{2n+1}, x_{2n})] \geq \gamma d(x_{2n+1}, x_{2n+2}),$$

hence

$$d(x_{2n+1}, x_{2n+2}) \leq \frac{1+\beta}{\gamma-\beta} d(x_{2n+1}, x_{2n}),\ \forall\, n = 0, 1, \cdots . \tag{3.10}$$

Similarly, taking $x = x_{2n+3}, y = x_{2n+2}$ and using (3.9), we obtain

$$d(x_{2n+2}, x_{2n+1}) + \alpha d(x_{2n+3}, x_{2n+1}) \geq \gamma d(x_{2n+3}, x_{2n+2}),$$

which implies that

$$d(x_{2n+2}, x_{2n+1}) + \alpha[d(x_{2n+3}, x_{2n+2}) + d(x_{2n+2}, x_{2n+1})] \geq \gamma d(x_{2n+3}, x_{2n+2}),$$

hence

$$d(x_{2n+3}, x_{2n+2}) \leq \frac{1+\alpha}{\gamma-\alpha} d(x_{2n+2}, x_{2n+1}),\ \forall\, n = 0, 1, \cdots . \tag{3.11}$$

Let $h = \max\{\frac{1+\alpha}{\gamma-\alpha}, \frac{1+\beta}{\gamma-\beta}\}$, then $0 < h < 1$, and from (3.10) and (3.11),

$$d(x_{n+2}, x_{n+1}) \leq h d(x_{n+1}, x_n),\ \forall\, n = 0, 1, \cdots . \tag{3.12}$$

Hence $\{x_n\}$ is a Cauchy sequence by Lemma 2.6. Since X is complete, there exists $u \in X$ such that $x_n \to u$. Since T is onto, there exists $z \in X$ such that $u = Tz$, $x_{2n+1} \to Tz$ and $x_{2n} \to Tz$. Taking $x = x_{2n+1}, y = z$ and using (3.9), we obtain

$$d(x_{2n}, Tz) + \alpha\, d(x_{2n+1}, Tz) + \beta\, d(z, x_{2n}) \geq \gamma\, d(x_{2n+1}, z),$$

hence

$$|d(x_{2n}, Tz)| + \alpha\, |d(x_{2n+1}, Tz)| + \beta\, |d(z, x_{2n})| \geq \gamma\, |d(x_{2n+1}, z)|.$$

Let $n \to \infty$, then by Lemma 2.4, the above inequality becomes

$$\beta\, |d(z, Tz)| \geq \gamma\, |d(Tz, z)|.$$

Hence $|d(z, Tz)| = 0$ since $0 \leq \beta < \gamma$, that is, $Tz = z$.

Similarly, Since S is onto, there exists $w \in X$ such that $u = Sw$, $x_{2n+1} \to Sw$ and $x_{2n} \to Sw$. Taking $x = w$, $y = x_{2n+2}$ and using (3.9), we obtain

$$d(Sw, x_{2n+1}) + \alpha\, d(w, x_{2n+1}) + \beta\, d(x_{2n+2}, Sw) \geq \gamma\, d(w, x_{2n+2}),$$

hence

$$|d(Sw, x_{2n+1})| + \alpha\, |d(w, x_{2n+1})| + \beta\, |d(x_{2n+2}, Sw)| \geq \gamma\, |d(w, x_{2n+2})|.$$

Let $n \to \infty$, then by Lemma 2.4, the above inequality becomes

$$\alpha\,|d(w,Sw)| \geq \gamma\,|d(w,Sw)|.$$

Hence $|d(w,Sw)| = 0$ since $0 \leq \alpha < \gamma$, that is, $Sw = w$.

Taking $x = w, y = z$ and using (3.9), we obtain

$$d(Sw,Tz) + \alpha\,d(w,Tz) + \beta\,d(z,Sw) \geq \gamma\,d(w,z),$$

that is,

$$(1+\alpha+\beta)\,d(z,w) \geq \gamma\,d(w,z). \tag{3.13}$$

Hence $d(w,z) = 0$ since $\gamma > 2\max\{\alpha,\beta\}+1 \geq 1+\alpha+\beta$, that is, $w = z$. Therefore $z = Tz = Sz$, i.e., z is a common fixed point of S and T.

Suppose that v is also a common fixed point of S and T, that is, $v = Sv = Tv$ holds. Taking $x = z, y = v$ and using (3.9), we obtain

$$(1+\alpha+\beta)d(z,v) \geq \gamma d(z,v).$$

Hence $z = v$, So z is the unique common fixed point of S and T. □

Using Theorem 3.11, we give the following particular forms:

Corollary 3.12. *Let (X,d) be a complete complex valued metric space, $S,T : X \to X$ two onto mappings. If for each $x,y \in x$ with $x \neq y$,*

$$d(Sx,Ty) \geq hd(x,y),$$

where $h > 1$. Then S and T have a unique common fixed point.

Corollary 3.13. *Let (X,d) be a complete complex valued metric space, $T : X \to X$ an onto mapping. If for each $x,y \in X$ with $x \neq y$,*

$$d(Tx,Ty) + \alpha\,d(x,Ty) + \beta\,d(y,Tx) \geq \gamma\,d(x,y),$$

where $\alpha,\beta,\gamma \geq 0$ and $\gamma > 2\max\{\alpha,\beta\}+1$. Then T has a unique fixed point.

Finally, we give a fixed point theorem for an expansive map which is weaker than I-expansive condition but stronger than III-expansive condition.

Theorem 3.14. *Let (X,d) be a complete complex valued metric space, $T : X \to X$ an onto mapping. Suppose that $d(y,Ty)$ and $d(x,y)$ are comparable for $x,y \in X$ with $x \neq y$, and the following holds*

$$d(Tx,Ty) \geq h\,\min\{d(y,Ty), d(x,y)\}, \tag{3.14}$$

where $h > 1$. If T is continuous, then T has a fixed point.

Proof. Take any element $x_0 \in X$, and construct a sequence $\{x_n\}$ in X satisfying

$$x_n = Tx_{n+1},\ n = 0, 1, 2, \cdots.$$

If there exists n such that $x_n = x_{n+1}$, then x_n is the fixed point of T. So we assume that

$$x_n \neq x_{n+1}, \forall\, n = 0, 1, 2, \cdots$$

For each $n = 0, 1, 2, \cdots$, taking $x = x_{n+1}, y = x_{n+2}$ and using (3.14), we obtain

$$d(x_n, x_{n+1}) \geq h\, d(x_{n+1}, x_{n+2}),$$

hence

$$d(x_{n+1}, x_{n+2}) \leq M\, d(x_n, x_{n+1}),$$

where $M = \frac{1}{h} < 1$, Hence $\{x_n\}$ is a Cauchy sequence by Lemma 2.6. Since X is complete, there exists $x^* \in X$ such that $x_n = Tx_{n+1} \to x^*$. Since T is continuous, so $x_n = Tx_{n+1} \to Tx^*$ by Lemma 2.5. Hence $Tx^* = x^*$ by Lemma 2.3, i.e., x^* is a fixed point of T. □

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