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EXISTENCE AND CONCENTRATION RESULTS FOR KIRCHHOFF-TYPE SCHRÖDINGER SYSTEMS WITH STEEP POTENTIAL WELL

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Abstract. In this paper, we consider the following Kirchhoff-type Schrödinger system

$$\begin{cases} -\Big(a_1 + b_1 \displaystyle\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)\Delta u + \gamma V(x)u = \dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta} & \text{in } \mathbb{R}^3,\\ -\Big(a_2 + b_2 \displaystyle\int_{\mathbb{R}^3} |\nabla v|^2 dx\Big)\Delta v + \gamma W(x)v = \dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \mathbb{R}^3,\\ u,\ v \in H^1(\mathbb{R}^3), \end{cases}$$

where a_i and b_i are positive constants for $i = 1, 2$, $\gamma > 0$ is a parameter, $V(x)$ and $W(x)$ are nonnegative continuous potential functions. By applying the Nehari manifold method and the concentration-compactness principle, we obtain the existence and concentration of ground state solutions when the parameter γ is sufficiently large.

1. Introduction and main results

Let us consider the following Kirchhoff-type Schrödinger system in $\mathbb{R}^3$:
$(\mathcal{KS})_\gamma$

$$\begin{cases} -\Big(a_1 + b_1 \displaystyle\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)\Delta u + \gamma V(x)u = \dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta} & \text{in } \mathbb{R}^3,\\ -\Big(a_2 + b_2 \displaystyle\int_{\mathbb{R}^3} |\nabla v|^2 dx\Big)\Delta v + \gamma W(x)v = \dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \mathbb{R}^3,\\ u,\ v \in H^1(\mathbb{R}^3), \end{cases}$$

where a_i and b_i are positive constants for $i = 1, 2$, $\gamma > 0$ is a parameter, $\alpha > 2$, $\beta > 2$ satisfy $\alpha + \beta < 2^* = 6$, and $V(x), W(x)$ are nonnegative continuous potential functions on $\mathbb{R}^3$.

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In recent years, many papers have extensively considered the scalar Kirchhoff-type equation

$$(1.1)\qquad \begin{cases} -\Big(a+b\displaystyle\int_\Omega |\nabla u|^2dx\Big)\Delta u = f(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases}$$

where a and b are positive constants and $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain. For example, Ma and Rivera [13] obtained positive solutions of (1.1) by using variational methods. Alves, Corrêa and Ma [1] studied problem (1.1) and obtained positive solutions via the mountain pass theorem. Perera and Zhang [15] obtained a nontrivial solution of (1.1) via Yang index and critical group. Mao and Zhang [14] obtained three solutions by the invariant sets of descent flow. He and Zou [7] showed existence of infinitely many solutions by using the local minimum methods and the fountain theorems. Cheng and Wu [5] studied the existence of positive solutions for problem (1.1) when the nonlinearity f is asymptotically t^3-growth at infinity. We also note that problem (1.1) is related to the stationary analogue of the equation

$$(1.2)\qquad \rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0,$$

where ρ, P_0, h, E, L are constants. The equation (1.2) was proposed by Kirchhoff in [9] as an extension of the classical D'Alembert wave equation for free vibrations of elastic strings. Kirchhoff's model considers the changes in length of the string produced by transverse vibrations. Problem (1.2) began to call attention of several researchers after the pioneer work of Lions [10], where a functional analysis approach was proposed. It is pointed in [1] that the problem (1.2) model may describe some physical and biological systems, where u denotes a process which depends on the average of itself.

On the other hand, the following Kirchhoff-type equation

$$(1.3)\qquad \begin{cases} -\Big(a+b\displaystyle\int_{\mathbb{R}^3} |\nabla u|^2dx\Big)\Delta u + V(x)u = f(x,u) & \text{in } \mathbb{R}^3,\\ u\in H^1(\mathbb{R}^3), & \end{cases}$$

has also been investigated by many authors. In [18], by applying symmetric mountain pass theorem, the author obtained the existence results for nontrivial solutions and a sequence of high energy solutions for problem (1.3). Subsequently, Liu and He [12] proved the existence of infinitely many high energy solutions for (1.3) when f is a subcritical nonlinearity which does't need to satisfy the usual Ambrosetti-Rabinowitz-type growth conditions. More recently, by using variational methods, Sun and Wu [16] obtained the existence and concentration of ground state solutions for (1.3) when $V(x)$ was replaced by $\lambda V(x)$, where λ is a positive parameter. We would also mention the recent papers [19, 20] where the existence of high energy solutions for Kirchhoff-type

Schrödinger systems was established. For more related works, one can also see [3, 4, 8, 17] and the references therein.

Motivated by the works mentioned above, in the present paper we will study a class of Kirchhoff-type Schrödinger systems with steep potential well in $\mathbb{R}^3$. Such problems are often referred to as being nonlocal because of the presence of the terms $(\int_{\mathbb{R}^3}|\nabla u|^2dx)\Delta u$ and $(\int_{\mathbb{R}^3}|\nabla v|^2dx)\Delta v$ which imply that problem $(\mathcal{KS})_\gamma$ is no longer pointwise identity. This phenomenon provokes some mathematical difficulties, which motivate the study of such a class of particularly interesting problems. The existence and concentration of ground state solutions of $(\mathcal{KS})_\gamma$ are obtained by applying the Nehari manifold method and concentration-compactness principle.

Before stating our main results, we need to introduce some assumptions and notations:

$(\mathcal{H}_1)$ $V(x), W(x) \in \mathcal{C}(\mathbb{R}^3, [0,+\infty))$ and $\Omega := \text{int}(V^{-1}(0)) = \text{int}(W^{-1}(0))$ is nonempty with smooth boundary and $\overline{\Omega} = V^{-1}(0) = W^{-1}(0)$;

$(\mathcal{H}_2)$ there exist $M_1, M_2 > 0$ such that $\mathcal{L}(\{x \in \mathbb{R}^3 | V(x) \leq M_1\}) < \infty$, $\mathcal{L}(\{x \in \mathbb{R}^3 | W(x) \leq M_2\}) < \infty$, where $\mathcal{L}$ denotes the Lebesgue measure in $\mathbb{R}^3$.

The conditions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ imply that $\gamma V(x)$ and $\gamma W(x)$ represent potential well whose depth is controlled by γ. $\gamma V(x)$ and $\gamma W(x)$ are called steep potential well if γ is sufficiently large, and one expects to find solutions which localize near its bottom Ω. The hypothesis $(\mathcal{H}_2)$ was first introduced by Bartsch and Wang [2] in the study of a nonlinear Schrödinger equation. Let $E_V := \{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2dx < +\infty\}$ and $E_W := \{v \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} W(x)v^2dx < +\infty\}$ with the norms $\|u\|^2_{\gamma,V} = \int_{\mathbb{R}^3}(a_1|\nabla u|^2 + \gamma V(x)u^2)dx$ and $\|v\|^2_{\gamma,W} = \int_{\mathbb{R}^3}(a_2|\nabla v|^2 + \gamma W(x)v^2)dx$ respectively. For any given $\gamma > 0$, we consider the Hilbert space $E := E_V \times E_W$ endowed with the norm

$$\|(u,v)\|_\gamma = \left(\|u\|^2_{\gamma,V} + \|v\|^2_{\gamma,W}\right)^{\frac{1}{2}}.$$

The energy functional associated with $(\mathcal{KS})_\gamma$ is defined on E by

$$\mathcal{E}_\gamma(u,v) = \frac{1}{2}\|(u,v)\|^2_\gamma + \frac{1}{4}(b_1\Upsilon^2(u) + b_2\Upsilon^2(v)) - \frac{2}{\alpha+\beta}\int_{\mathbb{R}^3}|u|^\alpha|v|^\beta dx,$$

where $\Upsilon(w) = \int_{\mathbb{R}^3}|\nabla w|^2dx$. In view of the assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, the energy functional $\mathcal{E}_\gamma(u,v)$ is well defined and belongs to $\mathcal{C}^1(E,\mathbb{R})$, and that

$$\begin{aligned}\langle \mathcal{E}'_\gamma(u,v),(\varphi,\psi)\rangle = &\Big(a_1 + b_1\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)\int_{\mathbb{R}^3}\nabla u\nabla\varphi dx \\ &+ \Big(a_2 + b_2\int_{\mathbb{R}^3}|\nabla v|^2dx\Big)\int_{\mathbb{R}^3}\nabla v\nabla\psi dx \\ &+ \gamma\int_{\mathbb{R}^3}\big(V(x)u\varphi + W(x)v\psi\big)dx \\ &- \frac{2}{\alpha+\beta}\int_{\mathbb{R}^3}\big(\alpha|u|^{\alpha-2}|v|^\beta u\varphi + \beta|u|^\alpha|v|^{\beta-2}v\psi\big)dx.\end{aligned}$$

Hence, if $(u,v)\in E$ is a critical point of $\mathcal{E}_\gamma(u,v)$, then (u,v) is a solution of problem $(\mathcal{KS})_\gamma$.

We define the minimax c_γ as

$$c_\gamma=\inf_{(u,v)\in\mathcal{N}_\gamma}\mathcal{E}_\gamma(u,v), \tag{1.4}$$

where

$$\mathcal{N}_\gamma=\big\{(u,v)\in E\setminus\{(0,0)\}:\langle\mathcal{E}'_\gamma(u,v),(u,v)\rangle=0\big\},$$

$\langle\cdot,\cdot\rangle$ is the duality product between E and its dual space E^{-1}. Note that $\mathcal{N}_\gamma$ contains every nonzero solution of problem $(\mathcal{KS})_\gamma$.

A ground state solution for $(\mathcal{KS})_\gamma$ is a critical point (u_0,v_0) of $\mathcal{E}_\gamma(u,v)$ in E which solves the following minimization problem

$$\inf_{(u,v)\in\mathcal{N}_\gamma}\mathcal{E}_\gamma(u,v)=\mathcal{E}_\gamma(u_0,v_0).$$

The main results we get are the following:

Theorem 1.1. *Suppose that $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ hold, then there is $\gamma^*>0$ such that, for all $\gamma\geq\gamma^*$, the system $(\mathcal{KS})_\gamma$ has at least a ground state solution (u_γ,v_γ) in $H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$.*

Theorem 1.2. *For each $\gamma>0$ large, let (u_γ,v_γ) be the solutions obtained in Theorem 1.1. Then $(u_\gamma,v_\gamma)\to(u,v)$ as $\gamma\to\infty$, where (u,v) is a nontrivial solution of*

$$\begin{cases}-\Big(a_1+b_1\displaystyle\int_\Omega|\nabla u|^2dx\Big)\Delta u=\dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^\beta & \text{in } \Omega,\\ -\Big(a_2+b_2\displaystyle\int_\Omega|\nabla v|^2dx\Big)\Delta v=\dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega,\\ u(x)=0,\quad v(x)=0 & \text{on } \partial\Omega.\end{cases}$$

As far as we know, problem $(\mathcal{KS})_\gamma$ has not been considered before. In order to prove the main results, we have to overcome some difficulties in using variational methods. The main difficulties lie in the appearance of the nonlocal term and the lack of compactness due to the unboundedness of the domain $\mathbb{R}^3$. Since we neither assume that the potentials are radially symmetric nor impose any other hypotheses on the behavior of the potentials for $|x|\to\infty$, we can not use the usual way to recover compactness. To recover the compactness, we adopt the idea used in [2] and establish the compactness conditions dependent of parameter. Let us point out that the adaptation of the idea to our problem is not trivial at all because of the presence of the nonlocal terms $(\int_{\mathbb{R}^3}|\nabla u|^2dx)\Delta u$, $(\int_{\mathbb{R}^3}|\nabla v|^2dx)\Delta v$ and the coupled terms $|u|^{\alpha-2}u|v|^\beta$, $|u|^\alpha|v|^{\beta-2}v$.

This paper is organized as follows. In Section 2, we will prove some important lemmas that will be used for the proofs of the main results. Section 3 is devoted to the proofs of Theorems 1.1 and 1.2.

2. Preliminaries

Notation. In this paper we will use the following notations:

- $C, C_1, C_2, \ldots$ denote positive(possibly different) constants.
- $\to$ (respectively $\rightharpoonup$) denotes strong (respectively weak) convergence.
- $o_n(1)$ denotes $o_n(1) \to 0$ as $n \to \infty$.
- $L^s(\Omega)(1 \leq s < +\infty)$ denote Lebesgue spaces, the norm L^s is denoted by $|\cdot|_s$ for $1 \leq s < +\infty$.
- For given set $K \subset \mathbb{R}^3$, we set $K^c = \mathbb{R}^3 \backslash K$.
- B_r denotes a ball centered at the origin with radius $r > 0$.
- The dual space of a Banach space E will be denoted by E^{-1}.
- $\langle \cdot, \cdot \rangle$ denote the duality pairing between E^{-1} and E.

The functional $I \in C^1(E, \mathbb{R})$ is said to satisfy the $(PS)_c$ condition if any sequence $\{z_n\} \subset E$ such that as $n \to \infty$, $I(z_n) \to c, I'(z_n) \to 0$ strongly in E^{-1} contains a subsequence converging in E to a critical point of I. In this paper, we will take $I = \mathcal{E}_\gamma(u, v)$ and $E = E_V \times E_W$.

Lemma 2.1. *Under the conditions* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$*, the following hold*:

(i) *Let* $(u, v) \in \mathcal{N}_\gamma$*, then there exists* $\sigma > 0$ *which is independent of* γ *such that* $\|(u, v)\|_\gamma \geq \sigma$.

(ii) *For each* $(u, v) \in E \setminus \{(0, 0)\}$*, there is a unique* $t_{(u,v)} > 0$ *such that* $t_{(u,v)}(u, v) \in \mathcal{N}_\gamma$*. Moreover,* $\mathcal{E}_\gamma(t_{(u,v)}(u, v)) = \max_{t \geq 0} \mathcal{E}_\gamma(t(u, v))$.

Proof. (i) First, by Young inequality, we get

$$|u|^\alpha |v|^\beta \leq \frac{\alpha}{\alpha+\beta}|u|^{\alpha+\beta} + \frac{\beta}{\alpha+\beta}|v|^{\alpha+\beta},$$

then by the continuity of the Sobolev embedding $E_V \hookrightarrow L^s(\mathbb{R}^3)$ and $E_W \hookrightarrow L^s(\mathbb{R}^3)$ for $2 \leq s \leq 6$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^3} |u|^\alpha |v|^\beta dx &\leq \frac{\alpha}{\alpha+\beta} \int_{\mathbb{R}^3} |u|^{\alpha+\beta} dx + \frac{\beta}{\alpha+\beta} \int_{\mathbb{R}^3} |v|^{\alpha+\beta} dx \\ &\leq C_1 \|u\|_V^{\alpha+\beta} + C_2 \|v\|_W^{\alpha+\beta} \leq C \|(u, v)\|_\gamma^{\alpha+\beta}, \end{aligned} \tag{2.1}$$

where $C > 0$ is independent of γ. So, by (2.1), for any $(u, v) \in \mathcal{N}_\gamma$, we have

$$\begin{aligned} 0 = \langle \mathcal{E}'_\gamma(u, v), (u, v) \rangle &= \|(u, v)\|_\gamma^2 + b_1 \Upsilon^2(u) + b_2 \Upsilon^2(v) - 2 \int_{\mathbb{R}^3} |u|^\alpha |v|^\beta dx \\ &\geq \|(u, v)\|_\gamma^2 - 2C \|(u, v)\|_\gamma^{\alpha+\beta}. \end{aligned} \tag{2.2}$$

Note that $\alpha + \beta > 4$, thus there exists $\sigma > 0$ such that $\|(u, v)\|_\gamma \geq \sigma$.

(ii) Let $(u, v) \in E \setminus \{(0, 0)\}$ be fixed. For $t > 0$, we consider the fibering maps $\phi : t \to \mathcal{E}_\gamma(t(u, v))$ defined by

$$\phi(t) := \mathcal{E}_\gamma(t(u, v)) = \frac{t^2}{2}\|(u, v)\|_\gamma^2 + \frac{t^4}{4}(b_1 \Upsilon^2(u) + b_2 \Upsilon^2(v)) - \frac{2t^{\alpha+\beta}}{\alpha+\beta} \int_{\mathbb{R}^3} |u|^\alpha |v|^\beta dx.$$

We observe that $\phi'(t) = \langle \mathcal{E}'_\gamma(t(u,v)), (u,v)\rangle = 0$ if and only if $t(u,v) \in \mathcal{N}_\gamma$. First we claim that $\phi(t) > 0$ for $t > 0$ small. Indeed, by (2.1), we have that

$$\begin{aligned}\phi(t) &\geq \frac{t^2}{2}\|(u,v)\|_\gamma^2 - \frac{2t^{\alpha+\beta}}{\alpha+\beta}\int_{\mathbb{R}^3}|u|^\alpha|v|^\beta dx \\ &\geq \frac{t^2}{2}\|(u,v)\|_\gamma^2 - C_3 t^{\alpha+\beta}\|(u,v)\|_\gamma^{\alpha+\beta},\end{aligned}$$

since $\alpha+\beta > 4$, so $\phi(t) > 0$ whenever $t > 0$ is small enough. It is easy to see that $\phi(t) \to -\infty$ as $t \to +\infty$. Hence there exists $t_{(u,v)} > 0$ such that $\phi'(t_{(u,v)}) = 0$, that is $t_{(u,v)}(u,v) \in \mathcal{N}_\gamma$. Moreover, $\mathcal{E}_\gamma(t_{(u,v)}(u,v)) = \max\limits_{t\geq 0}\mathcal{E}_\gamma(t(u,v))$.

In addition, the condition $\phi'(t) = 0$ is equivalent to

$$b_1\Upsilon^2(u) + b_2\Upsilon^2(v) = -\frac{1}{t^2}\|(u,v)\|_\gamma^2 + 2t^{\alpha+\beta-4}\int_{\mathbb{R}^3}|u|^\alpha|v|^\beta dx. \tag{2.3}$$

The right side of (2.3) is strictly increasing for $t > 0$ recalling that $\alpha+\beta > 4$, so there exists a unique $t_{(u,v)} > 0$ such that $\phi'(t_{(u,v)}) = 0$, and the second conclusion follows. □

Lemma 2.2. *Let $(\mathcal{H}_1) - (\mathcal{H}_2)$ hold and $\{(u_n, v_n)\}$ is a $(PS)_c$ sequence for $\mathcal{E}_\gamma(u,v)$. Then we have*

(i) *$\{(u_n,v_n)\}$ is bounded in E;*

(ii) *if $c \neq 0$, then $c \geq c_0$, for some $c_0 > 0$ is independent of γ.*

Proof. Let $\{(u_n,v_n)\}$ be a $(PS)_c$ sequence for $\mathcal{E}_\gamma(u,v)$, that is,

$$\mathcal{E}_\gamma(u_n,v_n) = c + o_n(1) \text{ and } \mathcal{E}'_\gamma(u_n,v_n) = o_n(1).$$

Then we have,

$$\begin{aligned}c + o_n(1) - \frac{1}{4}o_n(\|(u_n,v_n)\|_\gamma) &= \mathcal{E}_\gamma(u_n,v_n) - \frac{1}{4}\langle \mathcal{E}'_\gamma(u_n,v_n),(u_n,v_n)\rangle \\ &= \frac{1}{2}\|(u_n,v_n)\|_\gamma^2 + \Big(\frac{1}{2} - \frac{2}{\alpha+\beta}\Big)\int_{\mathbb{R}^3}|u_n|^\alpha|v_n|^\beta dx \\ &\geq \frac{1}{2}\|(u_n,v_n)\|_\gamma^2, \end{aligned}\tag{2.4}$$

which implies that $\{(u_n,v_n)\}$ is bounded in E.

On the other hand, we have

$$\begin{aligned}o_n(\|(u_n,v_n)\|_\gamma) &= \langle \mathcal{E}'_\gamma(u_n,v_n),(u_n,v_n)\rangle \\ &= \|(u_n,v_n)\|_\gamma^2 + b_1\Upsilon^2(u_n) + b_2\Upsilon^2(v_n) - 2\int_{\mathbb{R}^3}|u_n|^\alpha|v_n|^\beta dx \\ &\geq \|(u_n,v_n)\|_\gamma^2 - 2C\|(u_n,v_n)\|_\gamma^{\alpha+\beta},\end{aligned}$$

since $\alpha+\beta>4$, there exists $0<\sigma_1<1$ such that

$$\langle \mathcal{E}'_\gamma(u_n,v_n),(u_n,v_n)\rangle \geq \frac{1}{4}\|(u_n,v_n)\|_\gamma^2 \text{ for } \|(u_n,v_n)\|_\gamma<\sigma_1. \tag{2.5}$$

Now, if $c<\frac{\sigma_1^2}{2}$ and $\{(u_n,v_n)\}$ is a $(PS)_c$-sequence of $\mathcal{E}_\gamma$, then by (2.4)

$$\lim_{n\to\infty}\|(u_n,v_n)\|_\gamma^2 \leq 2c<\sigma_1^2.$$

Hence, $\|(u_n,v_n)\|_\gamma<\sigma_1$ for n large, then by (2.5)

$$\frac{1}{4}\|(u_n,v_n)\|_\gamma^2 \leq \langle \mathcal{E}'_\gamma(u_n,v_n),(u_n,v_n)\rangle = o_n(1)\|(u_n,v_n)\|_\gamma,$$

which implies $\|(u_n,v_n)\|_\gamma\to 0$ as $n\to\infty$ and $c=0$. It follows that (ii) holds for $c_0=\frac{\sigma_1^2}{2}$. □

Lemma 2.3. *Let C^* be fixed. Given $\varepsilon>0$ there exist $\Gamma_\varepsilon=\Gamma(\varepsilon,C^*)>0$ and $\rho_\varepsilon=\rho(\varepsilon,C^*)>0$ such that, if $\{(u_n,v_n)\}$ is a $(PS)_c$-sequence of $\mathcal{E}_\gamma(u,v)$ with $c\leq C^*,\gamma\geq\Gamma_\varepsilon$, then*

$$\limsup_{n\to\infty}\int_{B^c_{\rho_\varepsilon}}|u_n|^\alpha|v_n|^\beta dx\leq\varepsilon. \tag{2.6}$$

Proof. For $\rho>0$, we set

$A(\rho):=\{x\in\mathbb{R}^3:|x|\geq\rho, V(x)\geq M_1\},\ B(\rho):=\{x\in\mathbb{R}^3:|x|\geq\rho, V(x)<M_1\}$,

then

$$\begin{aligned}\int_{A(\rho)}|u_n|^2dx &\leq \frac{1}{\gamma M_1}\int_{\mathbb{R}^3}\gamma V(x)u_n^2dx\\ &\leq \frac{1}{\gamma M_1}\int_{\mathbb{R}^3}(a_1|\nabla u_n|^2+\gamma V(x)u_n^2)dx\\ &\leq \frac{1}{\gamma M_1}\Big(2c+o_n(\|(u_n,v_n)\|_\gamma)\Big)\\ &\leq \frac{1}{\gamma M_1}\Big(2C^*+o_n(\|(u_n,v_n)\|_\gamma)\Big)\to 0 \quad \text{as } \gamma\to\infty. \end{aligned}\tag{2.7}$$

Using the Hölder inequality and (2.4), for $1<q<3$ we obtain

$$\begin{aligned}\int_{B(\rho)}|u_n|^2dx &\leq \Big(\int_{\mathbb{R}^3}|u_n|^{2q}dx\Big)^{\frac{1}{q}}\cdot\mathcal{L}(B(\rho))^{\frac{q-1}{q}}\\ &\leq C_4\|u_n\|^2_{H^1(\mathbb{R}^3)}\cdot\mathcal{L}(B(\rho))^{\frac{q-1}{q}}\\ &\leq C_4\cdot 2C^*\cdot\mathcal{L}(B(\rho))^{\frac{q-1}{q}}\to 0 \quad \text{as } \rho\to\infty,\end{aligned}\tag{2.8}$$

where $C_4 = C_4(q)$ is a positive constant. Setting $\theta = \frac{3(\alpha+\beta-2)}{2(\alpha+\beta)}$, by using the Gagliardo-Nirenberg inequality, (2.7) and (2.8), we obtain that

$$\begin{aligned}\int_{B_\rho^c} |u_n|^{\alpha+\beta}dx &\le C\Big(\int_{B_\rho^c} |\nabla u_n|^2 dx\Big)^{\frac{(\alpha+\beta)\theta}{2}} \cdot \Big(\int_{B_\rho^c} |u_n|^2 dx\Big)^{\frac{(\alpha+\beta)(1-\theta)}{2}}\\
&\le C_5 \|(u_n,v_n)\|_\gamma^{(\alpha+\beta)\theta} \cdot \Big(\int_{A(\rho)} |u_n|^2 dx + \int_{B(\rho)} |u_n|^2 dx\Big)^{\frac{(\alpha+\beta)(1-\theta)}{2}}\\
&\le C_6 \Big(\int_{A(\rho)} |u_n|^2 dx + \int_{B(\rho)} |u_n|^2 dx\Big)^{\frac{(\alpha+\beta)(1-\theta)}{2}} \to 0 \text{ as } \gamma,\rho \to \infty. \qquad (2.9)\end{aligned}$$

Similarly,

$$\int_{B_\rho^c} |v_n|^{\alpha+\beta} dx \le \varepsilon \quad \text{for } \gamma, \rho \text{ large.} \qquad (2.10)$$

At last, using the Hölder inequality, (2.9) and (2.10) we have that

$$\begin{aligned}&\limsup_{n\to\infty} \int_{B_{\rho_\varepsilon}^c} |u_n|^\alpha |v_n|^\beta dx\\
\le\ &\limsup_{n\to\infty} \Big(\int_{B_{\rho_\varepsilon}^c} |u_n|^{\alpha+\beta} dx\Big)^{\frac{\alpha}{\alpha+\beta}} \Big(\int_{B_{\rho_\varepsilon}^c} |v_n|^{\alpha+\beta} dx\Big)^{\frac{\beta}{\alpha+\beta}} \le \varepsilon.\end{aligned}$$

This concludes the proof of Lemma 2.3. □

The following Brézis-Lieb type lemma is proved in [6].

Lemma 2.4. *Let $\alpha + \beta < 2^*$ and $\{(u_n, v_n)\} \subset E$ is a sequence such that $(u_n, v_n) \rightharpoonup (u, v)$ weakly in E. Then we have*

$$\int_{\mathbb{R}^3} |u_n|^\alpha |v_n|^\beta dx - \int_{\mathbb{R}^3} |u_n - u|^\alpha |v_n - v|^\beta dx = \int_{\mathbb{R}^3} |u|^\alpha |v|^\beta dx + o_n(1).$$

Lemma 2.5. *Let $\gamma > 0$ be fixed and $\{(u_n, v_n)\}$ is a $(PS)_c$-sequence of $\mathcal{E}_\gamma$. Then up to a subsequence $(u_n, v_n) \rightharpoonup (u, v)$ in E with (u, v) being a weak solution of $(\mathcal{KS})_\gamma$. Moreover, $\{(u_n - u, v_n - v)\}$ is a $(PS)_d$-sequence for $\mathcal{E}_\gamma$ with $d = c - \mathcal{E}_\gamma(u, v)$.*

Proof. Since $\{(u_n, v_n)\}$ is bounded in E (see Lemma 2.2(i)), there is a subsequence of $\{(u_n, v_n)\}$ such that $(u_n, v_n) \rightharpoonup (u, v)$ in E as $n \to \infty$. In order to see that (u, v) is a critical point of $\mathcal{E}_\gamma$ we recall that

$$\begin{cases} (u_n, v_n) \rightharpoonup (u, v), & \text{in } E,\\ (u_n, v_n) \to (u, v), & \text{a.e. in } \mathbb{R}^3,\\ (u_n, v_n) \to (u, v), & \text{in } L^{s_1}_{loc}(\mathbb{R}^3) \times L^{s_2}_{loc}(\mathbb{R}^3),\ 2 \le s_1, s_2 < 6.\end{cases}$$

Moreover, there exist $A, B \in \mathbb{R}$, such that

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \to A, \qquad \int_{\mathbb{R}^3} |\nabla v_n|^2 dx \to B,$$

then by the Fatou's lemma we get that

$$\int_{\mathbb{R}^3} |\nabla u|^2 dx \leq A, \qquad \int_{\mathbb{R}^3} |\nabla v|^2 dx \leq B.$$

We claim that $\int_{\mathbb{R}^3} |\nabla u|^2 dx = A$, and $\int_{\mathbb{R}^3} |\nabla v|^2 dx = B$. Arguing by contradiction, we assume that $\int_{\mathbb{R}^3} |\nabla u|^2 dx < A$ or $\int_{\mathbb{R}^3} |\nabla v|^2 dx < B$. By $\mathcal{E}'_\gamma(u_n, v_n) \to 0$ and $(u_n, v_n) \rightharpoonup (u, v)$ in E, for any $(\varphi, \psi) \in E$, we have

$$\int_{\mathbb{R}^3} (a_1 \nabla u \nabla \varphi + a_2 \nabla v \nabla \psi) dx + \gamma \int_{\mathbb{R}^3} V(x) u \varphi + W(x) v \psi dx + A b_1 \int_{\mathbb{R}^3} \nabla u \nabla \varphi dx$$
$$+ B b_2 \int_{\mathbb{R}^3} \nabla v \nabla \psi dx - \frac{2}{\alpha + \beta} \int_{\mathbb{R}^3} (\alpha |u|^{\alpha - 2} u |v|^\beta \varphi + \beta |u|^\alpha |v|^{\beta - 2} v \psi) dx = 0.$$

Then $\langle \mathcal{E}'_\gamma(u, v), (u, v) \rangle < 0$. On the other hand, by Lemma 2.1(ii) it is easy to get that $\langle \mathcal{E}'_\gamma(t(u, v)), t(u, v) \rangle > 0$ for $t > 0$ is small enough. Hence there exists $t_0 \in (0, 1)$ satisfying $\langle \mathcal{E}'_\gamma(t_0(u, v)), t_0(u, v) \rangle = 0$. Moreover, $\mathcal{E}_\gamma(t_0(u, v)) = \max_{0 \leq t \leq 1} \mathcal{E}_\gamma(t(u, v))$, so

$$\begin{aligned}
c_\gamma &\leq \mathcal{E}_\gamma(t_0(u, v)) = \mathcal{E}_\gamma(t_0(u, v)) - \frac{1}{4} \langle \mathcal{E}'_\gamma(t_0(u, v)), t_0(u, v) \rangle \\
&= \frac{t_0^2}{4} \|(u, v)\|_\gamma^2 + \Big(\frac{1}{2} - \frac{2}{\alpha + \beta}\Big) t_0^{\alpha + \beta} \int_{\mathbb{R}^3} |u|^\alpha |v|^\beta dx \\
&< \frac{1}{4} \|(u, v)\|_\gamma^2 + \Big(\frac{1}{2} - \frac{2}{\alpha + \beta}\Big) \int_{\mathbb{R}^3} |u|^\alpha |v|^\beta dx \\
&\leq \liminf_{n \to \infty} \left(\frac{1}{4} \|(u_n, v_n)\|_\gamma^2 + \Big(\frac{1}{2} - \frac{2}{\alpha + \beta}\Big) \int_{\mathbb{R}^3} |u_n|^\alpha |v_n|^\beta dx \right) \\
&= \liminf_{n \to \infty} \Big(\mathcal{E}_\gamma(u_n, v_n) - \frac{1}{4} \langle \mathcal{E}'_\gamma(u_n, v_n), (u_n, v_n) \rangle \Big) = c_\gamma,
\end{aligned}$$

which is a contradiction. Then

$$\int_{\mathbb{R}^3} |\nabla u|^2 dx = A = \lim_{n \to \infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx, \quad \int_{\mathbb{R}^3} |\nabla v|^2 dx = B = \lim_{n \to \infty} \int_{\mathbb{R}^3} |\nabla v_n|^2 dx.$$

Thus for any $(\varphi, \psi) \in E$, we have

$$\langle \mathcal{E}'_\gamma(u, v), (\varphi, \psi) \rangle = \lim_{n \to \infty} \langle \mathcal{E}'_\gamma(u_n, v_n), (\varphi, \psi) \rangle = 0.$$

So (u, v) is a critical point of of $\mathcal{E}_\gamma$, that is (u, v) is a weak solution of $(\mathcal{KS})_\gamma$.

We consider a new sequence $(\widetilde{u}_n, \widetilde{v}_n) = (u_n - u, v_n - v)$. Now we verify that

$$\mathcal{E}_\gamma(\widetilde{u}_n, \widetilde{v}_n) = c - \mathcal{E}_\gamma(u, v) \quad \text{as } n \to \infty \tag{2.11}$$

and

$$\mathcal{E}'_\gamma(\widetilde{u}_n, \widetilde{v}_n) \to 0 \quad \text{as } n \to \infty. \tag{2.12}$$

By the Brézis-Lieb lemma, we have that

$$\|(\widetilde{u}_n, \widetilde{v}_n)\|_\gamma^2 = \|(u_n, v_n)\|_\gamma^2 - \|(u, v)\|_\gamma^2 + o_n(1),$$

$$\Big(\int_{\mathbb{R}^3}|\nabla\widetilde{u}_n|^2dx\Big)^2=\Big(\int_{\mathbb{R}^3}|\nabla u_n|^2dx\Big)^2-\Big(\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)^2+o_n(1),$$

$$\Big(\int_{\mathbb{R}^3}|\nabla\widetilde{v}_n|^2dx\Big)^2=\Big(\int_{\mathbb{R}^3}|\nabla v_n|^2dx\Big)^2-\Big(\int_{\mathbb{R}^3}|\nabla v|^2dx\Big)^2+o_n(1).$$

To show (2.11) we observe

$$\begin{aligned}\mathcal{E}_\gamma(\widetilde{u}_n,\widetilde{v}_n)&=\frac{1}{2}\int_{\mathbb{R}^3}(a_1|\nabla\widetilde{u}_n|^2+\gamma V(x)|\widetilde{u}_n|^2)dx\\&\quad+\frac{1}{2}\int_{\mathbb{R}^3}(a_2|\nabla\widetilde{v}_n|^2+\gamma W(x)|\widetilde{v}_n|^2)dx\\&\quad+\frac{1}{4}\bigg(b_1\Big(\int_{\mathbb{R}^3}|\nabla\widetilde{u}_n|^2dx\Big)^2+b_2\Big(\int_{\mathbb{R}^3}|\nabla\widetilde{v}_n|^2dx\Big)^2\bigg)\\&\quad-\frac{2}{\alpha+\beta}\int_{\mathbb{R}^3}|\widetilde{u}_n|^\alpha|\widetilde{v}_n|^\beta dx\\&=\mathcal{E}_\gamma(u_n,v_n)-\mathcal{E}_\gamma(u,v)+o_n(1)\\&\quad+\frac{2}{\alpha+\beta}\Big(\int_{\mathbb{R}^3}|u_n|^\alpha|v_n|^\beta dx-\int_{\mathbb{R}^3}|u|^\alpha|v|^\beta dx-\int_{\mathbb{R}^3}|\widetilde{u}_n|^\alpha\widetilde{v}_n|^\beta dx\Big).\end{aligned}\tag{2.13}$$

From Lemma 2.4, $\int_{\mathbb{R}^3}|u_n|^\alpha|v_n|^\beta dx-\int_{\mathbb{R}^3}|u|^\alpha|v|^\beta dx-\int_{\mathbb{R}^3}|\widetilde{u}_n|^\alpha|\widetilde{v}_n|^\beta dx\to 0$ as $n\to\infty$. Thus from (2.13) we obtain (2.11).

In order to show (2.12), let $(\varphi,\psi)\in E$. We note that

$$\begin{aligned}\langle\mathcal{E}'_\gamma(\widetilde{u}_n,\widetilde{v}_n),(\varphi,\psi)\rangle&=\langle\mathcal{E}'_\gamma(u_n,v_n),(\varphi,\psi)\rangle-\langle\mathcal{E}'_\gamma(u,v),(\varphi,\psi)\rangle\\&\quad-\frac{2\alpha}{\alpha+\beta}\int_{\mathbb{R}^3}|\widetilde{u}_n|^{\alpha-2}|\widetilde{v}_n|^\beta\widetilde{u}_n\varphi dx\\&\quad-\frac{2\beta}{\alpha+\beta}\int_{\mathbb{R}^3}|\widetilde{u}_n|^{\alpha}|\widetilde{v}_n|^{\beta-2}\widetilde{v}_n\psi dx\\&\quad+\frac{2\alpha}{\alpha+\beta}\int_{\mathbb{R}^3}|u_n|^{\alpha-2}|v_n|^\beta u_n\varphi dx\\&\quad+\frac{2\beta}{\alpha+\beta}\int_{\mathbb{R}^3}|u_n|^{\alpha}|v_n|^{\beta-2}v_n\psi dx\\&\quad-\frac{2\alpha}{\alpha+\beta}\int_{\mathbb{R}^3}|u|^{\alpha-2}|v|^\beta u\varphi dx\\&\quad-\frac{2\beta}{\alpha+\beta}\int_{\mathbb{R}^3}|u|^{\alpha}|v|^{\beta-2}v\psi dx.\end{aligned}\tag{2.14}$$

Since $\mathcal{E}'_\gamma(u_n,v_n)\to 0$ and $u_n\to u, v_n\to v$ in $L^s(\mathbb{R}^3)(2\le s<6)$, we have

$$\lim_{n\to\infty}\sup_{\|\varphi\|_{\gamma,V}\le 1}\int_{\mathbb{R}^3}\Big(|\widetilde{u}_n|^{\alpha-2}|\widetilde{v}_n|^\beta\widetilde{u}_n-|u_n|^{\alpha-2}|v_n|^\beta u_n+|u|^{\alpha-2}|v|^\beta u\Big)\varphi dx=0,$$

$$\lim_{n\to\infty}\sup_{\|\psi\|_{\gamma,W}\le 1}\int_{\mathbb{R}^3}\Big(|\widetilde{u}_n|^{\alpha}|\widetilde{v}_n|^{\beta-2}\widetilde{v}_n-|u_n|^{\alpha}|v_n|^{\beta-2}v_n+|u|^{\alpha}|v|^{\beta-2}v\Big)\psi dx=0.$$

Thus from (2.14) we obtain that

$$\lim_{n\to\infty}\langle \mathcal{E}'_\gamma(\widetilde{u}_n,\widetilde{v}_n),(\varphi,\psi)\rangle = 0,\ \forall\ (\varphi,\psi)\in E,$$

which implies (2.12) and this completes the proof of Lemma 2.5. □

Then we have the following compactness result.

Lemma 2.6. *Suppose that* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$ *hold. Then for any* $C_0 > 0$*, there exists* $\Gamma_0 > 0$ *such that* $\mathcal{E}_\gamma$ *satisfies the* $(PS)_c$*-condition for all* $c \le C_0$ *and* $\gamma \ge \Gamma_0$*.*

Proof. Let $c_0 > 0$ be given by Lemma 2.2(ii) and choose $\varepsilon > 0$ such that $\varepsilon < \frac{c_0(\alpha+\beta)}{\alpha+\beta-2}$. Then for the given $C_0 > 0$, we choose $\Gamma_\varepsilon > 0$ and $\rho_\varepsilon > 0$ in Lemma 2.3. We claim that $\Gamma_0 = \Gamma_\varepsilon$ is as required in Lemma 2.6. Let $\{(u_n,v_n)\}\subset E$ be a $(PS)_c$-sequence of $\mathcal{E}_\gamma(u,v)$ with $c \le C_0$ and $\gamma \ge \Gamma_0$. By Lemma 2.5, we may suppose that $(u_n,v_n)\rightharpoonup(u,v)$ weakly in E and then $\{(\widetilde{u}_n,\widetilde{v}_n)\} = \{(u_n-u,v_n-v)\}$ is a $(PS)_d$-sequence of $\mathcal{E}_\gamma$ with $d = c-\mathcal{E}_\gamma(u,v)$. We claim that $d = 0$. Arguing by contradiction, assume that $d \ne 0$. Lemma 2.2(ii) implies that $d \ge c_0 > 0$. Since $(\widetilde{u}_n,\widetilde{v}_n)$ is a $(PS)_d$-sequence of $\mathcal{E}_\gamma$, we have

$$\mathcal{E}_\gamma(\widetilde{u}_n,\widetilde{v}_n) = d + o_n(1),\qquad \mathcal{E}'_\gamma(\widetilde{u}_n,\widetilde{v}_n) = o_n(1).$$

Then we get

$$\begin{aligned} d + o_n(1) - \frac{1}{2}o_n(\|(u_n,v_n)\|_\gamma) &= \mathcal{E}_\gamma(\widetilde{u}_n,\widetilde{v}_n) - \frac{1}{2}\langle \mathcal{E}'_\gamma(\widetilde{u}_n,\widetilde{v}_n),(\widetilde{u}_n,\widetilde{v}_n)\rangle \\ &= -\frac{1}{4}(b_1\Upsilon^2(\widetilde{u}_n) + b_2\Upsilon^2(\widetilde{v}_n)) \\ &\quad + \Big(1-\frac{2}{\alpha+\beta}\Big)\int_{\mathbb{R}^3}|\widetilde{u}_n|^\alpha|\widetilde{v}_n|^\beta dx \\ &\le \Big(1-\frac{2}{\alpha+\beta}\Big)\int_{\mathbb{R}^3}|\widetilde{u}_n|^\alpha|\widetilde{v}_n|^\beta dx. \end{aligned}\tag{2.15}$$

From (2.15) we deduce that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}|\widetilde{u}_n|^\alpha|\widetilde{v}_n|^\beta dx \ge d\Big(1-\frac{2}{\alpha+\beta}\Big)^{-1} \ge \frac{\alpha+\beta}{\alpha+\beta-2}c_0.$$

On the other hand, Lemma 2.3 implies

$$\limsup_{n\to\infty}\int_{B^c_{\rho_\varepsilon}}|\widetilde{u}_n|^\alpha|\widetilde{v}_n|^\beta dx \le \varepsilon < \frac{c_0(\alpha+\beta)}{\alpha+\beta-2}.$$

This implies $(\widetilde{u}_n,\widetilde{v}_n)\rightharpoonup(\tilde{u},\tilde{v})$ in E with $(\tilde{u},\tilde{v})\ne(0,0)$, which is a contradiction. Therefore $d = 0$ and it follows from (2.4) that

$$\lim_{n\to\infty}\|(\widetilde{u}_n,\widetilde{v}_n)\|^2_\gamma \le 2d = 0,$$

hence $(\widetilde{u}_n,\widetilde{v}_n)\to(0,0)$ in E, that is, $(u_n,v_n)\to(u,v)$ in E. This completes the proof of Lemma 2.6. □

3. Proof of the main results

In this section we give the proofs of our main results. First, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. First, we can check that $\mathcal{E}_\gamma$ satisfies the mountain-pass geometry. Then using a version of the mountain-pass theorem without (PS) condition, there exists $\{(u_n, v_n)\} \subset E$ satisfying

$$\mathcal{E}_\gamma(u_n, v_n) \to c_\gamma \text{ and } \mathcal{E}'_\gamma(u_n, v_n) \to 0.$$

Moreover, by Lemma 2.2(i) $\{(u_n, v_n)\}$ is bounded in E. Then, up to a subsequence, $(u_n, v_n) \rightharpoonup (u, v)$ weakly in E and $(u_n, v_n) \to (u, v)$ a.e. in $x \in \mathbb{R}^3$. By Lemma 2.6, there exists $\gamma^* > 0$, such that $(u_n, v_n) \to (u, v)$ in E for $\gamma \geq \gamma^*$. Furthermore, by Lemma 2.5 we have that $\mathcal{E}'_\gamma(u, v) = 0$. By Lemma 2.1(i), we know that $(u, v) \neq (0, 0)$, then $(u, v) \in \mathcal{N}_\gamma$, and using the Fatou's lemma we get

$$\begin{aligned}
\mathcal{E}_\gamma(u, v) &= \mathcal{E}_\gamma(u, v) - \frac{1}{4}\langle \mathcal{E}'_\gamma(u, v), (u, v)\rangle \\
&= \frac{1}{4}\|(u, v)\|_\gamma^2 + \Big(\frac{1}{2} - \frac{2}{\alpha+\beta}\Big) \int_{\mathbb{R}^3} |u|^\alpha |v|^\beta dx \\
&\leq \liminf_{n\to\infty} \left(\frac{1}{4}\|(u_n, v_n)\|_\gamma^2 + \Big(\frac{1}{2} - \frac{2}{\alpha+\beta}\Big) \int_{\mathbb{R}^3} |u_n|^\alpha |v_n|^\beta dx)\right) \\
&= \liminf_{n\to\infty} \Big(\mathcal{E}_\gamma(u_n, v_n) - \frac{1}{4}\langle \mathcal{E}'_\gamma(u_n, v_n), (u_n, v_n)\rangle\Big) \\
&= c_\gamma.
\end{aligned}$$

Hence, $\mathcal{E}_\gamma(u, v) \leq c_\gamma$. On the other hand, from the definition of c_γ, we have $c_\gamma \leq \mathcal{E}_\gamma(u, v)$. So, $\mathcal{E}_\gamma(u, v) = c_\gamma$, that is (u, v) is a ground state solution of problem $(\mathcal{KS})_\gamma$. □

In order to investigate the concentration for the solutions obtained in Theorem 1.1, we consider the following Kirchhoff-type system:

$$(\mathcal{KS})_\infty \qquad \begin{cases} -\Big(a_1 + b_1 \displaystyle\int_\Omega |\nabla u|^2 dx\Big)\Delta u = \dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^\beta & \text{in } \Omega, \\ -\Big(a_2 + b_2 \displaystyle\int_\Omega |\nabla v|^2 dx\Big)\Delta v = \dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ u(x) = 0, \quad v(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega = \mathrm{int}(V^{-1}(0)) = \mathrm{int}(W^{-1}(0))$. The energy functional associated with $(\mathcal{KS})_\infty$ is defined by

$$\begin{aligned}
\mathcal{E}_\infty(u, v) = {}& \frac{1}{2}\int_\Omega \big(a_1|\nabla u|^2 + a_2|\nabla v|^2)\big)dx \\
&+ \frac{1}{4}\left(b_1\Big(\int_\Omega |\nabla u|^2 dx\Big)^2 + b_2\Big(\int_\Omega |\nabla v|^2 dx\Big)^2\right) - \frac{2}{\alpha+\beta}\int_\Omega |u|^\alpha |v|^\beta dx.
\end{aligned}$$

Let

$$\mathcal{M} := \{(u,v) \in (H_0^1(\Omega) \times H_0^1(\Omega))\backslash\{(0,0)\} : \langle \mathcal{E}'_\infty(u,v),(u,v)\rangle = 0\}$$

be the manifold and $c^\infty = \inf\limits_{(u,v)\in\mathcal{M}} \mathcal{E}_\infty(u,v)$.

Let us point out that the same results hold with $\mathcal{E}_\infty$, c^∞, $\mathcal{M}$ in place of $\mathcal{E}_\gamma$, c_γ, $\mathcal{N}_\gamma$, respectively. We note that condition $(\mathcal{H}_2)$ implies that the Sobolev imbedding $H_0^1(\Omega) \times H_0^1(\Omega) \hookrightarrow L^{s_1}(\Omega) \times L^{s_2}(\Omega)$ is compact for $2 \le s_1, s_2 < 6$, and hence the following Lemma 3.1 is standard.

Lemma 3.1. *The infimum c^∞ is achieved by a pair of functions $(u,v) \in \mathcal{M}$ which is a ground state solution of $(\mathcal{KS})_\infty$.*

Lemma 3.2. $\lim\limits_{\gamma\to+\infty} c_\gamma = c^\infty$, *where c_γ is defined in* (1.4).

Proof. It is easy to see that $c_\gamma \le c^\infty$ for all $\gamma \ge 0$. We assume $\lim_{n\to\infty} c_{\gamma_n} = k < c^\infty$ for a sequence $\gamma_n \to \infty$ as $n \to \infty$. Lemma 2.2 implies $k > 0$. We assume that $\{(u_n,v_n)\}$ such that c_{γ_n} is achieved. By Lemma 2.2(i), $\{(u_n,v_n)\}$ is bounded in E, we may assume that

$$\begin{cases} (u_n,v_n) \rightharpoonup (u,v), & \text{in } E, \\ (u_n,v_n) \to (u,v), & \text{in } L^{s_1}_{loc}(\mathbb{R}^3) \times L^{s_2}_{loc}(\mathbb{R}^3), 2 \le s_1, s_2 < 6. \end{cases} \tag{3.1}$$

Now we claim that $(u,v)|_{\Omega^c} = (0,0)$. In fact, if $(u,v)|_{\Omega^c} \ne (0,0)$, there exists a compact subset $\mathcal{D} \subset \Omega^c$ with $\mathrm{dist}(\mathcal{D},\partial\Omega) > 0$ such that $(u,v)|_{\mathcal{D}} \ne (0,0)$. Then by (3.1)

$$\int_{\mathcal{D}} |u_n|^2 dx \to \int_{\mathcal{D}} |u|^2 dx > 0, \quad \int_{\mathcal{D}} |v_n|^2 dx \to \int_{\mathcal{D}} |v|^2 dx > 0.$$

Moreover, there exists $\varepsilon_0 > 0$ such that $V(x) \ge \varepsilon_0$, $W(x) \ge \varepsilon_0$ for any $x \in \mathcal{D}$. By the choice of $\{(u_n,v_n)\}$, we have

$$\begin{aligned} &\mathcal{E}_{\gamma_n}(u_n,v_n) \\ &= \frac{1}{2}\int_{\mathbb{R}^3} \Big(a_1|\nabla u_n|^2 + a_2|\nabla v_n|^2 + \gamma_n(V(x)u_n^2 + W(x)v_n^2)\Big)dx \\ &\quad + \frac{1}{4}\Big(b_1\Upsilon^2(u_n) + b_2\Upsilon^2(v_n)\Big) - \frac{2}{\alpha+\beta}\int_{\mathbb{R}^3} |u_n|^\alpha |v_n|^\beta dx \\ &= \Big(\frac{1}{2} - \frac{1}{\alpha+\beta}\Big)\int_{\mathbb{R}^3} \Big(a_1|\nabla u_n|^2 + a_2|\nabla v_n|^2 + \gamma_n(V(x)u_n^2 + W(x)v_n^2)\Big)dx \\ &\quad + \Big(\frac{1}{4} - \frac{1}{\alpha+\beta}\Big)\Big(b_1\Upsilon^2(u_n) + b_2\Upsilon^2(v_n)\Big) \\ &\ge \Big(\frac{1}{2} - \frac{1}{\alpha+\beta}\Big)\int_{\mathbb{R}^3} \Big(\gamma_n(V(x)u_n^2 + W(x)v_n^2)\Big)dx \\ &\ge \Big(\frac{1}{2} - \frac{1}{\alpha+\beta}\Big)\int_{\mathcal{D}} \gamma_n\varepsilon_0(u_n^2 + v_n^2)dx \to +\infty \ \text{ as } n \to \infty. \end{aligned}$$

This contradiction shows that $(u,v)|_{\Omega^c} = (0,0)$.

Next we show that $u_n \to u, v_n \to v$ in $L^q(\mathbb{R}^3)$ for $2 < q < 6$. Otherwise, by concentration-compactness principle of P. L. Lions [11], there exist $\delta > 0, \varrho > 0$, $y_n, \tilde{y}_n \in \mathbb{R}^3$ with $|y_n| \to +\infty, |\tilde{y}_n| \to +\infty$ such that

$$\int_{B_\varrho(y_n)} |u_n - u|^2 dx \geq \delta > 0,$$

$$\int_{B_\varrho(\tilde{y}_n)} |v_n - v|^2 dx \geq \delta > 0.$$

On the other hand, by the choice of $\{(u_n, v_n)\}$ and the facts that $\mathcal{L}(B_\varrho(y_n) \cap \{x|V(x) \leq M_1\}) \to 0, \mathcal{L}(B_\varrho(\tilde{y}_n) \cap \{x|W(x) \leq M_2\}) \to 0$ as $n \to \infty$, we obtain

$$\begin{aligned}
&\mathcal{E}_{\gamma_n}(u_n, v_n) \\
&\geq \Big(\frac{1}{2} - \frac{1}{\alpha+\beta}\Big) \int_{B_\varrho(y_n)\cap\{x|V(x)\geq M_1\}} (a_1|\nabla u_n|^2 + \gamma_n V(x) u_n^2) dx \\
&\quad + \Big(\frac{1}{2} - \frac{1}{\alpha+\beta}\Big) \int_{B_\varrho(\tilde{y}_n)\cap\{x|W(x)\geq M_2\}} (a_2|\nabla v_n|^2 + \gamma_n W(x) v_n^2) dx \\
&\geq \Big(\frac{1}{2} - \frac{1}{\alpha+\beta}\Big)\gamma_n\Big(M_1 \int_{B_\varrho(y_n)} |u_n - u|^2 dx - \int_{B_\varrho(y_n)\cap\{x|V(x)\leq M_1\}} |u_n - u|^2 dx\Big) \\
&\quad + \Big(\frac{1}{2} - \frac{1}{\alpha+\beta}\Big)\gamma_n\Big(M_2 \int_{B_\varrho(\tilde{y}_n)} |v_n - v|^2 dx - \int_{B_\varrho(\tilde{y}_n)\cap\{x|W(x)\leq M_2\}} |v_n - v|^2 dx\Big) \\
&= \Big(\frac{1}{2} - \frac{1}{\alpha+\beta}\Big)\gamma_n\Big(M_1 \int_{B_\varrho(y_n)} |u_n - u|^2 dx - o_n(1)\Big) \\
&\quad + \Big(\frac{1}{2} - \frac{1}{\alpha+\beta}\Big)\gamma_n\Big(M_2 \int_{B_\varrho(\tilde{y}_n)} |v_n - v|^2 dx - o_n(1)\Big) \to +\infty, \text{ as } n \to \infty.
\end{aligned}$$

This contradiction implies $u_n \to u, v_n \to v$ in $L^q(\mathbb{R}^3)$ for $2 < q < 6$.

Since $\{(u_n, v_n)\}$ is bounded in E, by the Fatou's lemma, we have that

$$\liminf_{n\to\infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx \geq \int_{\mathbb{R}^3} |\nabla u|^2 dx,$$

$$\liminf_{n\to\infty} \int_{\mathbb{R}^3} |\nabla v_n|^2 dx \geq \int_{\mathbb{R}^3} |\nabla v|^2 dx.$$

On the other hand, by the choice of $\{(u_n, v_n)\}$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} \Big(a_1|\nabla u_n|^2 + a_2|\nabla v_n|^2 + \gamma_n(V(x)u_n^2 + W(x)v_n^2)\Big) dx \\
&\quad + b_1\Upsilon^2(u_n) + b_2\Upsilon^2(v_n) \\
&= 2\int_{\mathbb{R}^3} |u_n|^\alpha |v_n|^\beta dx.
\end{aligned} \tag{3.2}$$

By (3.2) it follows that

$$\int_{\mathbb{R}^3} (a_1|\nabla u|^2 + a_2|\nabla v|^2) dx + b_1\Upsilon^2(u) + b_2\Upsilon^2(v)$$

$$\leq \lim_{n\to\infty}\left[\int_{\mathbb{R}^3}(a_1|\nabla u_n|^2+a_2|\nabla v_n|^2)dx+b_1\Upsilon^2(u_n)+b_2\Upsilon^2(v_n)\right]$$

$$\leq 2\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^\alpha|v_n|^\beta dx. \tag{3.3}$$

Next, we first prove that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^\alpha|v_n|^\beta dx=\int_{\mathbb{R}^3}|u|^\alpha|v|^\beta dx. \tag{3.4}$$

Given $\varepsilon>0$, we can argue as in the proof of Lemma 2.3 to conclude that, for some $\rho>0$ large, there holds

$$\limsup_{n\to\infty}\int_{B_\rho^c(0)}|u_n|^\alpha|v_n|^\beta dx\leq\varepsilon.$$

By taking ρ larger if necessary, we may assume that $\int_{B_\rho^c(0)}|u|^\alpha|v|^\beta dx\leq\varepsilon$. Moreover, the local convergence in (3.1) and the Lebesgue dominated convergence theorem imply that

$$\int_{B_\rho(0)}|u_n|^\alpha|v_n|^\beta dx\to\int_{B_\rho(0)}|u|^\alpha|v|^\beta dx \text{ as } n\to\infty.$$

Since

$$\left|\int_{\mathbb{R}^3}(|u_n|^\alpha|v_n|^\beta-|u|^\alpha|v|^\beta)dx\right|$$
$$\leq\int_{B_\rho^c(0)}|u_n|^\alpha|v_n|^\beta dx+\int_{B_\rho^c(0)}|u|^\alpha|v|^\beta dx+\left|\int_{B_\rho(0)}(|u_n|^\alpha|v_n|^\beta-|u|^\alpha|v|^\beta)dx\right|,$$

it follows from the above estimates and convergences that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^3}(|u_n|^\alpha|v_n|^\beta-|u|^\alpha|v|^\beta)dx\leq 2\varepsilon,$$

and therefore (3.4) holds. Then by (3.3) and (3.4) we have

$$\int_{\mathbb{R}^3}(a_1|\nabla u|^2+a_2|\nabla v|^2)dx+b_1\Upsilon^2(u)+b_2\Upsilon^2(v)\leq 2\int_{\mathbb{R}^3}|u|^\alpha|v|^\beta dx.$$

As a consequence of $(u,v)|_{\Omega^c}=(0,0)$ we obtain

$$\int_\Omega(a_1|\nabla u|^2+a_2|\nabla v|^2)dx+b_1\Big(\int_\Omega|\nabla u|^2dx\Big)^2+b_2\Big(\int_\Omega|\nabla v|^2dx\Big)^2$$
$$\leq 2\int_\Omega|u|^\alpha|v|^\beta dx.$$

Thus there exists $t_0\in(0,1]$ such that $t_0(u,v)\in\mathcal{M}$ and

$$\mathcal{E}_\infty(t_0(u,v))\leq\mathcal{E}_\infty(u,v),$$

hence $c^\infty\leq\mathcal{E}_\infty(t_0(u,v))\leq\mathcal{E}_\infty(u,v)\leq\lim_{n\to\infty}\mathcal{E}_{\gamma_n}(u_n,v_n)=k<c^\infty$, which is a contradiction. □

Now we give the proof of Theorem 1.2.

Proof of Theorem 1.2. For any sequence $\gamma_n \to +\infty$, let $(u_n, v_n) := (u_{\gamma_n}, v_{\gamma_n})$ be the solutions of $(\mathcal{KS})_{\gamma_n}$ obtained in Theorem 1.1, that is $\gamma_n \to +\infty$ such that $(u_n, v_n) \in \mathcal{N}_{\gamma_n}$, $\mathcal{E}_{\gamma_n}(u_n, v_n) = c_{\gamma_n}$ and $\mathcal{E}'_{\gamma_n}(u_n, v_n) = 0$. By Lemma 2.2(i) we know that $\{(u_n, v_n)\}$ must be bounded in $H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$, we may assume that $(u_n, v_n) \rightharpoonup (u, v)$ in $H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ and $(u_n, v_n) \to (u, v)$ in $L^{s_1}_{loc}(\mathbb{R}^3) \times L^{s_2}_{loc}(\mathbb{R}^3)$ for $2 \le s_1, s_2 < 6$. We shall show that $(u, v) \in H^1_0(\Omega) \times H^1_0(\Omega)$ is a ground state solution of $(\mathcal{KS})_\infty$, that is $\mathcal{E}_\infty(u, v) = c^\infty, (u_n, v_n) \to (u, v)$ in $H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$. As in the proof of Lemma 3.2 we can prove that $(u, v)|_{\Omega^c} = (0, 0)$ whereas $(u, v) \in H^1_0(\Omega) \times H^1_0(\Omega)$ and $(u_n, v_n) \to (u, v)$ in $L^{s_1}(\mathbb{R}^3) \times L^{s_2}(\mathbb{R}^3)$ for $2 \le s_1, s_2 < 6$. Then it suffices to show that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx = \int_{\mathbb{R}^3} |\nabla u|^2 dx, \quad \lim_{n\to\infty} \int_{\mathbb{R}^3} |\nabla v_n|^2 dx = \int_{\mathbb{R}^3} |\nabla v|^2 dx,$$

$$\lim_{n\to\infty} \gamma_n \int_{\mathbb{R}^3} V(x) u_n^2 dx = 0, \quad \lim_{n\to\infty} \gamma_n \int_{\mathbb{R}^3} W(x) v_n^2 dx = 0.$$

In fact, if one of the above limits does not hold, by the Fatou's lemma, we have

$$\int_{\mathbb{R}^3} (a_1 |\nabla u|^2 + a_2 |\nabla v|^2) dx + b_1 \Upsilon^2(u) + b_2 \Upsilon^2(v) < 2 \int_{\mathbb{R}^3} |u|^\alpha |v|^\beta dx,$$

similar to the proof of Lemma 3.2, there exists $t_0 \in (0, 1)$ such that $t_0(u, v) \in \mathcal{M}$ and

$$c^\infty \le \mathcal{E}_\infty(t_0(u, v)) < \mathcal{E}_\infty(u, v) \le \lim_{n\to\infty} \mathcal{E}_{\gamma_n}(u_n, v_n) \le c^\infty,$$

which is a contradiction. This completes the proof of Theorem 1.2. □

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