

CENTROID OF TRIANGLES ASSOCIATED WITH A CURVE

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ABSTRACT. Archimedes showed that the area between a parabola and any chord AB on the parabola is four thirds of the area of triangle $\triangle ABP$, where P is the point on the parabola at which the tangent is parallel to the chord AB . Recently, this property of parabolas was proved to be a characteristic property of parabolas. With the aid of this characterization of parabolas, using centroid of triangles associated with a curve we present two conditions which are necessary and sufficient for a strictly locally convex curve in the plane to be a parabola.

1. Introduction

We study strictly locally convex plane curves. Recall that a regular plane curve $X : I \rightarrow \mathbb{R}^2$ in the plane \mathbb{R}^2 , where I is an open interval, is called *convex* if, for all $s \in I$ the trace $X(I)$ of X lies entirely on one side of the closed half-plane determined by the tangent line at $X(s)$ ([4]). A regular plane curve $X : I \rightarrow \mathbb{R}^2$ is called *locally convex* if, for each $s \in I$ there exists an open subinterval $I_0 \subset I$ containing s such that the curve $X|_{I_0}$ restricted to I_0 is a convex curve.

Henceforth, we will say that a locally convex curve X in the plane \mathbb{R}^2 is *strictly locally convex* if the curve is smooth (that is, of class $C^{(3)}$) and is of positive curvature κ with respect to the unit normal N pointing to the convex side. Hence, in this case we have $\kappa(s) = \langle X''(s), N(X(s)) \rangle > 0$, where $X(s)$ is an arc-length parametrization of X .

When $f : I \rightarrow \mathbb{R}$ is a smooth function defined on an open interval I , we will also say that f is *strictly convex* if the graph of f has positive curvature κ with respect to the upward unit normal N . This condition is equivalent to the positivity of $f''(x)$ on I .

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Suppose that X is a strictly locally convex curve in the plane \mathbb{R}^2 with the unit normal N pointing to the convex side. For a fixed point $P \in X$, and for a sufficiently small $h > 0$, we consider the line ℓ passing through $P + hN(P)$ which is parallel to the tangent line t of X at P and the points A and B where the line ℓ intersects the curve X . We denote by t_1, t_2 the tangent lines of X at the points A, B and by Q, A_1, B_1 the intersection points $t_1 \cap t_2, t_1 \cap \ell, t_2 \cap \ell$, respectively.

We let $L_P(h), g_P(h), j_P(h)$ and $k_P(h)$ denote the length of the chord AB , the distance from the centroid G of the section of X cut off by ℓ to the line ℓ , the distance from the centroid J of the triangle $\triangle QAB$ to the line ℓ and the distance from the centroid K of the triangle $\triangle QA_1B_1$ to the line ℓ , respectively.

Now, we consider $S_P(h)$ and $T_P(h)$ defined by the area of the region bounded by the curve X and chord AB , the area $|\triangle PAB|$ of triangle $\triangle PAB$, respectively. Then, obviously we have

$$(1.1) \quad T_P(h) = \frac{1}{2}hL_P(h)$$

and we get ([9])

$$(1.2) \quad \frac{d}{dh}S_P(h) = L_P(h).$$

It is well known that parabolas satisfy the following properties.

Proposition 1. *Suppose that X is an open part of a parabola. Then we have the following.*

1) *For arbitrary point $P \in X$ and sufficiently small $h > 0$, X satisfies*

$$(1.3) \quad S_P(h) = \frac{4}{3}T_P(h).$$

2) *For arbitrary point $P \in X$ and sufficiently small $h > 0$, X satisfies*

$$(1.4) \quad g_P(h) = \frac{2}{5}h.$$

3) *For arbitrary point $P \in X$ and sufficiently small $h > 0$, X satisfies*

$$(1.5) \quad j_P(h) = \frac{2}{3}h.$$

4) *For arbitrary point $P \in X$ and sufficiently small $h > 0$, X satisfies*

$$(1.6) \quad k_P(h) = \frac{4}{3}h.$$

Proof. For a proof of 1), see [16]. If we denote by V the point where the parallel line m through the point P to the axis of X meets the chord AB , then V is the mid point of AB and the point Q is on the line m with $PV = PQ$. This completes the proof of 2), 3) and 4). \square

Very recently, the first author of the present paper and Y. H. Kim showed that among strictly convex plane curves, the above area property (1.3) of parabolic sections characterize parabolas. More precisely, they proved as follows ([9]).

Proposition 2. *Let X be a strictly convex curve in the plane \mathbb{R}^2 . Then X is a parabola if and only if it satisfies*

(C) *For a point P on X and a chord AB of X parallel to the tangent of X at P , the area of the region bounded by the curve and AB is $4/3$ times the area of triangle $\triangle ABP$, that is,*

$$S_P(h) = \frac{4}{3}T_P(h).$$

Archimedes showed that parabolas satisfy (1.3) ([16]). Actually, in [9] the first author of the present paper with Y. H. Kim established five characterizations of parabolas, which are the converses of well-known properties of parabolas originally due to Archimedes ([16]). For some properties and characterizations of parabolas with respect to the area of triangles associated with a curve, see [3, 10, 13, 14]. For the higher dimensional analogues of some results in [9], see [7] and [8].

In [11], using Proposition 1, D.-S. Kim et al. proved the following characterization theorem for parabolas with respect to the function $g_P(h)$.

Proposition 3. *Let X be a strictly locally convex plane curve in the plane \mathbb{R}^2 . For a fixed point P on X and a sufficiently small $h > 0$, we denote by ℓ the parallel line through $P + hN(P)$ to the tangent t of the curve X at P . We let $g_P(h)$ the distance from the center G of gravity of the section of X cut off by ℓ to the line ℓ . Then X is an open part of a parabola if and only if it satisfies for a fixed point P on X and a sufficiently small $h > 0$*

$$g_P(h) = \frac{2}{5}h.$$

In [11], the distance from the center G of gravity of the section of X cut off by ℓ to the tangent t of X at P was denoted by $d_P(h)$. Hence, we see that $d_P(h) + g_P(h) = h$.

In this article, we study whether the remaining properties of parabolas in Proposition 1 characterize parabolas.

First of all, in Section 2 we prove the following:

Theorem 4. *Suppose that X denotes a strictly locally convex plane curve in the plane \mathbb{R}^2 . For a fixed point P on X and a sufficiently small $h > 0$, we denote by ℓ the parallel line through $P + hN(P)$ to the tangent t of the curve X at P . Then we have*

$$(1.7) \quad \lim_{h \rightarrow 0} \frac{j_P(h)}{h} = \frac{2}{3}$$

and

$$(1.8) \quad \lim_{h \rightarrow 0} \frac{k_P(h)}{h} = \frac{4}{3}.$$

Finally, with the aid of the characterization theorem of parabolas (Theorem 3 in [9]), in Section 4 we prove the following.

Theorem 5. *Suppose that X denotes a strictly locally convex $C^{(3)}$ curve in the plane \mathbb{R}^2 . Then the following are equivalent.*

1) *For all $P \in X$ and sufficiently small $h > 0$, X satisfies*

$$(1.9) \quad j_P(h) = \lambda(P)h^{\mu(P)},$$

where $\lambda(P)$ and $\mu(P)$ are some functions.

2) *For all $P \in X$ and sufficiently small $h > 0$, X satisfies*

$$j_P(h) = \frac{2}{3}h.$$

3) *X is an open part of a parabola.*

For the function $k_P(h)$, the similar argument as in the proof of Theorem 5 yields the following.

Theorem 6. *Suppose that X denotes a strictly locally convex $C^{(3)}$ curve in the plane \mathbb{R}^2 . Then the following are equivalent.*

1) *For all $P \in X$ and sufficiently small $h > 0$, X satisfies*

$$(1.10) \quad k_P(h) = \lambda(P)h^{\mu(P)},$$

where $\lambda(P)$ and $\mu(P)$ are some functions.

2) *For all $P \in X$ and sufficiently small $h > 0$, X satisfies*

$$k_P(h) = \frac{4}{3}h.$$

3) *X is an open part of a parabola.*

Remark. If we consider the distance $\delta_P(h)$ from the centroid of the triangle ΔPAB to the parallel line ℓ through the point $P + hN(P)$ to the tangent t of a strictly locally convex curve X at P , then X always satisfies $\delta_P(h) = \frac{1}{3}h$ for all sufficiently small $h > 0$.

For some characterizations of parabolas or conic sections by properties of tangent lines, see [5] and [12]. In [6], using curvature function κ and support function h of a plane curve, the first author of the present paper and Y. H. Kim gave a characterization of ellipses and hyperbolas centered at the origin.

Among the graphs of functions, Á. Bényi et al. proved some characterizations of parabolas ([1, 2]). In [15], B. Richmond and T. Richmond established a dozen necessary and sufficient conditions for the graph of a function to be a parabola by using elementary techniques.

Throughout this article, all curves are of class $C^{(3)}$ and connected, unless otherwise mentioned.

2. Proof of Theorem 4

In this section, we prove Theorem 4. First of all, we need the following lemma ([9]) which is useful in the proof of main theorems.

Lemma 7. *Suppose that X is a strictly locally convex $C^{(3)}$ curve in the plane \mathbb{R}^2 with the unit normal N pointing to the convex side. Then we have*

$$(2.1) \quad \lim_{h \rightarrow 0} \frac{1}{\sqrt{h}} L_P(h) = \frac{2\sqrt{2}}{\sqrt{\kappa(P)}},$$

where $\kappa(P)$ is the curvature of X at P with respect to the unit normal N pointing to the convex side.

Now, we prove Theorem 4 as follows.

Let us denote by X a strictly locally convex $C^{(3)}$ curve in the Euclidean plane \mathbb{R}^2 . We fix an arbitrary point P on X . Then, we may take a coordinate system (x, y) of \mathbb{R}^2 such that P is the origin $(0, 0)$ and x -axis is the tangent line t of X at P . Furthermore, we may regard X to be locally the graph of a non-negative strictly convex function $f : \mathbb{R} \rightarrow \mathbb{R}$ with $f(0) = f'(0) = 0$. Then N is the upward unit normal.

Since the curve X is of class $C^{(3)}$, the Taylor's formula of $f(x)$ is given by

$$(2.2) \quad f(x) = ax^2 + f_3(x),$$

where $2a = f''(0)$ and $f_3(x)$ is an $O(|x|^3)$ function. Noting that the curvature κ of X at P is given by $\kappa(P) = f''(0) > 0$, we see that a is positive.

For a sufficiently small $h > 0$, the line ℓ through $P + hN(P)$ and orthogonal to $N(P)$ is given by $y = h$. We denote by $A(s, f(s))$ and $B(t, f(t))$ the points where the line $\ell : y = h$ meets the curve X with $s < 0 < t$. Then we have $f(s) = f(t) = h$. The tangent lines t_1 and t_2 to X at A and B intersect at the point $Q = (x_0(h), y_0(h))$ with

$$(2.3) \quad x_0(h) = \frac{tf'(t) - sf'(s)}{f'(t) - f'(s)},$$

$$(2.4) \quad y_0(h) = h + \frac{(t-s)f'(t)f'(s)}{f'(t) - f'(s)} < 0$$

and they meet the x -axis (the tangent to X at P) at $B_1(s - h/f'(s), 0)$ and $B_2(t - h/f'(t), 0)$, respectively.

Noting $L_P(h) = t - s$, one gets

$$(2.5) \quad j_P(h) = h - \frac{1}{3}\{2h + y_0(h)\} = -\frac{1}{3} \frac{L_P(h)f'(t)f'(s)}{f'(t) - f'(s)}$$

and

$$(2.6) \quad k_P(h) = h - \frac{1}{3}y_0(h) = \frac{2}{3}h - \frac{1}{3} \frac{L_P(h)f'(t)f'(s)}{f'(t) - f'(s)}.$$

Hence we obtain

$$(2.7) \quad \frac{j_P(h)}{h} = \frac{1}{3} \frac{L_P(h)}{\sqrt{h}} \frac{1}{\alpha_P(h)}$$

and

$$(2.8) \quad \frac{k_P(h)}{h} = \frac{2}{3} + \frac{j_P(h)}{h},$$

where we use

$$(2.9) \quad \alpha_P(h) = \frac{(f'(s) - f'(t))}{f'(s)f'(t)} \sqrt{h}.$$

On the other hand, it follows from Lemma 5 in [13] that

$$(2.10) \quad \lim_{h \rightarrow 0} \alpha_P(h) = \frac{\sqrt{2}}{\sqrt{\kappa(P)}}.$$

Together with (2.7) and Lemma 7, this shows that

$$(2.11) \quad \lim_{h \rightarrow 0} \frac{j_P(h)}{h} = \frac{2}{3},$$

and hence from (2.8) we also get

$$(2.12) \quad \lim_{h \rightarrow 0} \frac{k_P(h)}{h} = \frac{4}{3}.$$

This completes the proof of Theorem 4.

3. Proofs of Theorems 5 and 6

In this section, we shall prove Theorem 5 and Theorem 6.

In order to prove Theorem 5 we use the main result of Theorem 3 in [9] and Theorem 4 stated in Section 1.

First, we prove:

Lemma 8. *Suppose that X is a strictly locally convex $C^{(3)}$ curve in the plane \mathbb{R}^2 with the unit normal N pointing to the convex side. Then we have*

$$(3.1) \quad \sqrt{h} \frac{d}{dh} L_P(h) = \alpha_P(h),$$

where $\alpha_P(h)$ is defined in (2.9).

Proof. Just as in the proof of Theorem 4 in Section 2, for an arbitrary point P on X we take a coordinate system (x, y) of \mathbb{R}^2 so that (2.2) holds with $f(0) = f'(0) = 0$ and $2a = f''(0) > 0$. Then, for sufficiently small $h > 0$, we put $f(t) = h$ with $t > 0$ and we denote by $A(s(t), h)$ and $B(t, h)$ the points where the line $\ell : y = h$ meets the curve X with $s = s(t) < 0 < t$. Then we have

$$(3.2) \quad f(s(t)) = f(t) = h$$

and

$$(3.3) \quad L_P(h) = t - s(t).$$

Noting $h = f(t)$, one obtains from (3.3) that

$$(3.4) \quad \frac{d}{dh}L_P(h) = (1 - s'(t))\frac{dt}{dh} = \frac{1 - s'(t)}{f'(t)}.$$

Therefore, it follows from (3.2) that

$$(3.5) \quad \frac{d}{dh}L_P(h) = \frac{1}{f'(t)} - \frac{1}{f'(s(t))} = \frac{f'(s) - f'(t)}{f'(t)f'(s)}.$$

Together with (2.9), this completes the proof of Lemma 8. □

It is obvious that any open part of parabolas satisfy 1) and 2) in Theorem 5.

Conversely, suppose that X is a strictly locally convex $C^{(3)}$ curve in the plane \mathbb{R}^2 which satisfies for all $P \in X$ and sufficiently small $h > 0$

$$j_P(h) = \lambda(P)h^{\mu(P)},$$

where $\lambda(P)$ and $\mu(P)$ are some functions. Using Theorem 4, by letting $h \rightarrow 0$ we see that

$$(3.6) \quad \lim_{h \rightarrow 0} h^{\mu(P)-1} = \frac{2}{3} \frac{1}{\lambda(P)},$$

which shows that $\mu(P) = 1$ and $\lambda(P) = \frac{2}{3}$. Therefore, the curve X satisfies for all $P \in X$ and sufficiently small $h > 0$

$$j_P(h) = \frac{2}{3}h.$$

Now, using Lemma 8 we get the following.

Lemma 9. *Suppose that X denotes a strictly locally convex $C^{(3)}$ curve in the plane \mathbb{R}^2 which satisfies (1.5) for all $P \in X$ and sufficiently small $h > 0$. Then for all $P \in X$ and sufficiently small $h > 0$ we have*

$$(3.7) \quad L_P(h) = \frac{2\sqrt{2}}{\sqrt{\kappa(P)}}\sqrt{h}.$$

Proof. It follows from (1.5) and (2.7) that

$$(3.8) \quad L_P(h) = 2\sqrt{h}\alpha_P(h).$$

Together with Lemma 8, this yields

$$(3.9) \quad 2h\frac{d}{dh}L_P(h) = L_P(h).$$

By integrating (3.9) with respect to h , we get for some constant $C = C(P)$

$$(3.10) \quad L_P(h) = C\sqrt{h}.$$

Thus, Lemma 7 completes the proof of Lemma 9. □

Finally, we prove Theorem 5 as follows.

It follows from (1.2) and $S_P(0) = 0$ that by integrating (3.7) we get

$$(3.11) \quad S_P(h) = \frac{4\sqrt{2}}{3\sqrt{\kappa(P)}}h\sqrt{h}.$$

Hence, together with (1.1) and (3.7), (3.11) yields that for all $P \in X$ and sufficiently small $h > 0$

$$(3.12) \quad S_P(h) = \frac{4}{3}T_P(h).$$

Thus, it follows from Proposition 2 that X is an open part of a parabola. This completes the proof of Theorem 5.

In order to prove Theorem 6, with the help of (2.8) we may use the similar argument as in the proof of Theorem 5.

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