Bull. Korean Math. Soc. **52** (2015), No. 2, pp. 581–591
http://dx.doi.org/10.4134/BKMS.2015.52.2.581

LIE TRIPLE DERIVATIONS ON FACTOR VON NEUMANN ALGEBRAS

Lei Liu

Abstract. Let $\mathcal{A}$ be a factor von Neumann algebra with dimension greater than 1. We prove that if a linear map $\delta : \mathcal{A} \to \mathcal{A}$ satisfies

$$\delta([[a, b], c]) = [[\delta(a), b], c] + [[a, \delta(b)], c] + [[a, b], \delta(c)]$$

for any $a, b, c \in \mathcal{A}$ with $ab = 0$ (resp. $ab = P$, where P is a fixed nontrivial projection of $\mathcal{A}$), then there exist an operator $T \in \mathcal{A}$ and a linear map $f : \mathcal{A} \to \mathbb{C}I$ vanishing at every second commutator $[[a, b], c]$ with $ab = 0$ (resp. $ab = P$) such that $\delta(a) = aT - Ta + f(a)$ for any $a \in \mathcal{A}$.

1. Introduction

Let $\mathcal{U}$ be an algebra over the complex field $\mathbb{C}$. Recall that a linear map δ from $\mathcal{U}$ into itself is called a derivation if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{U}$. δ is called a Lie derivation if $\delta([a, b]) = [\delta(a), b] + [a, \delta(b)]$ for all $a, b \in \mathcal{U}$, where $[a, b] = ab - ba$. More generally, δ is called a Lie triple derivation if $\delta([[a, b], c]) = [[\delta(a), b], c] + [[a, \delta(b)], c] + [[a, b], \delta(c)]$ for all $a, b, c \in \mathcal{U}$. Derivations, Lie derivations and Lie triple derivations are very important maps both in theory and applications.

In recent years the local actions of derivations and Lie derivations have been studied intensively. One direction is to study the conditions under which derivations and Lie derivations of operator algebras can be completely determined by the action on some elements concerning products. We say that a linear map $\delta : \mathcal{U} \to \mathcal{U}$ is derivable at a given point $Z \in \mathcal{U}$ if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{U}$ with $ab = Z$. This kind of maps were discussed by several authors (see [1, 2, 3, 6, 9, 12] and references therein). Similarly, a linear map $\delta : \mathcal{U} \to \mathcal{U}$ is said to be Lie derivable at a given point $Z \in \mathcal{U}$ if $\delta([a, b]) = [\delta(a), b] + [a, \delta(b)]$ for all $a, b \in \mathcal{U}$ with $ab = Z$. Lu and Jing [8] discussed such maps on $B(X)$ where X is a Banach space with $\dim X \geq 3$ and $B(X)$ is the algebra of all bounded

Received March 19, 2014; Revised August 26, 2014.
2010 *Mathematics Subject Classification.* Primary 16W25; Secondary 47B47.
Key words and phrases. Lie derivations, Lie triple derivations, factor von Neumann algebras.
Supported by NNSF of China(11326109 and 11401452) and the Fundamental Research Funds for the Central Universities.

linear operators acting on X. It is proved in [8] that if δ is Lie derivable at $Z = 0$ (resp., $Z = P$, where P is a fixed nontrivial idempotent of $B(X)$), then $\delta = d + \tau$, where d is a derivation of $B(X)$ and $\tau : B(X) \to \mathbb{C}I$ is a linear map vanishing at every commutator $[a, b]$ with $ab = 0$ (resp. $ab = P$). Ji [5] gave a similar result on factor von Neumann algebras with dimension great than 4. In [10], the result was generalized on prime rings.

But, so far, there have been no papers on the study of the local actions of Lie triple derivations on operator algebras. We say that a linear map $\delta : \mathcal{U} \to \mathcal{U}$ is Lie triple derivable at a given point $Z \in \mathcal{U}$ if $\delta([[a, b], c]) = [[\delta(a), b], c] + [[a, \delta(b)], c] + [[a, b], \delta(c)]$ for all $a, b, c \in \mathcal{U}$ with $ab = Z$. It is obvious that the condition of being a Lie triple derivable map at some point is much weaker than the condition of being a Lie triple derivation. It is the aim of the present article to investigate the linear maps δ on factor von Neumann algebras satisfying $\delta([[a, b], c]) = [[\delta(a), b], c] + [[a, \delta(b)], c] + [[a, b], \delta(c)]$ for any a, b, c with $ab = 0$ (resp., $ab = P$, where P is a fixed nontrivial projection).

Throughout the article, let H be a complex Hilbert space and $B(H)$ be the algebra of all bounded linear operator on H. We denote by $\mathcal{A} \subseteq B(H)$ the factor von Neumann algebra (i.e., the center of $\mathcal{A}$ is $\mathbb{C}I$, where I is the identity of $\mathcal{A}$). Recall that $\mathcal{A}$ is prime (i.e., for any $a, b \in \mathcal{A}$, $a\mathcal{A}b = 0$ implies $a = 0$ or $b = 0$). We refer the reader to [7] for the theory of von Neumann algebras.

2. Characterizing Lie triple derivations by acting on zero product

In this section, we consider the question of characterizing Lie triple derivations by action at zero product on factor von Neumann algebras.

Theorem 2.1. *Let $\mathcal{A}$ be a factor von Neumann algebra with dimension greater than 1 acting on a Hilbert space, and a linear map $\delta : \mathcal{A} \to \mathcal{A}$ satisfying*

$$\delta([[a, b], c]) = [[\delta(a), b], c] + [[a, \delta(b)], c] + [[a, b], \delta(c)]$$

for all $a, b, c \in \mathcal{A}$ with $ab = 0$. Then there exist an operator $T \in \mathcal{A}$ and a linear map $f : \mathcal{A} \to \mathbb{C}I$ vanishing at every second commutator $[[a, b], c]$ when $ab = 0$ such that

$$\delta(a) = aT - Ta + f(a), \ \forall a \in \mathcal{A}.$$

Proof. Fix a nontrivial projection $P_1 \in \mathcal{A}$ and let $P_2 = I - P_1$. We denote $\mathcal{A}_{ij} = P_i\mathcal{A}P_j$ for $i, j = 1, 2$. Then $\mathcal{A} = \mathcal{A}_{11} + \mathcal{A}_{12} + \mathcal{A}_{21} + \mathcal{A}_{22}$ and each operator $a \in \mathcal{A}$ can be written as $a = a_{11} + a_{12} + a_{21} + a_{22}$, where $a_{ij} \in \mathcal{A}_{ij}$, $i, j = 1, 2$.

We shall organize the proof of Theorem 2.1 in a series of claims.

Claim 1. *Let $a_{ii} \in \mathcal{A}_{ii}$, $i = 1, 2$. If $a_{11}b_{12} = b_{12}a_{22}$ for all $b_{12} \in \mathcal{A}_{12}$, then $a_{11} + a_{22} \in \mathbb{C}I$.*

For any $x_{11} \in \mathcal{A}_{11}$, $x_{12} \in \mathcal{A}_{12}$, we have $a_{11}x_{11}x_{12} = x_{11}x_{12}a_{22} = x_{11}a_{11}x_{12}$. Since $\mathcal{A}$ is prime, it follows that $a_{11}x_{11} = x_{11}a_{11}$. Clearly, $\mathcal{A}_{11}$ is a factor von Neumann algebra on P_1H. Then $a_{11} = \lambda_1 P_1$, $\lambda_1 \in \mathbb{C}$. Similarly, we have

$a_{22} = \lambda_2 P_2$, $\lambda_2 \in \mathbb{C}$. So $\lambda_1 b_{12} = a_{11} b_{12} = b_{12} a_{22} = \lambda_2 b_{12}$, which implies $\lambda_1 = \lambda_2$. Hence, $a_{11} + a_{22} \in \mathbb{C}I$ proving the claim.

Moreover, for any $a_{12} \in \mathcal{A}_{12}$, since $a_{12}P_1 = 0$, we have

$$\begin{aligned}\delta(a_{12}) &= \delta([[a_{12}, P_1], P_1]) \\ &= [[\delta(a_{12}), P_1], P_1] + [[a_{12}, \delta(P_1)], P_1] + [[a_{12}, P_1], \delta(P_1)] \\ &= P_1\delta(a_{12})P_2 + P_2\delta(a_{12})P_1 + P_1\delta(P_1)a_{12} - a_{12}\delta(P_1)P_2 \\ &\quad + \delta(P_1)a_{12} - a_{12}\delta(P_1).\end{aligned}$$

Multiplying P_1 from the left side and P_2 from the right side of the above equation, we arrive at $P_1\delta(P_1)a_{12} = a_{12}\delta(P_1)P_2$. It follows from Claim 1 that $P_1\delta(P_1)P_1 + P_2\delta(P_1)P_2 = \lambda I$, $\lambda \in \mathbb{C}$. Let $E = P_1\delta(P_1)P_2 - P_2\delta(P_1)P_1$, and $\varphi = \delta - \delta_E$, where δ_E is the inner derivation given by $\delta_E(x) = xE - Ex$ for all $x \in \mathcal{A}$. It is not difficult to verify

$$\varphi(P_1) = \delta(P_1) - \delta_E(P_1) = \lambda I,$$

and

$$\varphi([[a, b], c]) = [[\varphi(a), b], c] + [[a, \varphi(b)], c] + [[a, b], \varphi(c)]$$

for any $a, b, c \in \mathcal{A}$ with $ab = 0$.

Claim 2. $\varphi(P_2) \in \mathbb{C}I$.

Since $P_2P_1 = 0$ and $\varphi(P_1) = \lambda I$, we have

$$0 = \varphi([[P_2, P_1], P_1]) = [[\varphi(P_2), P_1], P_1] = P_1\varphi(P_2)P_2 + P_2\varphi(P_2)P_1.$$

For any $a_{12} \in \mathcal{A}_{12}$, since $P_2a_{12} = 0$, we get

$$\begin{aligned}-\varphi(a_{12}) &= \varphi([[P_2, a_{12}], P_2]) \\ &= [[\varphi(P_2), a_{12}], P_2] + [[P_2, \varphi(a_{12})], P_2] + [[P_2, a_{12}], \varphi(P_2)] \\ &= P_1\varphi(P_2)a_{12} - a_{12}\varphi(P_2)P_2 - P_1\varphi(a_{12})P_2 - P_2\varphi(a_{12})P_1 \\ &\quad - a_{12}\varphi(P_2) + \varphi(P_2)a_{12}.\end{aligned}$$

Multiplying the above equation by P_1 from the left and by P_2 from the right, we obtain

$$P_1\varphi(P_2)a_{12} = a_{12}\varphi(P_2)P_2.$$

Then it follows that $P_1\varphi(P_2)P_1 + P_2\varphi(P_2)P_2 \in \mathbb{C}I$ by Claim 1, and hence $\varphi(P_2) \in \mathbb{C}I$.

Claim 3. $\varphi(\mathcal{A}_{ij}) \subseteq \mathcal{A}_{ij}$, $1 \leq i \neq j \leq 2$.

Since $a_{12}P_1 = 0$ and $\varphi(P_1) = \lambda I$, we get

$$\varphi(a_{12}) = \varphi([[a_{12}, P_1], P_1]) = P_1\varphi(a_{12})P_2 + P_2\varphi(a_{12})P_1,$$

which implies $P_1\varphi(a_{12})P_1 = P_2\varphi(a_{12})P_2 = 0$. Now it suffices to show that $P_2\varphi(a_{12})P_1 = 0$. Indeed, for any $b_{12} \in \mathcal{A}_{12}$, $x \in \mathcal{A}$, it is easy to check that

$$0 = \varphi([[a_{12}, b_{12}], x]) = [[\varphi(a_{12}), b_{12}], x] + [[a_{12}, \varphi(b_{12})], x],$$

which leads to $[\varphi(a_{12}), b_{12}] + [a_{12}, \varphi(b_{12})] = \gamma I \in \mathbb{C}I$. Then we have

$$[\varphi(a_{12}), b_{12}] = \gamma I - [a_{12}, \varphi(b_{12})]$$

$$
\begin{aligned}
&= \gamma I + [[a_{12}, P_1], \varphi(b_{12})] \\
&= \gamma I + \varphi([[a_{12}, P_1], b_{12}]) - [[\varphi(a_{12}), P_1], b_{12}] \\
&= \gamma I - [[\varphi(a_{12}), P_1], b_{12}] \\
&= \gamma I - [P_2\varphi(a_{12})P_1, b_{12}].
\end{aligned}
$$

This together with $P_1\varphi(a_{12})P_1 = P_2\varphi(a_{12})P_2 = 0$ entails that $[P_2\varphi(a_{12})P_1, b_{12}] \in \mathbb{C}I$. It follows from [4, Problem 230] that $[P_2\varphi(a_{12})P_1, b_{12}] = 0$. Then $P_2\varphi(a_{12})b_{12} = 0$. Since $\mathcal{A}$ is prime, we have $P_2\varphi(a_{12})P_1 = 0$. Consequently, $\varphi(\mathcal{A}_{12}) \subseteq \mathcal{A}_{12}$.

Similarly, we can obtain $\varphi(\mathcal{A}_{21}) \subseteq \mathcal{A}_{21}$.

Claim 4. *There exist linear functionals f_i on $\mathcal{A}_{ii}$ such that $\varphi(a_{ii}) - f_i(a_{ii})I \in \mathcal{A}_{ii}$ for any $a_{ii} \in \mathcal{A}_{ii}$, $i = 1, 2$.*

Since $a_{11}P_2 = 0$ and from Claim 2, we have

$$0 = \varphi([[a_{11}, P_2], P_2]) = [[\varphi(a_{11}), P_2], P_2] = P_1\varphi(a_{11})P_2 + P_2\varphi(a_{11})P_1.$$

Moreover, for any $b_{22} \in \mathcal{A}_{22}$ and $x \in \mathcal{A}$, it is easy to check that

$$0 = \varphi([[a_{11}, b_{22}], x]) = [[\varphi(a_{11}), b_{22}], x] + [[a_{11}, \varphi(b_{22})], x],$$

which implies that $[\varphi(a_{11}), b_{22}] + [a_{11}, \varphi(b_{22})] = \mu I \in \mathbb{C}I$. Multiplying the above equation from both sides by P_2, we arrive at $[P_2\varphi(a_{11})P_2, b_{22}] = \mu P_2$, which leads to $[P_2\varphi(a_{11})P_2, b_{22}] = 0$ by [4, Problem 230]. So there exists $\tilde{\mu} \in \mathbb{C}$ such that $P_2\varphi(a_{11})P_2 = \tilde{\mu}P_2$.

Thus we have

$$\varphi(a_{11}) = P_1\varphi(a_{11})P_1 + P_2\varphi(a_{11})P_2 = P_1\varphi(a_{11})P_1 - \tilde{\mu}P_1 + \tilde{\mu}I.$$

We can define a linear functional f_1 on $\mathcal{A}_{11}$ by $f_1(a_{11}) = \tilde{\mu}$. Combining with the above equation, we have $\varphi(a_{11}) - f_1(a_{11})I = P_1\varphi(a_{11})P_1 - \tilde{\mu}P_1 \in \mathcal{A}_{11}$ for any $a_{11} \in \mathcal{A}_{11}$.

With the similar argument, we can get a linear functional f_2 on $\mathcal{A}_{22}$ such that $f_2(a_{22}) = \bar{\mu} \in \mathbb{C}$, and $\varphi(a_{22}) - f_2(a_{22})I \in \mathcal{A}_{22}$ for any $a_{22} \in \mathcal{A}_{22}$.

Now, we define a linear map $\omega : \mathcal{A} \to \mathcal{A}$ by

$$\omega(a) = \varphi(a) - f_1(P_1aP_1)I - f_2(P_2aP_2)I, \quad \forall a \in \mathcal{A}.$$

By Claims 3-4, we have $\omega(P_i) = 0$, $\omega(\mathcal{A}_{ij}) \subseteq \mathcal{A}_{ij}$, $i, j = 1, 2$, and $\omega(a_{ij}) = \varphi(a_{ij})$ for any $a_{ij} \in \mathcal{A}_{ij}$, $1 \le i \ne j \le 2$. In the following we shall show ω is a derivation.

Claim 5. *$\omega(a_{ii}b_{ij}) = a_{ii}\omega(b_{ij}) + \omega(a_{ii})b_{ij}$ for any $a_{ii} \in \mathcal{A}_{ii}$, $b_{ij} \in \mathcal{A}_{ij}$, $1 \le i \ne j \le 2$.*

Due to $b_{ij}a_{ii} = 0$, the following equations hold.

$$
\begin{aligned}
\omega(a_{ii}b_{ij}) &= \varphi(a_{ii}b_{ij}) \\
&= \varphi([[b_{ij}, a_{ii}], P_i]) \\
&= [[\varphi(b_{ij}), a_{ii}], P_i] + [[b_{ij}, \varphi(a_{ii})], P_i] \\
&= [[\omega(b_{ij}), a_{ii}], P_i] + [[b_{ij}, \omega(a_{ii})], P_i]
\end{aligned}
$$

$$= a_{ii}\omega(b_{ij}) + \omega(a_{ii})b_{ij}.$$

With the similar argument in Claim 5, we have the following claim.

Claim 6. $\omega(a_{ij}b_{jj}) = a_{ij}\omega(b_{jj}) + \omega(a_{ij})b_{jj}$ *for any* $a_{ij} \in \mathcal{A}_{ij}$, $b_{jj} \in \mathcal{A}_{jj}$, $1 \leq i \neq j \leq 2$.

Claim 7. $\omega(a_{ii}b_{ii}) = a_{ii}\omega(b_{ii}) + \omega(a_{ii})b_{ii}$ *for any* $a_{ii}, b_{ii} \in \mathcal{A}_{ii}$, $i = 1, 2$.

For any $b_{ij} \in \mathcal{A}_{ij}$, we have, from Claim 5, that

$$\omega(a_{ii}b_{ii}b_{ij}) = a_{ii}b_{ii}\omega(b_{ij}) + \omega(a_{ii}b_{ii})b_{ij}.$$

At the same time,

$$\begin{aligned}\omega(a_{ii}b_{ii}b_{ij}) &= a_{ii}\omega(b_{ii}b_{ij}) + \omega(a_{ii})b_{ii}b_{ij}\\ &= a_{ii}b_{ii}\omega(b_{ij}) + a_{ii}\omega(b_{ii})b_{ij} + \omega(a_{ii})b_{ii}b_{ij}.\end{aligned}$$

Comparing the above two equations, we get

$$\omega(a_{ii}b_{ii})b_{ij} = a_{ii}\omega(b_{ii})b_{ij} + \omega(a_{ii})b_{ii}b_{ij}.$$

Since $\mathcal{A}$ is prime, we obtain $\omega(a_{ii}b_{ii}) = a_{ii}\omega(b_{ii}) + \omega(a_{ii})b_{ii}$.

Claim 8. $\omega(a_{ij}b_{ji}) = a_{ij}\omega(b_{ji}) + \omega(a_{ij})b_{ji}$ *for any* $a_{ij} \in \mathcal{A}_{ij}$, $b_{ji} \in \mathcal{A}_{ji}$, $1 \leq i \neq j \leq 2$.

Since $P_2a_{12} = 0$ and $\varphi(P_2) \in \mathbb{C}I$, we have

$$\begin{aligned}\varphi([[P_2, a_{12}], b_{21}]) &= [[P_2, \varphi(a_{12})], b_{21}] + [[P_2, a_{12}], \varphi(b_{21})]\\ &= [[P_2, \omega(a_{12})], b_{21}] + [[P_2, a_{12}], \omega(b_{21})]\\ &= b_{21}\omega(a_{12}) + \omega(b_{21})a_{12} - a_{12}\omega(b_{21}) - \omega(a_{12})b_{21}.\end{aligned}$$

Since $\omega(a) = \varphi(a) - f_1(P_1aP_1)I - f_2(P_2aP_2)I$, $\forall a \in \mathcal{A}$,

$$\begin{aligned}&\omega(b_{21}a_{12} - a_{12}b_{21}) - f_1(a_{12}b_{21})I + f_2(b_{21}a_{12})I\\ &= b_{21}\omega(a_{12}) + \omega(b_{21})a_{12} - a_{12}\omega(b_{21}) - \omega(a_{12})b_{21}.\end{aligned}$$

We shall show $f_1(a_{12}b_{21})I - f_2(b_{21}a_{12})I = 0$. Multiplying the above equation by a_{12} to the left side and right side respectively, we obtain the following two equations

$$(1)\quad a_{12}\omega(b_{21}a_{12}) - (f_1(a_{12}b_{21}) - f_2(b_{21}a_{12}))a_{12} = a_{12}b_{21}\omega(a_{12}) + a_{12}\omega(b_{21})a_{12}$$

and

$$(2)\quad \omega(a_{12})b_{21}a_{12} + a_{12}\omega(b_{21})a_{12} = \omega(a_{12}b_{21})a_{12} + (f_1(a_{12}b_{21}) - f_2(b_{21}a_{12}))a_{12}.$$

Computing (1)+(2), we get

$$\begin{aligned}&a_{12}\omega(b_{21}a_{12}) + \omega(a_{12})b_{21}a_{12} - (f_1(a_{12}b_{21}) - f_2(b_{21}a_{12}))a_{12}\\ &= a_{12}b_{21}\omega(a_{12}) + \omega(a_{12}b_{21})a_{12} + (f_1(a_{12}b_{21}) - f_2(b_{21}a_{12}))a_{12}.\end{aligned}$$

It follows from Claims 5-6 that

$$a_{12}\omega(b_{21}a_{12}) + \omega(a_{12})b_{21}a_{12} = \omega(a_{12}b_{21}a_{12}) = a_{12}b_{21}\omega(a_{12}) + \omega(a_{12}b_{21})a_{12},$$

which combining with the above equation implies $(f_1(a_{12}b_{21})-f_2(b_{21}a_{12}))a_{12} = 0$. Then $f_1(a_{12}b_{21})I - f_2(b_{21}a_{12})I = 0$. So we arrive at

$$\omega(b_{21}a_{12} - a_{12}b_{21}) = \omega(b_{21})a_{12} + b_{21}\omega(a_{12}) - \omega(a_{12})b_{21} - a_{12}\omega(b_{21}).$$

This is equivalent to $\omega(b_{21}a_{12})=\omega(b_{21})a_{12}+b_{21}\omega(a_{12})$ and $\omega(a_{12}b_{21})=\omega(a_{12})b_{21} +a_{12}\omega(b_{21})$.

So ω is a derivation by Claims 5-8.

Hence we have $\delta(a) = \varphi(a)+\delta_E(a) = \omega(a)+f_1(P_1aP_1)I+f_2(P_2aP_2)I+\delta_E(a)$, $\forall a \in \mathcal{A}$. Denote $\phi(a) = \omega(a) + \delta_E(a)$ and $f(a) = f_1(P_1aP_1)I + f_2(P_2aP_2)I$. Clearly, ϕ is a derivation on $\mathcal{A}$ and f is a linear map from $\mathcal{A}$ to $\mathbb{C}I$. By [11], we know every derivation of $\mathcal{A}$ is inner, that is, there exists an operator $T \in \mathcal{A}$ such that $\phi(a) = aT - Ta$ for any $a \in \mathcal{A}$. Then $\delta(a) = aT - Ta + f(a)$ for any $a \in \mathcal{A}$.

For $ab = 0$, it follows that

$$\begin{aligned} f([[a,b],c]) &= \delta([[a,b],c]) - \phi([[a,b],c]) \\ &= [[\delta(a),b],c] + [[a,\delta(b)],c] + [[a,b],\delta(c)] - \phi([[a,b],c]) \\ &= [[\phi(a),b],c] + [[a,\phi(b)],c] + [[a,b],\phi(c)] - \phi([[a,b],c]) \\ &= 0. \end{aligned}$$

□

3. Characterizing Lie triple derivations by acting on projection product

In this section, we consider the question of characterizing Lie triple derivations by action at projection product on factor von Neumann algebras. The proof of the following theorem shares the similar outline as that of Theorem 2.1, but it needs different techniques.

Theorem 3.1. *Let $\mathcal{A}$ be a factor von Neumann algebra with dimension greater than 1 acting on a Hilbert space, and a linear map $\delta : \mathcal{A} \to \mathcal{A}$ satisfying*

$$\delta([[a,b],c]) = [[\delta(a),b],c] + [[a,\delta(b)],c] + [[a,b],\delta(c)]$$

for all $a,b,c \in \mathcal{A}$ with $ab = P$, where $P \in \mathcal{A}$ is a fixed nontrivial projection. Then there exist an operator $T \in \mathcal{A}$ and a linear map $f : \mathcal{A} \to \mathbb{C}I$ vanishing at every second commutator $[[a,b],c]$ when $ab = P$ such that

$$\delta(a) = aT - Ta + f(a),\ \forall a \in \mathcal{A}.$$

Proof. We denote $P_1 = P$, $P_2 = I - P_1$, and $\mathcal{A}_{ij} = P_i\mathcal{A}P_j$ for $i,j = 1,2$.

For any $a_{12} \in \mathcal{A}_{12}$, since $(P_1 + a_{12})P_1 = P_1$, we obtain

$$\begin{aligned} \delta(a_{12}) &= \delta([[P_1 + a_{12}, P_1], P_1]) \\ &= [[\delta(P_1 + a_{12}), P_1], P_1] + [[P_1 + a_{12}, \delta(P_1)], P_1] + [[P_1 + a_{12}, P_1], \delta(P_1)] \\ &= [[\delta(a_{12}), P_1], P_1] + [[a_{12}, \delta(P_1)], P_1] + [[a_{12}, P_1], \delta(P_1)] \\ &= P_1\delta(a_{12})P_2 + P_2\delta(a_{12})P_1 + P_1\delta(P_1)a_{12} - a_{12}\delta(P_1)P_2 \\ &\quad + \delta(P_1)a_{12} - a_{12}\delta(P_1). \end{aligned}$$

Multiplying P_1 from the left side and P_2 from the right side of the above equation, we arrive at $P_1\delta(P_1)a_{12} = a_{12}\delta(P_1)P_2$. It follows from Claim 1 of the proof of Theorem 2.1 that $P_1\delta(P_1)P_1 + P_2\delta(P_1)P_2 = \lambda I$, $\lambda \in \mathbb{C}$. Let $E = P_1\delta(P_1)P_2 - P_2\delta(P_1)P_1$, and $\varphi = \delta - \delta_E$, where δ_E is the inner derivation. It is not difficult to verify that

$$\varphi(P_1) = \delta(P_1) - \delta_E(P_1) = \lambda I,$$

and

$$\varphi([[a,b],c]) = [[\varphi(a),b],c] + [[a,\varphi(b)],c] + [[a,b],\varphi(c)]$$

for any $a, b, c \in \mathcal{A}$ with $ab = P_1$.

Now we organize the proof in a series of claims.

Claim 1. *$\varphi(P_2) \in \mathbb{C}I$.*

Since $(P_1 + P_2)P_1 = P_1$ and $\varphi(P_1) = \lambda I$, we have

$$0 = \varphi([[P_1 + P_2, P_1], P_1]) = [[\varphi(P_1 + P_2), P_1], P_1] = P_1\varphi(P_2)P_2 + P_2\varphi(P_2)P_1.$$

For any $a_{12} \in \mathcal{A}_{12}$, since $(P_1 + a_{12})(P_1 + P_2 - a_{12}) = P_1$, we get

$$\begin{aligned}\varphi(a_{12}) &= \varphi([[P_1 + a_{12}, P_1 + P_2 - a_{12}], P_1]) \\ &= [[\varphi(a_{12}), -a_{12}], P_1] + [[P_1 + a_{12}, \varphi(P_2) - \varphi(a_{12})], P_1] \\ &= P_2\varphi(a_{12})P_1 + P_1\varphi(a_{12})P_2 + P_1\varphi(P_2)a_{12} - a_{12}\varphi(P_2)P_2.\end{aligned}$$

Multiplying the above equation by P_1 from the left and by P_2 from the right, we obtain

$$P_1\varphi(P_2)a_{12} = a_{12}\varphi(P_2)P_2.$$

It follows from Claim 1 of the proof of Theorem 2.1 that $P_1\varphi(P_2)P_1 + P_2\varphi(P_2)P_2 \in \mathbb{C}I$. Hence $\varphi(P_2) \in \mathbb{C}I$.

Claim 2. *$\varphi(\mathcal{A}_{ij}) \subseteq \mathcal{A}_{ij}$, $1 \le i \ne j \le 2$.*

Since $(P_1 + a_{12})P_1 = P_1$ and $\varphi(P_1) = \lambda I$, we get

$$\varphi(a_{12}) = \varphi([[P_1 + a_{12}, P_1], P_1]) = P_1\varphi(a_{12})P_2 + P_2\varphi(a_{12})P_1,$$

which implies $P_1\varphi(a_{12})P_1 = P_2\varphi(a_{12})P_2 = 0$. Now, for any $b_{12} \in \mathcal{A}_{12}$, we have

$$\begin{aligned}0 &= \varphi([[P_1 + b_{12}, P_1], b_{12}]) \\ &= [[\varphi(b_{12}), P_1], b_{12}] + [[b_{12}, P_1], \varphi(b_{12})] \\ &= P_2\varphi(b_{12})b_{12} - b_{12}\varphi(b_{12})P_1 + \varphi(b_{12})b_{12} - b_{12}\varphi(b_{12}).\end{aligned}$$

Multiplying the above equation from both side by P_2, we arrive at $P_2\varphi(b_{12})b_{12} = 0$. Moreover, it follows that

$$\begin{aligned}0 &= \varphi([[P_1 + a_{12}, P_1], b_{12}]) \\ &= [[\varphi(a_{12}), P_1], b_{12}] + [[a_{12}, P_1], \varphi(b_{12})] \\ &= P_2\varphi(a_{12})b_{12} - b_{12}\varphi(a_{12})P_1 + \varphi(b_{12})a_{12} - a_{12}\varphi(b_{12}).\end{aligned}$$

Multiplying the equation by b_{12} from the right and for the fact $P_2\varphi(b_{12})b_{12} = 0$, we obtain $b_{12}\varphi(a_{12})b_{12} = 0$. By linearizing, we get $b_{12}\varphi(a_{12})d_{12} + d_{12}\varphi(a_{12})b_{12} = 0$ for any b_{12}, $d_{12} \in \mathcal{A}_{12}$. It is not difficult to check

$$P_2\varphi(a_{12})b_{12}\varphi(a_{12})[b_{12}\varphi(a_{12})d_{12}]\varphi(a_{12})P_1 \\ + P_2\varphi(a_{12})b_{12}\varphi(a_{12})[d_{12}\varphi(a_{12})b_{12}]\varphi(a_{12})P_1 = 0,$$

that is,

$$P_2\varphi(a_{12})b_{12}\varphi(a_{12})d_{12}\varphi(a_{12})b_{12}\varphi(a_{12})P_1 = 0.$$

Since $\mathcal{A}$ is prime, we have $P_2\varphi(a_{12})b_{12}\varphi(a_{12})P_1 = 0$. Then $P_2\varphi(a_{12})P_1 = 0$. Consequently, $\varphi(\mathcal{A}_{12}) \subseteq \mathcal{A}_{12}$.

Similarly, we can obtain $\varphi(\mathcal{A}_{21}) \subseteq \mathcal{A}_{21}$.

Claim 3. *There exists a linear functional f_1 on $\mathcal{A}_{11}$ such that $\varphi(a_{11}) - f_1(a_{11})I \in \mathcal{A}_{11}$ for all $a_{11} \in \mathcal{A}_{11}$.*

First suppose that a_{11} is invertible in $\mathcal{A}_{11}$, i.e., there exists $a_{11}^{-1} \in \mathcal{A}_{11}$ such that $a_{11}^{-1}a_{11} = a_{11}a_{11}^{-1} = P_1$. Since $a_{11}^{-1}a_{11} = P_1$, we have

$$0 = \varphi([[a_{11}^{-1}, a_{11}], P_1]) = [[\varphi(a_{11}^{-1}), a_{11}], P_1] + [[a_{11}^{-1}, \varphi(a_{11})], P_1].$$

It follows from $(P_2 + a_{11}^{-1})a_{11} = P_1$ and Claim 1 that

$$\begin{aligned} 0 &= \varphi([[P_2 + a_{11}^{-1}, a_{11}], P_1]) \\ &= [[\varphi(a_{11}^{-1}), a_{11}], P_1] + [[P_2 + a_{11}^{-1}, \varphi(a_{11})], P_1] \\ &= P_1\varphi(a_{11})P_2 + P_2\varphi(a_{11})P_1. \end{aligned}$$

Moreover, for any $b_{22} \in \mathcal{A}_{22}$ and $x \in \mathcal{A}$, since $(a_{11}^{-1} + b_{22})a_{11} = P_1$, it is easy to check that

$$\begin{aligned} 0 &= \varphi([[a_{11}^{-1} + b_{22}, a_{11}], x]) \\ &= [[\varphi(a_{11}^{-1} + b_{22}), a_{11}], x] + [[a_{11}^{-1} + b_{22}, \varphi(a_{11})], x] \\ &= [[\varphi(b_{22}), a_{11}], x] + [b_{22}, \varphi(a_{11})], x], \end{aligned}$$

which implies that $[\varphi(b_{22}), a_{11}] + [b_{22}, \varphi(a_{11})] = \mu I \in \mathbb{C}I$. Multiplying the above equation from both sides by P_2, we arrive at $[b_{22}, P_2\varphi(a_{11})P_2] = \mu P_2$. By [4, Problem 230], we get $[b_{22}, P_2\varphi(a_{11})P_2] = 0$. So there exists $\tilde{\mu} \in \mathbb{C}$ such that $P_2\varphi(a_{11})P_2 = \tilde{\mu}P_2$.

If a_{11} is not invertible in $\mathcal{A}_{11}$, we may find a sufficiently big number n such that $nP_1 - a_{11}$ is invertible in $\mathcal{A}_{11}$. It follows from the preceding case that $P_1\varphi(nP_1 - a_{11})P_2 + P_2\varphi(nP_1 - a_{11})P_1 = 0$, and $P_2\varphi(nP_1 - a_{11})P_2 = \tilde{\mu}P_2$. Since $\varphi(P_1) = \lambda I$, we also have $P_1\varphi(a_{11})P_2 + P_2\varphi(a_{11})P_1 = 0$ and $P_2\varphi(a_{11})P_2 = \bar{\mu}P_2$, where $\bar{\mu} = n\lambda - \tilde{\mu}$. Without loss of generality, we still denote $P_2\varphi(a_{11})P_2 = \tilde{\mu}P_2$.

Thus for any $a_{11} \in \mathcal{A}_{11}$, we have

$$\varphi(a_{11}) = P_1\varphi(a_{11})P_1 + P_2\varphi(a_{11})P_2 = P_1\varphi(a_{11})P_1 - \tilde{\mu}P_1 + \tilde{\mu}I.$$

We define a linear functional f_1 on $\mathcal{A}_{11}$ by $f_1(a_{11}) = \tilde{\mu}$. Then combining with the above equation, we get $\varphi(a_{11}) - f(a_{11})I = P_1\varphi(a_{11})P_1 - \tilde{\mu}P_1 \in \mathcal{A}_{11}$ for any $a_{11} \in \mathcal{A}_{11}$.

Claim 4. *There exists a linear functional f_2 on $\mathcal{A}_{22}$ such that $\varphi(a_{22}) - f_2(a_{22})I \in \mathcal{A}_{22}$ for any $a_{22} \in \mathcal{A}_{22}$.*

For any $a_{22} \in \mathcal{A}_{22}$, since $(P_1 + a_{22})P_1 = P_1$, we have

$$0 = \varphi([[P_1 + a_{22}, P_1], P_1]) = P_1\varphi(a_{22})P_2 + P_2\varphi(a_{22})P_1.$$

The rest step is similar to the proof of Claim 3.

Now, we define a linear map $\omega : \mathcal{A} \to \mathcal{A}$ by

$$\omega(a) = \varphi(a) - f_1(P_1aP_1)I - f_2(P_2aP_2)I, \quad \forall a \in \mathcal{A}.$$

By Claim 2-4, we have $\omega(P_i) = 0$, $\omega(\mathcal{A}_{ij}) \subseteq \mathcal{A}_{ij}$, $i, j = 1, 2$, and $\omega(a_{ij}) = \varphi(a_{ij})$ for any $a_{ij} \in \mathcal{A}_{ij}$, $1 \le i \ne j \le 2$.

In the following we shall show ω is a derivation.

Claim 5. $\omega(a_{11}b_{12}) = a_{11}\omega(b_{12}) + \omega(a_{11})b_{12}$ *for any* $a_{11} \in \mathcal{A}_{11}$, $b_{12} \in \mathcal{A}_{12}$.

If a_{11} is invertible in $\mathcal{A}_{11}$, then for any $x_{12} \in \mathcal{A}_{12}$ we have $(a_{11}^{-1}x_{12} + a_{11}^{-1})a_{11} = P_1$. It follows that

$$\begin{aligned}
\omega(a_{12}) &= \omega([[a_{11}^{-1}x_{12} + a_{11}^{-1}, a_{11}], P_1]) \\
&= \varphi([[a_{11}^{-1}x_{12} + a_{11}^{-1}, a_{11}], P_1]) \\
&= [[\varphi(a_{11}^{-1}x_{12} + a_{11}^{-1}), a_{11}], P_1] + [[a_{11}^{-1}x_{12} + a_{11}^{-1}, \varphi(a_{11})], P_1] \\
&= [[\omega(a_{11}^{-1}x_{12} + a_{11}^{-1}), a_{11}], P_1] + [[a_{11}^{-1}x_{12} + a_{11}^{-1}, \omega(a_{11})], P_1] \\
&= [[\omega(a_{11}^{-1}x_{12}), a_{11}], P_1] + [[a_{11}^{-1}x_{12}, \omega(a_{11})], P_1] \\
&= a_{11}\omega(a_{11}^{-1}x_{12}) + \omega(a_{11})a_{11}^{-1}x_{12}.
\end{aligned}$$

Replacing b_{12} with $a_{11}^{-1}x_{12}$, we have $\omega(a_{11}b_{12}) = a_{11}\omega(b_{12}) + \omega(a_{11})b_{12}$.

If a_{11} is not invertible in $\mathcal{A}_{11}$, we may find a sufficiently big number n such that $nP_1 - a_{11}$ is invertible in $\mathcal{A}_{11}$. Then $\omega((nP_1 - a_{11})a_{12}) = (nP_1 - a_{11})\omega(a_{12}) + \omega(nP_1 - a_{11})a_{12}$. Clearly, P_1 is invertible in $\mathcal{A}_{11}$, so we get $\omega(a_{11}b_{12}) = a_{11}\omega(b_{12}) + \omega(a_{11})b_{12}$ from the above equation.

Claim 6. $\omega(a_{21}b_{11}) = a_{21}\omega(b_{11}) + \omega(a_{21})b_{11}$ *for any* $a_{21} \in \mathcal{A}_{21}$, $b_{11} \in \mathcal{A}_{11}$.

Considering $a_{11}(x_{21}a_{11}^{-1} + a_{11}^{-1}) = P_1$ and using the same approach in Claim 5, we know that Claim 6 is true.

Claim 7. $\omega(a_{22}b_{21}) = a_{22}\omega(b_{21}) + \omega(a_{22})b_{21}$ *for any* $a_{22} \in \mathcal{A}_{22}$, $b_{21} \in \mathcal{A}_{21}$.

Due to $(P_1 + a_{22} - a_{22}b_{21})(P_1 + b_{21}) = P_1$, we compute

$$\begin{aligned}
-\omega(b_{21}) &= \omega([[P_1 + a_{22} - a_{22}b_{21}, P_1 + b_{21}], P_1]) \\
&= \varphi([[P_1 + a_{22} - a_{22}b_{21}, P_1 + b_{21}], P_1]) \\
&= [[\omega(P_1 + a_{22} - a_{22}b_{21}), P_1 + b_{21}], P_1] \\
&\quad + [[P_1 + a_{22} - a_{22}b_{21}, \omega(P_1 + b_{21})], P_1] \\
&= a_{22}\omega(b_{21}) - \omega(a_{22}b_{21}) + \omega(a_{22})b_{21} - \omega(b_{21}),
\end{aligned}$$

that is, $\omega(a_{22}b_{21}) = a_{22}\omega(b_{21}) + \omega(a_{22})b_{21}$.

Considering $(P_1 + a_{12})(P_1 - b_{22} + a_{12}b_{22}) = P_1$, we arrive at:

Claim 8. $\omega(a_{12}b_{22}) = a_{12}\omega(b_{22}) + \omega(a_{12})b_{22}$ *for any* $a_{12} \in \mathcal{A}_{12}$, $b_{22} \in \mathcal{A}_{22}$.

Claim 9. $\omega(a_{ii}b_{ii}) = a_{ii}\omega(b_{ii}) + \omega(a_{ii})b_{ii}$, $i = 1, 2$.

It is similar to Claim 7 in the proof of Theorem 2.1.

Claim 10. $\omega(a_{ij}b_{ji}) = a_{ij}\omega(b_{ji}) + \omega(a_{ij})b_{ji}$ *for any* $a_{ij}, b_{ij} \in \mathcal{A}_{ij}$, $1 \leq i \neq j \leq 2$.

Since $(a_{12} + P_1)P_1 = P_1$, we have

$$\begin{aligned}\varphi(b_{21}a_{12} - a_{12}b_{21}) &= \varphi([[a_{12} + P_1, P_1], b_{21}]) \\ &= [[\omega(a_{12} + P_1), P_1], b_{21}] + [[a_{12} + P_1, P_1], \omega(b_{21})] \\ &= b_{21}\omega(a_{12}) + \omega(b_{21})a_{12} - a_{12}\omega(b_{21}) - \omega(a_{12})b_{21}.\end{aligned}$$

Since $\omega(a) = \varphi(a) - f_1(P_1aP_1)I - f_2(P_2aP_2)I$, $\forall a \in \mathcal{A}$,

$$\begin{aligned}&\omega(b_{21}a_{12} - a_{12}b_{21}) - f_1(a_{12}b_{21})I + f_2(b_{21}a_{12})I \\ &= b_{21}\omega(a_{12}) + \omega(b_{21})a_{12} - a_{12}\omega(b_{21}) - \omega(a_{12})b_{21}.\end{aligned}$$

With the same approach as in Claim 8 in the proof of Theorem 2.1, we can get $f_1(a_{12}b_{21})I - f_2(b_{21}a_{12})I = 0$. So we arrive at

$$\omega(b_{21}a_{12} - a_{12}b_{21}) = \omega(b_{21})a_{12} + b_{21}\omega(a_{12}) - \omega(a_{12})b_{21} - a_{12}\omega(b_{21}).$$

This is equivalent to $\omega(b_{21}a_{12}) = \omega(b_{21})a_{12} + b_{21}\omega(a_{12})$ and $\omega(a_{12}b_{21}) = \omega(a_{12})b_{21} + a_{12}\omega(b_{21})$. Consequently, Claim 10 is true.

So we can conclude that ω is a derivation by Claims 5-10.

With the similar argument as in the proof of Theorem 2.1, we can verify there exist an operator $T \in \mathcal{A}$ and a linear map $f : \mathcal{A} \to \mathbb{C}I$ vanishing at every second commutator $[[a, b], c]$ when $ab = P$ such that $\delta(a) = aT - Ta + f(a), \forall a \in \mathcal{A}$. □

References

[1] J. Alaminos, J. Extremera, A. R. Villena, and M. Bresar, *Characterizing homomorphisms and derivations on C^*-algebras*, Proc. Roy. Soc. Edinburgh Sect. A **137** (2007), no. 1, 1–7.

[2] M. Bresar, *Characterizing homomorphisms, derivations and multipliers in rings with idempotents*, Proc. Roy. Soc. Edinburgh Sect. A **137** (2007), no. 1, 9–21.

[3] M. A. Chebotar, W.-F. Ke, and P.-H. Lee, *Maps characterized by action on zero products*, Pacific J. Math. **216** (2004), no. 2, 217–228.

[4] P. Halmos, *A Hilbert Space Problem Book*, 2nd edn. Springer-Verlag, NewYork, 1982.

[5] P. Ji, W. Qi, and X. Sun, *Characterizations of Lie derivations of factor von Neumann algebras*, Linear Multilinear Algebra **61** (2013), no. 3, 417–428.

[6] W. Jing, S. J. Lu, and P. T. Li, *Characterisations of derivations on some operator algebras*, Bull. Austral. Math. Soc. **66** (2002), no. 2, 227–232.

[7] R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras. I, II*, Academic Press, New York, 1983; 1986.

[8] F. Lu and W. Jing, *Characterizations of Lie derivations of $B(X)$*, Linear Algebra Appl. **432** (2009), 89–99.

[9] X. Qi and J. Hou, *Characterizations of derivations of Banach space nest algebras: All-derivable points*, Linear Algebra Appl. **432** (2010), no. 12, 3183–3200.

[10] ———, *Characterization of Lie derivations on prime rings*, Comm. Algebra **39** (2011), no. 10, 3824–3835.

[11] S. Sakai, *Derivations of W^*-algebras*. Ann. Math. **83** (1966), 273–279.

[12] J. Zhu and S. Zhao, *Characterizations of all-derivable points in nest algebras*, Proc. Amer. Math. Soc. **141** (2013), no. 7, 2343–2350.

School of Mathematics and Statistics
Xidian University
Xi'an 710071, P. R. China
E-mail address: leiliu@mail.xidian.edu.cn