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# UNIVARIATE LEFT FRACTIONAL POLYNOMIAL HIGH ORDER MONOTONE APPROXIMATION

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Abstract. Let $f \in C^r([-1,1])$, $r \geq 0$ and let $L^*$ be a linear left fractional differential operator such that $L^*(f) \geq 0$ throughout $[0,1]$. We can find a sequence of polynomials $Q_n$ of degree $\leq n$ such that $L^*(Q_n) \geq 0$ over $[0,1]$, furthermore $f$ is approximated left fractionally and simultaneously by $Q_n$ on $[-1,1]$. The degree of these restricted approximations is given via inequalities using a higher order modulus of smoothness for $f^{(r)}$.

## 1. Introduction

The topic of monotone approximation started in [6] has become a major trend in approximation theory. A typical problem in this subject is: given a positive integer $k$, approximate a given function whose $k$th derivative is $\geq 0$ by polynomials having this property.

In [3] the authors replaced the $k$th derivative with a linear differential operator of order $k$. We mention this motivating result.

**Theorem 1.** *Let $h, k, p$ be integers, $0 \leq h \leq k \leq p$ and let $f$ be a real function, $f^{(p)}$ continuous in $[-1,1]$ with modulus of continuity $\omega_1(f^{(p)}, x)$ there. Let $a_j(x)$, $j = h, h+1, \ldots, k$ be real functions, defined and bounded on $[-1,1]$ and assume $a_h(x)$ is either $\geq$ some number $\alpha > 0$ or $\leq$ some number $\beta < 0$ throughout $[-1,1]$. Consider the operator*

$$L = \sum_{j=h}^{k} a_j(x) \left[\frac{d^j}{dx^j}\right]$$

*and suppose, throughout $[-1,1]$,*

$$L(f) \geq 0. \quad (1)$$

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*Then, for every integer* $n \geq 1$, *there is a real polynomial* $Q_n(x)$ *of degree* $\leq n$ *such that*

$$L(Q_n) \geq 0 \text{ throughout } [-1,1]$$

*and*

$$\max_{-1\leq x\leq 1} |f(x) - Q_n(x)| \leq C n^{k-p} \omega_1 \left( f^{(p)}, \frac{1}{n} \right),$$

*where* $C$ *is independent of* $n$ *or* $f$.

We use also the notation $I = [-1,1]$.

We would like to mention:

**Theorem 2** (Gonska and Hinnemann [5])**.** *Let* $r \geq 0$ *and* $s \geq 1$. *Then there exists a sequence* $Q_n = Q_n^{(r,s)}$ *of linear polynomial operators mapping* $C^r(I)$ *into* $P_n$ *(space of polynomials of degree* $\leq n$*), such that for all* $f \in C^r(I)$*, all* $|x| \leq 1$ *and all* $n \geq \max(4(r+1), r+s)$ *we have*

$$\left| f^{(k)}(x) - (Q_n f)^{(k)}(x) \right| \leq M_{r,s} (\Delta_n(x))^{r-k} \omega_s \left( f^{(r)}, \Delta_n(x) \right), \ 0 \leq k \leq r, \tag{2}$$

*where* $\Delta_n(x) = \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}$, *and* $M_{r,s}$ *is a constant independent of* $f$, $x$, *and* $n$. *Above* $\omega_s$ *is the usual modulus of smoothnees of order* $s$ *with respect to the supremum norm.*

Theorem 2 implies the useful:

**Corollary 3** ([2])**.** *Let* $r \geq 0$ *and* $s \geq 1$. *Then there exists a sequence* $Q_n = Q_n^{(r,s)}$ *of linear polynomial operators mapping* $C^r(I)$ *into* $P_n$*, such that for all* $f \in C^r(I)$ *and all* $n \geq \max(4(r+1), r+s)$ *we have*

$$\left\| f^{(k)} - (Q_n f)^{(k)} \right\|_\infty \leq \frac{C_{r,s}}{n^{r-k}} \omega_s \left( f^{(r)}, \frac{1}{n} \right), \quad k = 0, 1, \ldots, r, \tag{3}$$

*where* $C_{r,s}$ *is a constant independent of* $f$ *and* $n$.

In [2] we proved the motivational:

**Theorem 4.** *Let* $h, v, r$ *be integers,* $0 \leq h \leq v \leq r$ *and let* $f \in C^r(I)$, *with* $f^{(r)}$ *having modulus of smoothness* $\omega_s\left(f^{(r)}, \delta\right)$ *there,* $s \geq 1$. *Let* $\alpha_j(x)$, $j = h, h+1, \ldots, v$ *be real functions, defined and bounded on* $I$ *and suppose* $\alpha_h$ *is either* $\geq \alpha > 0$ *or* $\leq \beta < 0$ *throughout* $I$. *Take the operator*

$$L = \sum_{j=h}^{v} \alpha_j(x) \left[ \frac{d^j}{dx^j} \right] \tag{4}$$

*and assume, throughout* $I$,

$$L(f) \geq 0. \tag{5}$$

*Then for every integer* $n \geq \max(4(r+1), r+s)$, *there exists a real polynomial* $Q_n(x)$ *of degree* $\leq n$ *such that*

$$L(Q_n) \geq 0 \text{ throughout } I, \tag{6}$$

*and*

$$\left\|f^{(k)}-Q_n^{(k)}\right\|_{\infty} \leq \frac{C}{n^{r-v}} \omega_s\left(f^{(r)}, \frac{1}{n}\right), \quad 0 \leq k \leq h. \tag{7}$$

*Moreover, we get*

$$\left\|f^{(k)}-Q_n^{(k)}\right\|_{\infty} \leq \frac{C}{n^{r-k}} \omega_s\left(f^{(r)}, \frac{1}{n}\right), \quad h+1 \leq k \leq r, \tag{8}$$

*were $C$ is a constant independent of $f$ and $n$.*

In this article we extend Theorem 4 to the fractional level. Indeed here $L$ is replaced by $L^*$, a linear left Caputo fractional differential operator. Now the monotonicity property is only true on the critical interval $[0,1]$. Simultaneous and fractional convergence remains true on all of $I$.

We are also inspired by [1].

We make:

**Definition 5** ([4], p. 50)**.** Let $\alpha > 0$ and $\lceil \alpha \rceil = m$, ($\lceil \cdot \rceil$ ceiling of the number). Consider $f \in C^m([-1,1])$. We define the left Caputo fractional derivative of $f$ of order $\alpha$ as follows:

$$\left(D_{*-1}^{\alpha} f\right)(x)=\frac{1}{\Gamma(m-\alpha)} \int_{-1}^{x}(x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \tag{9}$$

for any $x \in [-1,1]$, where $\Gamma$ is the gamma function.

We set

$$D_{*-1}^{0} f(x)=f(x),$$

$$D_{*-1}^{m} f(x)=f^{(m)}(x), \ \ \forall\, x \in[-1,1]. \tag{10}$$

## 2. Main result

We present:

**Theorem 6.** *Let $h, v, r$ be integers, $1 \leq h \leq v \leq r$ and let $f \in C^r([-1,1])$, with $f^{(r)}$ having modulus of smoothness $\omega_s\left(f^{(r)}, \delta\right)$ there, $s \geq 1$. Let $\alpha_j(x)$, $j = h, h+1, \ldots, v$ be real functions, defined and bounded on $[-1,1]$ and suppose $\alpha_h(x)$ is either $\geq \alpha > 0$ or $\leq \beta < 0$ on $[0,1]$. Let the real numbers $\alpha_0 = 0 < \alpha_1 \leq 1 < \alpha_2 \leq 2 < \cdots < \alpha_r \leq r$. Here $D_{*-1}^{\alpha_j} f$ stands for the left Caputo fractional derivative of $f$ of order $\alpha_j$ anchored at $-1$. Consider the linear left fractional differential operator*

$$L^* := \sum_{j=h}^{k} \alpha_j(x)\left[D_{*-1}^{\alpha_j}\right] \tag{11}$$

*and suppose, throughout $[0,1]$,*

$$L^*(f) \geq 0. \tag{12}$$

*Then, for any $n \in \mathbb{N}$ such that $n \geq \max(4(r+1), r+s)$, there exists a real polynomial $Q_n(x)$ of degree $\leq n$ such that*

$$L^*(Q_n) \geq 0 \quad \text{throughout} \quad [0,1], \tag{13}$$

*and*

$$\sup_{-1\leq x\leq 1} \left|\left(D^{\alpha_j}_{*-1} f\right)(x) - \left(D^{\alpha_j}_{*-1} Q_n\right)(x)\right| \leq \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_{r,s}}{n^{r-j}} \omega_s\left(f^{(r)}, \frac{1}{n}\right), \tag{14}$$

*$j = h+1, \ldots, r$; $C_{r,s}$ is a constant independent of $f$ and $n$.*

*Set*

$$l_j :\equiv \sup_{x\in[-1,1]} \left|\alpha_h^{-1}(x)\,\alpha_j(x)\right|, \quad h \leq j \leq v. \tag{15}$$

*When $j = 1, \ldots, h$ we derive*

$$\sup_{-1\leq x\leq 1} \left|\left(D^{\alpha_j}_{*-1} f\right)(x) - \left(D^{\alpha_j}_{*-1} Q_n\right)(x)\right| \leq \frac{C_{r,s}}{n^{r-v}} \omega_s\left(f^{(r)}, \frac{1}{n}\right) \cdot \tag{16}$$

$$\left[\left(\sum_{\tau=h}^{v} l_\tau \frac{2^{\tau-\alpha_\tau}}{\Gamma(\tau-\alpha_\tau+1)}\right)\left(\sum_{\lambda=0}^{h-j} \frac{2^{h-\alpha_j-\lambda}}{\lambda!\,\Gamma(h-\alpha_j-\lambda+1)}\right) + \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\right].$$

*Finally it holds*

$$\sup_{-1\leq x\leq 1} |f(x) - Q_n(x)| \leq \frac{C_{r,s}}{n^{r-v}} \omega_s\left(f^{(r)}, \frac{1}{n}\right)\left[\frac{1}{h!}\sum_{\tau=h}^{v} l_\tau \frac{2^{\tau-\alpha_\tau}}{\Gamma(\tau-\alpha_\tau+1)} + 1\right]. \tag{17}$$

*Proof.* Here let $Q_n$ as in Corollary 3. Let $\alpha_j > 0$, $j = 1, \ldots, r$, such that $0 < \alpha_1 \leq 1 < \alpha_2 \leq 2 < \alpha_3 \leq 3 < \cdots < \alpha_r \leq r$. That is $\lceil \alpha_j \rceil = j$, $j = 1, \ldots, r$.

We consider the left Caputo fractional derivatives

$$\left(D^{\alpha_j}_{*-1} f\right)(x) = \frac{1}{\Gamma(j-\alpha_j)} \int_{-1}^{x} (x-t)^{j-\alpha_j-1} f^{(j)}(t)\,dt, \tag{18}$$

and

$$\left(D^{j}_{*-1} f\right)(x) = f^{(j)}(x),$$

and

$$\left(D^{\alpha_j}_{*-1} Q_n\right)(x) = \frac{1}{\Gamma(j-\alpha_j)} \int_{-1}^{x} (x-t)^{j-\alpha_j-1} Q_n^{(j)}(t)\,dt, \tag{19}$$

$$\left(D^{j}_{*-1} Q_n\right)(x) = Q_n^{(j)}(x); \; j = 1, \ldots, r.$$

We notice that

$$\left|\left(D^{\alpha_j}_{*-1} f\right)(x) - \left(D^{\alpha_j}_{*-1} Q_n\right)(x)\right|$$

$$
\begin{aligned}
(20)\quad &= \frac{1}{\Gamma(j-\alpha_j)} \left| \int_{-1}^{x} (x-t)^{j-\alpha_j-1} f^{(j)}(t)\,dt - \int_{-1}^{x} (x-t)^{j-\alpha_j-1} Q_n^{(j)}(t)\,dt \right| \\
&= \frac{1}{\Gamma(j-\alpha_j)} \left| \int_{-1}^{x} (x-t)^{j-\alpha_j-1} \left( f^{(j)}(t) - Q_n^{(j)}(t) \right) dt \right| \\
(21)\quad &\leq \frac{1}{\Gamma(j-\alpha_j)} \int_{-1}^{x} (x-t)^{j-\alpha_j-1} \left| f^{(j)}(t) - Q_n^{(j)}(t) \right| dt \\
&\overset{(3)}{\leq} \frac{1}{\Gamma(j-\alpha_j)} \left( \int_{-1}^{x} (x-t)^{j-\alpha_j-1} dt \right) \frac{C_{r,s}}{n^{r-j}} \omega_s \left( f^{(r)}, \frac{1}{n} \right) \\
(22)\quad &= \frac{1}{\Gamma(j-\alpha_j)} \frac{(x+1)^{j-\alpha_j}}{(j-\alpha_j)} \frac{C_{r,s}}{n^{r-j}} \omega_s \left( f^{(r)}, \frac{1}{n} \right) \\
&= \frac{(x+1)^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_{r,s}}{n^{r-j}} \omega_s \left( f^{(r)}, \frac{1}{n} \right) \\
&\leq \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_{r,s}}{n^{r-j}} \omega_s \left( f^{(r)}, \frac{1}{n} \right).
\end{aligned}
$$

We proved for any $x \in [-1,1]$ that

$$
(23)\qquad \left| \left( D_{*-1}^{\alpha_j} f \right)(x) - \left( D_{*-1}^{\alpha_j} Q_n \right)(x) \right| \leq \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_{r,s}}{n^{r-j}} \omega_s \left( f^{(r)}, \frac{1}{n} \right).
$$

Hence it holds

(24)

$$
\sup_{-1 \leq x \leq 1} \left| \left( D_{*-1}^{\alpha_j} f \right)(x) - \left( D_{*-1}^{\alpha_j} Q_n \right)(x) \right| \leq \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_{r,s}}{n^{r-j}} \omega_s \left( f^{(r)}, \frac{1}{n} \right),
$$

$j = 0, 1, \ldots, r$.

Above we set $D_{*-1}^{0} f(x) = f(x)$, $D_{*-1}^{0} Q_n(x) = Q_n(x)$, $\forall\ x \in [-1,1]$, and $\alpha_0 = 0$, i.e., $\lceil \alpha_0 \rceil = 0$.

Set also

$$
(25)\qquad \rho_n := C_{r,s} \omega_s \left( f^{(r)}, \frac{1}{n} \right) \left( \sum_{j=h}^{v} l_j \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} n^{j-r} \right).
$$

I. Suppose, throughout $[0,1]$, $\alpha_h(x) \geq \alpha > 0$. Let $Q_n(x)$, $x \in [-1,1]$, be a real polynomial of degree $\leq n$ so that

$$
(26)\qquad \begin{aligned}
&\max_{-1 \leq x \leq 1} \left| D_{*-1}^{\alpha_j} \left( f(x) + \rho_n \frac{x^h}{h!} \right) - \left( D_{*-1}^{\alpha_j} Q_n \right)(x) \right| \\
&\leq \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_{r,s}}{n^{r-j}} \omega_s \left( f^{(r)}, \frac{1}{n} \right),\ j = 0, 1, \ldots, r.
\end{aligned}
$$

When $j = h+1, \ldots, r$, then

$$\max_{-1\le x\le 1}\left|\left(D_{*-1}^{\alpha_j}f\right)(x)-\left(D_{*-1}^{\alpha_j}Q_n\right)(x)\right| \le \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{C_{r,s}}{n^{r-j}}\omega_s\left(f^{(r)},\frac{1}{n}\right), \tag{27}$$

proving (14).

For $j = 1, \ldots, h$ we get

$$D_{*-1}^{\alpha_j}\left(\frac{x^h}{h!}\right)=\frac{1}{\Gamma(j-\alpha_j)}\int_{-1}^{x}(x-t)^{j-\alpha_j-1}\frac{t^{h-j}}{(h-j)!}dt \tag{28}$$

(we see that $t = t+1-1$, and

$$t^{h-j} = ((t+1)-1)^{h-j}=\sum_{\lambda=0}^{h-j}\binom{h-j}{\lambda}(t+1)^{h-j-\lambda}(-1)^{\lambda})$$

$$= \frac{1}{(h-j)!\Gamma(j-\alpha_j)} \cdot\sum_{\lambda=0}^{h-j}(-1)^{\lambda}\binom{h-j}{\lambda}\int_{-1}^{x}(x-t)^{j-\alpha_j-1}(t+1)^{h-j-\lambda+1-1}dt$$

$$= \frac{1}{(h-j)!\Gamma(j-\alpha_j)} \cdot\sum_{\lambda=0}^{h-j}(-1)^{\lambda}\frac{(h-j)!}{\lambda!(h-j-\lambda)!}\frac{\Gamma(j-\alpha_j)\Gamma(h-j-\lambda+1)}{\Gamma(h-\alpha_j-\lambda+1)}(x+1)^{h-\alpha_j-\lambda}$$

$$= \sum_{\lambda=0}^{h-j}\frac{(-1)^{\lambda}}{\lambda!\Gamma(h-\alpha_j-\lambda+1)}(x+1)^{h-\alpha_j-\lambda}. \tag{29}$$

Hence for $j = 1, \ldots, h$ we found that

$$D_{*-1}^{\alpha_j}\left(\frac{x^h}{h!}\right)=\sum_{\lambda=0}^{h-j}\frac{(-1)^{\lambda}(x+1)^{h-\alpha_j-\lambda}}{\lambda!\Gamma(h-\alpha_j-\lambda+1)}. \tag{30}$$

Therefore we get from (26) that

$$\max_{-1\le x\le 1}\left|\left(D_{*-1}^{\alpha_j}f\right)(x)+\rho_n\left(\sum_{\lambda=0}^{h-j}\frac{(-1)^{\lambda}(x+1)^{h-\alpha_j-\lambda}}{\lambda!\Gamma(h-\alpha_j-\lambda+1)}\right)-\left(D_{*-1}^{\alpha_j}Q_n\right)(x)\right| \le \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{C_{r,s}}{n^{r-j}}\omega_s\left(f^{(r)},\frac{1}{n}\right),\ j=1,\ldots,h. \tag{31}$$

Therefore we get for $j = 1, \ldots, h$, that

$$\max_{-1\le x\le 1}\left|\left(D_{*-1}^{\alpha_j}f\right)(x)-\left(D_{*-1}^{\alpha_j}Q_n\right)(x)\right|$$

$$\leq \rho_n \left( \sum_{\lambda=0}^{h-j} \frac{2^{h-\alpha_j-\lambda}}{\lambda!\Gamma(h-\alpha_j-\lambda+1)} \right) + \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_{r,s}}{n^{r-j}} \omega_s \left( f^{(r)}, \frac{1}{n} \right) \tag{32}$$

$$= C_{r,s}\omega_s \left( f^{(r)}, \frac{1}{n} \right) \left( \sum_{\overline{j}=h}^{k} l_{\overline{j}} \frac{2^{\overline{j}-\alpha_{\overline{j}}}}{\Gamma\left(\overline{j}-\alpha_{\overline{j}}+1\right)} n^{\overline{j}-r} \right)$$

$$\cdot \left( \sum_{\lambda=0}^{h-j} \frac{2^{h-\alpha_j-\lambda}}{\lambda!\Gamma(h-\alpha_j-\lambda+1)} \right) + \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_{r,s}}{n^{r-j}} \omega_s \left( f^{(r)}, \frac{1}{n} \right)$$

$$= C_{r,s}\omega_s \left( f^{(r)}, \frac{1}{n} \right) \left[ \left( \sum_{\overline{j}=h}^{k} l_{\overline{j}} \frac{2^{\overline{j}-\alpha_{\overline{j}}}}{\Gamma\left(\overline{j}-\alpha_{\overline{j}}+1\right)} \frac{1}{n^{r-\overline{j}}} \right) \right. \tag{33}$$

$$\left. \cdot \left( \sum_{\lambda=0}^{h-j} \frac{2^{h-\alpha_j-\lambda}}{\lambda!\Gamma(h-\alpha_j-\lambda+1)} \right) + \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{1}{n^{r-j}} \right]$$

$$\leq C_{r,s}\omega_s \left( f^{(r)}, \frac{1}{n} \right) \frac{1}{n^{r-v}} \left[ \left( \sum_{\overline{j}=h}^{v} l_{\overline{j}} \frac{2^{\overline{j}-\alpha_{\overline{j}}}}{\Gamma\left(\overline{j}-\alpha_{\overline{j}}+1\right)} \right) \right. \tag{34}$$

$$\left. \cdot \left( \sum_{\lambda=0}^{h-j} \frac{2^{h-\alpha_j-\lambda}}{\lambda!\Gamma(h-\alpha_j-\lambda+1)} \right) + \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \right].$$

Hence for $j = 1, \ldots, h$ we derived (16):

$$\max_{-1 \leq x \leq 1} \left| \left( D_{*-1}^{\alpha_j} f \right)(x) - \left( D_{*-1}^{\alpha_j} Q_n \right)(x) \right| \leq \frac{C_{r,s}}{n^{r-v}} \omega_s \left( f^{(r)}, \frac{1}{n} \right) \cdot \tag{35}$$

$$\left[ \left( \sum_{\tau=h}^{v} l_\tau \frac{2^{\tau-\alpha_\tau}}{\Gamma(\tau-\alpha_\tau+1)} \right) \left( \sum_{\lambda=0}^{h-j} \frac{2^{h-\alpha_j-\lambda}}{\lambda!\Gamma(h-\alpha_j-\lambda+1)} \right) + \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \right].$$

From (26) when $j = 0$ we obtain

$$\max_{-1 \leq x \leq 1} \left| f(x) + \rho_n \frac{x^h}{h!} - Q_n(x) \right| \leq \frac{C_{r,s}}{n^r} \omega_s \left( f^{(r)}, \frac{1}{n} \right). \tag{36}$$

And

$$\max_{-1 \leq x \leq 1} \left| f(x) - Q_n(x) \right| \leq \frac{\rho_n}{h!} + \frac{C_{r,s}}{n^r} \omega_s \left( f^{(r)}, \frac{1}{n} \right) \tag{37}$$

$$= \frac{C_{r,s}}{h!} \omega_s \left( f^{(r)}, \frac{1}{n} \right) \left( \sum_{\tau=h}^{v} l_\tau \frac{2^{\tau-\alpha_\tau}}{\Gamma(\tau-\alpha_\tau+1)} n^{\tau-r} \right) + \frac{C_{r,s}}{n^r} \omega_s \left( f^{(r)}, \frac{1}{n} \right)$$

$$= C_{r,s}\omega_s \left( f^{(r)}, \frac{1}{n} \right) \left[ \frac{1}{h!} \sum_{\tau=h}^{v} l_\tau \frac{2^{\tau-\alpha_\tau}}{\Gamma(\tau-\alpha_\tau+1) n^{r-\tau}} + \frac{1}{n^r} \right]$$

$$\leq \frac{C_{r,s}}{n^{r-v}}\omega_s\left(f^{(r)},\frac{1}{n}\right)\left[\frac{1}{h!}\sum_{\tau=h}^{k} l_\tau \frac{2^{\tau-\alpha_\tau}}{\Gamma(\tau-\alpha_\tau+1)}+1\right], \tag{38}$$

that is proving (17).

Also if $0 \leq x \leq 1$, then

$$\begin{aligned} &\alpha_h^{-1}(x) L^*(Q_n(x)) \\ &= \alpha_h^{-1}(x) L^*(f(x)) + \rho_n \frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} \\ &\quad + \sum_{j=h}^{v} \alpha_h^{-1}(x)\alpha_j(x)\left[D_{*-1}^{\alpha_j} Q_n(x) - D_{*-1}^{\alpha_j} f(x) - \frac{\rho_n}{h!} D_{*-1}^{\alpha_j} x^h\right] \\ &\overset{(26)}{\geq} \rho_n \frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} - \left(\sum_{j=h}^{v} l_j \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{C_{r,s}}{n^{r-j}}\omega_s\left(f^{(r)},\frac{1}{n}\right)\right) \end{aligned} \tag{39}$$

$$= \rho_n \frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} - \rho_n = \rho_n\left[\frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} - 1\right] \tag{40}$$

$$= \rho_n\left[\frac{(x+1)^{h-\alpha_h} - \Gamma(h-\alpha_h+1)}{\Gamma(h-\alpha_h+1)}\right] \geq \rho_n\left[\frac{1-\Gamma(h-\alpha_h+1)}{\Gamma(h-\alpha_h+1)}\right] \geq 0. \tag{41}$$

Explanation: We know that $\Gamma(1) = 1$, $\Gamma(2) = 1$, and $\Gamma$ is convex and positive on $(0,\infty)$. Here $0 \leq h-\alpha_h < 1$ and $1 \leq h-\alpha_h+1 < 2$. Thus $\Gamma(h-\alpha_h+1) \leq 1$ and $1-\Gamma(h-\alpha_h+1) \geq 0$. Hence $L^*(Q_n(x)) \geq 0$, $x \in [0,1]$.

II. Suppose on $[0,1]$ that $\alpha_h(x) \leq \beta < 0$. Let $Q_n(x)$, $x \in [-1,1]$, be a real polynomial of degree $\leq n$ so that

$$\begin{aligned} &\max_{-1\leq x\leq 1}\left|D_{*-1}^{\alpha_j}\left(f(x) - \rho_n \frac{x^h}{h!}\right) - \left(D_{*-1}^{\alpha_j} Q_n\right)(x)\right| \\ &\leq \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{C_{r,s}}{n^{r-j}}\omega_s\left(f^{(r)},\frac{1}{n}\right),\ j = 0,1,\dots,r. \end{aligned} \tag{42}$$

Similarly we obtain again inequalities of convergence, see (14), (16) and (17).

Also if $0 \leq x \leq 1$, then

$$\begin{aligned} &\alpha_h^{-1}(x) L^*(Q_n(x)) \\ &= \alpha_h^{-1}(x) L^*(f(x)) - \rho_n \frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} \\ &\quad + \sum_{j=h}^{v} \alpha_h^{-1}(x)\alpha_j(x)\left[D_{*-1}^{\alpha_j} Q_n(x) - D_{*-1}^{\alpha_j} f(x) + \frac{\rho_n}{h!}\left(D_{*-1}^{\alpha_j} x^h\right)\right] \\ &\overset{(42)}{\leq} -\rho_n \frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} + \sum_{j=h}^{v} l_j \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{C_{r,s}}{n^{r-j}}\omega_s\left(f^{(r)},\frac{1}{n}\right) \end{aligned} \tag{43}$$

$$= \rho_n \left( 1 - \frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} \right) = \rho_n \left( \frac{\Gamma(h-\alpha_h+1) - (x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} \right) \tag{44}$$

$$\leq \rho_n \left( \frac{1-(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} \right) \leq 0, \tag{45}$$

and hence on $[0,1]$ again holds $L^*(Q_n(x)) \geq 0$. □

*Remark* 7 (to Theorem 6). Suppose that $\alpha_j(x)$, $j = h, h+1, \ldots, v$ are continuous functions on $[-1,1]$, and we have on $[0,1]$ only $L^*(f) > 0$. Relax the condition $\alpha_h(x)$ is either $\geq \alpha > 0$ or $\leq \beta < 0$ on $[0,1]$. Let $Q_n$ be the polynomial of degree $\leq n$ corresponding to $f$ from (24).

Then $D_{*-1}^{\alpha_j} Q_n$ converges uniformly to $D_{*-1}^{\alpha_j} f$ at a higher rate given by inequality (24), in particular for $0 \leq j \leq h$. Moreover, because $L^*(Q_n)$ converges uniformly to $L^*(f)$ on $[-1,1]$, $L^*(Q_n) > 0$ on $[0,1]$ for sufficiently large $n$.

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