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SOME REMARKS ON TOTAL CURVATURE OF A MINIMAL GRAPH

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Abstract. In this paper we discuss bounds for the total curvature of nonparametric minimal surfaces by using the properties of planar harmonic mappings.

1. Introduction

A complex–valued function $f(z) = u(z) + iv(z)$, defined on some domain $D \subset \mathbb{C}$ is a planar harmonic mapping if the components u and v are real–valued harmonic functions which need not be conjugate. Throughout this article we will discuss harmonic mappings of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. A harmonic mapping f defined on $\mathbb{D}$ can be uniquely written as $f = h + \overline{g}$, $g(0) = 0$, where h and g belong to the linear space $\mathcal{H}(\mathbb{D})$ of all holomorphic functions on $\mathbb{D}$. The mapping f is locally univalent if and only if its Jacobian $|h'|^2 - |g'|^2$ does not vanish. If we require that f is orientation-preserving, then the second complex dilatation $\omega(z) = g'(z)/h'(z)$ belongs to $\mathcal{H}(\mathbb{D})$ and $|\omega(z)| < 1$ on $\mathbb{D}$. References for this material include [2] and [5].

Harmonic univalent mappings were first studied in connection with minimal surfaces by E. Heinz (see [7]). This relationship between a univalent harmonic mapping and a minimal graph $\mathcal{M}$ comes from conformal representation of $\mathcal{M}$ via the Weierstrass-Enneper formulas (see e.g. [4]). Let $\mathcal{M} = \{(u, v, F(u, v)) : (u, v) \in \Omega\}$ be a nonparametric surface lying over a simply connected proper subdomain Ω of the complex plane $\mathbb{C}$. If $\mathcal{M}$ is parametrized by orientation-preserving isothermal parameters $z = x + iy \in \mathbb{D}$, the projection onto its base plane gives a univalent harmonic mapping $f(z) = u + iv$ of $\mathbb{D}$ onto Ω whose dilatation ω is the square of an holomorphic function with $|\omega(z)| < 1$ on $\mathbb{D}$. Conversely, if $f = h + \overline{g}$ is an orientation-preserving univalent harmonic mapping of $\mathbb{D}$ onto Ω with dilatation $\omega = p^2$ for some function $p \in \mathcal{H}(\mathbb{D})$, then

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the formulas

$$u(z) = \mathrm{Re}\{f(z)\}, \quad v(z) = \mathrm{Im}\{f(z)\}, \quad F(z) = 2\mathrm{Im}\left\{\int_{z_0}^{z} h'(\zeta)p(\zeta)d\zeta\right\}$$

define by isothermal parameters a nonparametric minimal surface $\mathcal{M}$ whose projection is f, so the first fundamental form of $\mathcal{M}$ is $ds^2 = \lambda^2|dz|^2$, where $\lambda = |h'| + |g'|$ is the conformal metric. Recall, that the general formula for Gauss curvature is

$$K = -\frac{1}{\lambda^2}\Delta(\log\lambda),$$

where Δ denotes the Laplacian. Therefore, in terms of the harmonic mapping $f = h + \overline{g}$, the Gauss curvature is found to be

$$K(\xi) = -\frac{|\omega'(z)|^2}{|h'(z)g'(z)|(1+|\omega(z)|)^4}, \qquad z \in \mathbb{D}$$

at a point $\xi \in \mathcal{M}$ that lies above $f(z)$. The equality $\omega = p^2$ implies that

$$K = -\frac{4|p'|^2}{(|h'|+|g'|)^2(1+|p|^2)^2}. \tag{1}$$

2. Main results

Suppose that $\mathcal{M}$ is a minimal graph given by orientation-preserving isothermal parameters over a simply connected domain Ω, $\Omega \subset \mathbb{C}$, and $f : \mathbb{D} \to \Omega$ is an orientation-preserving univalent harmonic mapping with dilatation $\omega = p^2$ corresponding to $\mathcal{M}$. If we now let $\mathcal{G}$ to be a domain whose closure is in $\mathbb{D}$, the surface over $f(\mathcal{G}) \subset \Omega$ has total curvature $T_{f,p}(\mathcal{G})$ given by

$$T_{f,p}(\mathcal{G}) = \iint_{\mathcal{G}} K\lambda^2 dA(z), \quad z = x + iy, \tag{2}$$

where $dA(z) = dxdy$ is the Euclidean area element. From (1) and (2) we obtain

$$T_{f,p}(\mathcal{G}) = -4\iint_{\mathcal{G}} \frac{|p'(z)|^2}{(1+|p(z)|^2)^2} dA(z), \tag{3}$$

which is due to the fact that f has dilatation $\omega = p^2$. It should be noted that $p^\sharp(z) = \frac{|p'(z)|}{1+|p(z)|^2}$ is the spherical derivative of p, and consequently $\frac{1}{4}|T_{f,p}(\mathcal{G})|$ is the area of the spherical image of $p(\mathcal{G})$ counted with multiplicity. By the above $|T_{f,p}(\mathcal{G})| \geq 4\mathcal{A}_s(p(\mathcal{G}))$, where $\mathcal{A}_s(p(\mathcal{G}))$ is the spherical area of the set $p(\mathcal{G})$ identified with its projection to the Riemann sphere.

Since p is a holomorphic function mapping $\mathbb{D}$ into itself, the Schwarz-Pick lemma gives

$$|T_{f,p}(\mathcal{G})| \leq 4\iint_{\mathcal{G}} \left(\frac{1-|p(z)|^2}{1+|p(z)|^2}\right)^2 \frac{dA(z)}{(1-|z|^2)^2} \leq 4\iint_{\mathcal{G}} \frac{dA(z)}{(1-|z|^2)^2}$$

as is easy to check. An estimate due to Goluzin yields a refinement of the above inequality.

Lemma 2.1. *Suppose that* $p \in \mathcal{H}(\mathbb{D})$ *has the expansion* $p(z) = a_0 + \sum_{n=k}^{\infty} a_n z^n$, *where* $k \geq 1$, *and* f *is an orientation-preserving univalent harmonic mapping with dilatation* $\omega = p^2$. *Then*

$$|T_{f,p}(\mathcal{G})| \leq 4k^2 \iint_{\mathcal{G}} \frac{|z|^{2k-2}}{(1-|z|^{2k})^2} dA(z).$$

Proof. We have used only the fact (see [6, page 333]) that

$$(1-|z|^{2k})|p'(z)| \leq k|z|^{k-1}(1-|p(z)|^2), \quad z \in \mathbb{D} \tag{4}$$

for a function $p(z) = a_0 + \sum_{n=k}^{\infty} a_n z^n$, where $k \geq 1$, holomorphic and $|p(z)| < 1$ on $\mathbb{D}$. □

Given any $r \in (0,1)$ suppose that $D_r = \{z \in \mathbb{C} : |z| < r\}$. Taking $\mathcal{G} = D_r$ we can now rephrase Lemma 2.1 as follows.

Corollary 2.2. *For any* $r \in (0,1)$ *we have*

$$|T_{f,p}(D_r)| \leq 4\pi \frac{r^2}{1-r^2}.$$

Theorem 2.3. *Let* f *be a harmonic univalent mapping of* $\mathbb{D}$ *with dilatation* $\omega = p^2$, *where* $p \in \mathcal{H}(\mathbb{D})$, $p(\mathbb{D}) \subset \mathbb{D}$ *and* $p(0) = 0$. *If* $r^2 \in [\frac{m-1}{m}, \frac{m}{m+1}]$, *where* $m \in \mathbb{N}$, *then*

$$|T_{f,p}(D_r)| \leq 4m\pi r^{2m}.$$

Proof. Assume that f has dilatation $\omega = p^2$ and observe that

$$|T_{f,p}(D_r)| \leq 4 \iint_{D_r} |p'(z)|^2 dA(z) = 4 \int_0^{2\pi} \int_0^r |p'(\rho e^{i\theta})| \rho d\rho d\theta$$

which is clear from (3). Then the assertion follows from [8, Corollary on page 262], since p is subordinate to the identity function $I(z) = z$. □

In particular we obtain:

Corollary 2.4. *Under the assumptions of Theorem 2.3, we have*

$$|T_{f,p}(D_r)| \leq \frac{4\pi}{e}(m+1), \quad r^2 \in \left[\frac{m-1}{m}, \frac{m}{m+1}\right]$$

for all $m \in \mathbb{N}$.

Proof. Theorem 2.3 yields $|T_{f,p}(D_r)| \leq 4\pi(m+1)(\frac{m}{m+1})^{m+1}$ for all $\frac{m-1}{m} \leq r^2 \leq \frac{m}{m+1}$. To prove the result we note that $((\frac{m}{m+1})^{m+1})_{m\in\mathbb{N}}$ is an increasing sequence and $\lim_{m\to\infty} (\frac{m}{m+1})^{m+1} = e^{-1}$. □

Theorem 2.5. *Suppose that* f *has the dilatation* $\omega = p^2$, *where* $p \in \mathcal{H}(\mathbb{D})$, $p(0) = 0$ *and* $|p(z)| < 1$ *on* $\mathbb{D}$. *For any fixed* $r \in (0,1)$ *we have*

$$A_{f,p}(r) = \int_0^r \frac{|T_{f,p}(D_t)|}{t} dt \leq 2\pi \log(1+r^2).$$

Equality holds if p is a function of the form $p(z) = e^{i\theta}z$, $\theta \in \mathbb{R}$. In particular, for any $r \in (0,1)$ we have $A_{f,p}(r) < 2\pi \log 2$.

Proof. We have

$$\Delta \log(1 + |p(z)|^2) = \frac{4|p'(z)|^2}{(1 + |p(z)|^2)^2},$$

where Δ denotes the Laplacian in the z-coordinate. On the account of this, an application of Green's theorem gives

$$|T_{f,p}(D_r)| = \int_{|z|=r} \frac{\partial}{\partial n} \log(1 + |p(z)|^2) ds = r\frac{\partial}{\partial r}\int_0^{2\pi} \log(1 + |p(re^{i\theta})|^2) d\theta,$$

and for $r \in (0,1)$ we get

$$\int_0^r \frac{|T_{f,p}(D_t)|}{t} dt = \int_0^{2\pi} \log(1 + |p(re^{i\theta})|^2) d\theta - 2\pi \log(1 + |p(0)|^2).$$

The Schwarz lemma enables us to write $|p(z)| \leq |z|$ on $\mathbb{D}$, and finally

$$\int_0^r \frac{|T_{f,p}(D_t)|}{t} dt \leq \int_0^{2\pi} \log(1 + r^2) d\theta = 2\pi \log(1 + r^2) < 2\pi \log 2$$

for a fixed $r \in (0,1)$. □

Now, we wish to investigate the case when p is a Bloch function. This means that the Bloch constant $\beta_p = \sup\{(1 - |z|^2)|p'(z)| : z \in \mathbb{D}\}$ is finite. Assume that f is an orientation-preserving univalent harmonic mapping of $\mathbb{D}$ and has a dilation $\omega = p^2$, where p is a Bloch function. Since $p(\mathbb{D}) \subset \mathbb{D}$, the constant β_p is no greater than one. This clearly forces

$$|T_{f,p}(D_r)| \leq 4\pi\beta_p^2 \frac{r^2}{1 - r^2}, \quad r \in (0,1). \tag{5}$$

Theorem 2.6. *If f is an orientation-preserving univalent harmonic mapping with dilatation $\omega = p^2$, where p is a non-vanishing inner function or an outer function, then*

$$|T_{f,p}(D_r)| \leq \frac{16\pi}{e^2} \frac{r^2}{1 - r^2}, \quad r \in (0,1).$$

Proof. In the case we have $\beta_p = \frac{2}{e}$ (see [3, Theorem 5]). □

Given $a \in \mathbb{D}$, let $\varphi_a(z) = (a - z)/(1 - \overline{a}z)$ be a Möbius transformation of $\mathbb{D}$, and let $\Delta(a, r) = \{z \in \mathbb{D} : |\varphi_a(z)| < r\}$ be the pseudohyperbolic disk with center a and radius $r \in (0,1)$. The composition $F_a = f \circ \varphi_a$ is again univalent harmonic and orientation-preserving on $\mathbb{D}$, but the important thing is that F_a has dilatation $\omega \circ \varphi_a = (p \circ \varphi_a)^2$. Therefore, we can consider a nonparametric minimal surface $\mathcal{M}_a$ corresponding to F_a for a fixed $a \in \mathbb{D}$. Then for a fixed $r \in (0,1)$ we have

$$A_{F_a, p\circ\varphi_a}(r) = 4\int_0^r \frac{dt}{t} \iint_{D_t} [(p \circ \varphi_a)^\sharp]^2 dA(z)$$

$$
\begin{aligned}
&= 4\int_0^r \frac{dt}{t}\iint_{\Delta(a,t)} [p^\sharp(z)]^2 dA(z)\\
&\le 4\beta_p^2 \int_0^r \frac{dt}{t}\iint_{\Delta(a,t)} \frac{dA(z)}{(1-|z|^2)^2}\\
&= 2\pi\beta_p^2 \log\frac{1}{1-r^2},
\end{aligned}
$$

provided that p is a Bloch function. In particular, given $r \in (0,1)$ we have $\sup_{a\in\mathbb{D}} A_{F_a,p\circ\varphi_a}(r) < \infty$.

3. Remarks

It follows from [9, Theorem 1] that $p^\sharp(z) \le \sqrt{\frac{\beta}{\pi-\beta}}(1-|z|^2)^{-1}$ at each $z \in \mathbb{D}$, provided $\mathcal{A}_s(p(\mathbb{D})) \le \beta < \pi$. Note that in our case we have $\mathcal{A}_s(p(\mathbb{D})) \le \beta \le \frac{\pi}{2}$, since $p(\mathbb{D}) \subset \mathbb{D}$, and this gives

$$
|T_{f,p}(D_r)| \le \frac{4\pi\beta}{\pi-\beta}\frac{r^2}{1-r^2}, \quad r \in (0,1).
$$

Assume that p is not necessarily Bloch and $\sup_{a\in\mathbb{D}} A_{F_a,p\circ\varphi_a}(r) < \infty$ for some $r \in (0,1)$. Then exists $r_* \in (0,r)$ such that

$$
\sup_{a\in\mathbb{D}} A_{F_a,p\circ\varphi_a}(r) < \log\frac{r}{r_*}
$$

and according to the above

$$
\begin{aligned}
A_{F_a,p\circ\varphi_a}(r) &\ge 4\int_{r_*}^r \frac{dt}{t}\iint_{\Delta(a,t)} [p^\sharp(z)]^2 dA(z)\\
&\ge 4\left(\log\frac{r}{r_*}\right)\iint_{\Delta(a,r_*)} [p^\sharp(z)]^2 dA(z),
\end{aligned}
$$

which yields

$$
\sup_{a\in\mathbb{D}} |T_{f,p}(\Delta(a,r_*))| \le \left(\log\frac{r}{r_*}\right)^{-1} \sup_{a\in\mathbb{D}} A_{F_a,p\circ\varphi_a}(r) < 1.
$$

According to [3, Theorem 6], if $p : \mathbb{D} \to \mathbb{D}$ is holomorphic on $\mathbb{D}$ with $\beta_p = 1$ then either p is conformal automorphism of $\mathbb{D}$ or all zeros of p define a convergent infinite Blaschke product B with $\beta_B = 1$.

If p is finite Blaschke product of degree greater than 1, then $\beta_p < 1$ (see [3, Corollary on page 96]) and (5) is better than the bound given in Corollary 2.2. If, in addition, $f(\mathbb{D})$ is a simple bounded polygonal domain, then $\mathcal{M}$ is a Jenkins–Serrin surface (see e.g. [1]).

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