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MODULES SATISFYING CERTAIN CHAIN CONDITIONS AND THEIR ENDOMORPHISMS

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Abstract. In this paper, we characterize w-Noetherian modules in terms of polynomial modules and w-Nagata modules. Then it is shown that for a finite type w-module M, every w-epimorphism of M onto itself is an isomorphism. We also define and study the concepts of w-Artinian modules and w-simple modules. By using these concepts, it is shown that for a w-Artinian module M, every w-monomorphism of M onto itself is an isomorphism and that for a w-simple module M, $\mathrm{End}_R M$ is a division ring.

1. Introduction

The question of when injective or surjective endomorphisms of certain modules over commutative rings are isomorphisms had been addressed in the literature. In [1], Bourbaki pointed out that if M is a Noetherian module, then every surjective endomorphism of M is an isomorphism. For the general case, Vasconcelos [5, 6] and Strooker [4] proved independently that if M is a finitely generated module, then every surjective endomorphism of M is an isomorphism. In [7], Vasconcelos also considered cases where an injective endomorphism of a finitely generated module is, actually, an isomorphism. It is a simple exercise that Artinian modules are endowed with this property [1, p. 23]. It is well known that if a module is simple, then its endomorphism ring is a division ring (this is sometimes called Schur's lemma).

Let D be an integral domain with quotient field $q(D)$. Following [11], a nonzero finitely generated ideal J of D is called a *GV-ideal*, denoted by $J \in \mathrm{GV}(D)$, if $J^{-1} = D$; and a torsion-free D-module M is called a *w-module* if $Jx \subseteq M$ for $x \in q(D) \otimes_D M$ and $J \in \mathrm{GV}(D)$ implies $x \in M$. A w-module M is called a *strong Mori module* if M satisfies the ACC on w-submodules of M. G. W. Chang characterized strong Mori modules in terms of polynomial modules and t-Nagata modules and also studied the above question in [2] as follows. It is shown that M is a strong Mori module over D if and only if the polynomial

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module $M[X]$ is a strong Mori module over $D[X]$; if and only if $M[X]_{N_v}$ is a Noetherian module over $D[X]_{N_v}$, where $N_v = \{f \in D[X] \mid c(f)_v = D\}$. And it is proved that if $\varphi : M \to M$ is an epimorphism, where M is a strong Mori module, then φ is an isomorphism. Certainly, this is the w-theoretic version of the aforementioned Bourbaki's theorem.

In this paper, we show that the two results above of G. W. Chang still hold for a commutative ring with zero divisors if we use a new extended definition of w-modules (see [9, 14]) under more weaker conditions (w-epimorphisms not epimorphisms). We also address the above questions on endomorphisms. To do this, we introduce and study the concepts of w-Artinian modules and w-simple modules.

Throughout this paper, R is a commutative ring with identity element and all modules are unitary. Following [14] a finitely generated ideal J of R is called a *GV-ideal*, if the natural homomorphism $R \to \mathrm{Hom}_R(J, R)$ is an isomorphism. Denote by $\mathrm{GV}(R)$ the set of GV-ideals of R. An R-module M is called GV-*torsion* if for any $x \in M$, there is a $J \in \mathrm{GV}(R)$ such that $Jx = 0$, and M is said to be GV-*torsion-free* if $Jx = 0$ for $J \in \mathrm{GV}(R)$ and $x \in M$ implies $x = 0$. Denote by $E(M)$ the injective envelope of M. For a GV-torsion-free R-module M, define

$$M_w = \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in \mathrm{GV}(R)\},$$

which is called the *w-envelope* of M. A GV-torsion-free module M is called a *w-module* if $M_w = M$, equivalently, $\mathrm{Ext}^1_R(R/J, M) = 0$ for any $J \in \mathrm{GV}(R)$. Then it is easy to see that the w-operation on R distributes over finite intersections since $\mathrm{GV}(R)$ is a multiplicative system of R. A w-ideal $\mathfrak{m}$ of R is called a *maximal w-ideal* if $\mathfrak{m}$ is maximal among proper integral w-ideals of R. It is shown that every maximal w-ideal of R is prime [14, Proposition 3.8].

Let M and N be R-modules. Following [9], a homomorphism $f : M \to N$ is called a *w-monomorphism* (*resp., w-epimorphism, w-isomorphism*) if $f_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is a monomorphism (resp., an epimorphism, an isomorphism) over $R_{\mathfrak{m}}$ for any maximal w-ideal $\mathfrak{m}$ of R. A sequence $A \to B \to C$ is said to be *w-exact* if the induced sequence $A_{\mathfrak{m}} \to B_{\mathfrak{m}} \to C_{\mathfrak{m}}$ is exact for any maximal w-ideal $\mathfrak{m}$ of R. An R-module M is said to be *of finite type* if there is a w-exact sequence $F \to M \to 0$, where F is finitely generated free. Thus, if M is of finite type, then $M_{\mathfrak{m}}$ is finitely generated over $R_{\mathfrak{m}}$ for any maximal w-ideal $\mathfrak{m}$ of R. A module M is said to be *w-Noetherian* if every submodule of M is of finite type. Certainly, when R is an integral domain, a torsion-free w-module M is a strong Mori module if and only if M is w-Noetherian.

2. Main results

Under the renewed notions we can not only generalize G. W. Chang's results to a w-Noetherian module but also give a proof with different approach. To do this, we need a couple of lemmas.

Lemma 2.1. *An R-module M is of finite type if and only if there is a finitely generated submodule N of M such that M/N is* GV*-torsion.*

Proof. See [9, Proposition 1.2]. □

Let X be an indeterminate over R. The *content* of a polynomial $f \in R[X]$, denoted by $c(f)$, is the ideal of R generated by the coefficients of f. Set $S_w = \{f \in R[X] \mid c(f)_w = R\}$ and $R\{X\} = R[X]_{S_w}$, which is called the *w-Nagata ring* of R. Let M be an R-module and $M[X] = M \otimes_R R[X]$. Then $M[X]_{S_w}$ is an $R[X]_{S_w}$-module and is called the *w-Nagata module* of M and set $M\{X\} = M[X]_{S_w}$. Note that if R is a domain, then $S_w = N_v$ and $R\{X\} = R[X]_{N_v}$.

Lemma 2.2. (1) *An R-module M is* GV*-torsion if and only if $M\{X\} = 0$.*

(2) *An R-sequence $A \to B \to C$ is w-exact if and only if the $R\{X\}$-sequence $A\{X\} \to B\{X\} \to C\{X\}$ is exact.*

(3) *Let $\alpha : M \to N$ be an R-homomorphism. Then α is a w-monomorphism* (*resp., w-epimorphism, w-isomorphism*) *if and only if the canonical extension $\overline{\alpha} : M\{X\} \to N\{X\}$ is a monomorphism* (*resp., an epimorphism, an isomorphism*)*.*

(4) *An R-module M is of finite type if and only if $M\{X\}$ is finitely generated over $R\{X\}$.*

Proof. See [10]. □

Lemma 2.3. *If J is a* GV*-ideal of $R[X]$, then there is $g \in J$ such that $c(g)_w = R$.*

Proof. See [13, Corollary 2.5]. □

Theorem 2.4. *The following statements are equivalent for an R-module M.*

(1) *M is a w-Noetherian module over R.*
(2) *$M[X]$ is a w-Noetherian module over $R[X]$.*
(3) *$M\{X\}$ is a Noetherian module over $R\{X\}$.*

Proof. (1) ⇒ (2). Similar to the proof of [14, Theorem 4.9].

(2)⇒(3). Let A be a submodule of $M\{X\}$. Then there is a submodule B of $M[X]$ such that $A = B_{S_w}$. Since $M[X]$ is w-Noetherian, B is of finite type over $R[X]$. Thus by Lemma 2.1, there is a finitely generated submodule C of B such that B/C is GV-torsion over $R[X]$. Let $u \in B$. Then there is a GV-ideal J of $R[X]$ such that $Ju \subseteq C$. By Lemma 2.3 there is $g \in J$ such that $c(g)_w = R$. Hence $c(g) \in \mathrm{GV}(R)$. From $gu \in C$ we have $\frac{u}{1} = \frac{gu}{g} \in C_{S_w}$. Therefore, $A = B_{S_w} = C_{S_w}$ is finitely generated over $R\{X\}$. So $M\{X\}$ is Noetherian.

(3) ⇒ (1). Let N be a submodule of M. Then $N\{X\}$ is a submodule of $M\{X\}$. Hence $N\{X\}$ is finitely generated by hypothesis. So N is of finite type by Lemma 2.2(4). Consequently, M is w-Noetherian. □

As a corollary, we can recover [13, Proposition 4.3] in the following.

Corollary 2.5. *The following statements are equivalent for a ring R.*

(1) *R is a w-Noetherian ring.*
(2) *$R[X]$ is a w-Noetherian ring.*
(3) *$R\{X\}$ is a Noetherian ring.*

Lemma 2.6. *Let M and N be w-modules and let $f : M \to N$ be a homomorphism. If f is a w-isomorphism, then f is an isomorphism.*

Proof. This is a simple corollary of [9, Theorem 1.2]. □

The following is the w-theoretic version of Vasconcelos-Strooker's theorem.

Theorem 2.7. *Let M be a finite type w-module and let $f : M \to M$ be a w-epimorphism. Then f is an isomorphism.*

Proof. Let $\mathfrak{m}$ be a maximal w-ideal of R. Then the induced map $f_{\mathfrak{m}} : M_{\mathfrak{m}} \to M_{\mathfrak{m}}$ is an epimorphism over $R_{\mathfrak{m}}$. By Vasconcelos-Strooker's theorem, $f_{\mathfrak{m}}$ is an isomorphism, that is, f is a w-isomorphism. By Lemma 2.6, f is an isomorphism. □

In [3], Orzech proved that if $f : N \to M$ is an epimorphism, where M is finitely generated and N is a submodule of M, then f is an isomorphism. This theorem is certainly a generalization of Vasconcelos' theorem. The following is a w-version of this theorem.

Theorem 2.8. *Let M be a finite type w-module and let N be a w-submodule of M. Suppose $f : N \to M$ is a w-epimorphism. Then f is an isomorphism.*

Proof. Similar to the proof of Theorem 2.7. □

Recall from [15] that a nonzero w-module M is said to be *w-simple* if M has no nontrivial w-submodules. It was shown in [15, Example 3.7] that simple modules and w-simple modules are two mutually exclusive concepts.

In [1], Bourbaki pointed out that any injective endomorphism of an Artinian module is always an isomorphism. Now we can give a w-version of this theorem by defining w-Artinian modules.

Definition 2.9. Let M be a w-module. If M has the DCC on w-submodules, then we say that M is a *w-Artinian module*.

It is natural that a w-simple module is certainly w-Artinian. Therefore, a w-Artinian module is not necessarily an Artinian module. Now we provide an explicit example of a module which is w-Artinian but not Artinian.

Example 2.10. Let K be a field and $R = K[X,Y]$. Then $M = (R/(X))_w$ is a w-simple, and therefore, is w-Artinian. Write $y = \overline{Y}$. Then $Ry \supset Ry^2 \supset \cdots \supset Ry^n \supset \cdots$ is a descending chain of submodules of M but not stationary. Therefore, M is not Artinian.

Proposition 2.11. *The following statements are equivalent for a w-module M.*

(1) *M is a w-Artinian module.*

(2) *Any nonempty subset of w-submodules of M has a minimum element.*

Proof. This is similar to the case of Artinian modules. □

Theorem 2.12. *Let A, B and C be w-modules and let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be w-exact. Then B is a w-Artinian module if and only if A and C are w-Artinian.*

Proof. Since A is GV-torsion-free and f is a w-monomorphism, f is a monomorphism. So we regard that A is a w-submodule of B. Suppose B is w-Artinian. Clearly A is w-Artinian. Let $C_1 \supseteq C_2 \supseteq \cdots \supseteq C_n \supseteq \cdots$ be a descending chain of w-submodules of C. Set $B_n = g^{-1}(C_n)$ for all n. It is routine to verify that B_n is a w-submodule of B and $B_n \supseteq B_{n+1}$. Therefore there is an integer m such that $B_n = B_m$ for all $n \geqslant m$. Note that $C = g(B)_w$ since g is a w-epimorphism. Hence $C_n = g(B_n)_w$. Consequently, $C_n = C_m$ for all $n \geqslant m$. It follows that C is w-Artinian.

Conversely, suppose A and C are w-Artinian. Let $B_1 \supseteq B_2 \supseteq \cdots \supseteq B_n \supseteq \cdots$ be a descending chain of w-submodules of B. Set $A_n = A \cap B_n$ and $C_n = g(B_n)_w$. Then $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$ and $C_1 \supseteq C_2 \supseteq \cdots \supseteq C_n \supseteq \cdots$ are descending chains of w-submodules of A and C, respectively. Thus there is an integer m such that $A_n = A_m$ and $C_n = C_m$ for all $n \geqslant m$. Let $b \in B_n$. Then $g(b) \in C_n = C_m$. Therefore there is a GV-ideal J of R such that $Jg(b) = g(Jb) \subseteq g(B_m)$. For $u \in J$, write $g(ub) = g(x)$, $x \in B_m$. Then $ub - x \in A_n = A_m$. Hence $Jb \subseteq B_m$. Since B_m is a w-module, we have $b \in B_m$. Thus we get that $B_n = B_m$ for all $n \geqslant m$. Consequently, B is w-Artinian. □

Corollary 2.13. *A direct sum $M_1 \oplus M_2 \oplus \cdots \oplus M_n$ is a w-Artinian module if and only if each M_i is a w-Artinian module.*

Proposition 2.14. *Let M be a w-Artinian module. Then $M_{\mathfrak{m}}$ is Artinian for each maximal w-ideal $\mathfrak{m}$ of R.*

Proof. Let $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$ be a descending chain of submodules of $M_{\mathfrak{m}}$. Let $\vartheta : M \to M_{\mathfrak{m}}$ be the natural map and set $B_n = \vartheta^{-1}(A_n)$. Then $(B_n)_{\mathfrak{m}} = A_n$ and $B_1 \supseteq B_2 \supseteq \cdots \supseteq B_n \supseteq \cdots$ is a descending chain of w-submodules of M. Thus there is an integer m such that $B_n = B_m$ for $n \geqslant m$. Therefore, $A_n = A_m$, whence $M_{\mathfrak{m}}$ is Artinian. □

Recall that a ring R is called a *DW ring* if every ideal of R is a w-ideal; equivalently, $\mathrm{GV}(R) = \{R\}$. By a slight modification of [8, Example 1.3(b)] we give a counterexample that the converse of Proposition 2.14 does not hold.

Example 2.15. Let E be a countable direct sum of copies of $\mathbb{Z}_2$ with addition and multiplication defined component-wise. Let $R = \mathbb{Z}_4 \times E$, and define addition and multiplication as follows:

$$(m,x)+(n,y)=(m+n,x+y)$$

and

$$(m,x)(n,y)=(mn,my+nx+xy),$$

where $m,n\in\mathbb{Z}_4$ and $x,y\in E$. Then R is a ring with identity $(1,0)$. For $\alpha=(2,0)\in R$, we have that $\operatorname{ann}(\alpha)=2\mathbb{Z}_4\times E$ is not finitely generated. Hence R is not a coherent ring. Therefore, R is not an Artinian ring. For any maximal ideal $\mathfrak{m}$ of R, it follows easily that $R_{\mathfrak{m}}=\mathbb{Z}_2$ or $R_{\mathfrak{m}}=\mathbb{Z}_4$. Thus $\dim(R)=0$, and hence R is a DW ring. Therefore R is not a w-Artinian R-module, but for any maximal w-ideal $\mathfrak{m}$, $R_{\mathfrak{m}}$ is an Artinian module over $R_{\mathfrak{m}}$.

Now we give a w-theoretic version of the other Bourbaki's Theorem aforementioned.

Theorem 2.16. *Let M be a w-Artinian module and let $f: M\to M$ be a w-monomorphism. Then f is an isomorphism.*

Proof. Since M is GV-torsion-free, f is actually a monomorphism. Consequently, f^n is also a monomorphism for all n. Thus $\operatorname{Im}(f)\supseteq\operatorname{Im}(f^2)\supseteq\cdots$ is a descending chain of w-submodules of M. Hence there is an integer n such that $\operatorname{Im}(f^n)=\operatorname{Im}(f^{n+1})$. Therefore, for each $x\in M$, there is an element $y\in M$ such that $f^n(x)=f^{n+1}(y)$. It follows $x=f(y)$. Consequently, $\operatorname{Im}(f)=M$. So f is an isomorphism. □

The following is a w-theoretic version of Schur's Lemma.

Corollary 2.17. *Let M be a w-simple module. Then* $\operatorname{End}_R M$ *is a division ring.*

Proof. Let f be a nonzero endomorphism of M. Thus $\ker(f)\neq M$. By [14, Theorem 2.7], $\ker(f)$ is a w-submodule of M. Hence $\ker(f)=0$. So f is a monomorphism. By Theorem 2.16, f is an isomorphism. Hence $\operatorname{End}_R M$ is a division ring. □

In order to give a new characterization of Artinian rings, we need a couple of lemmas.

Lemma 2.18. *Suppose that R satisfies the DCC on w-ideals. Then we have*:

(1) *Non-zero-divisors of R are units.*
(2) *R has only finitely many maximal w-ideals.*

Proof. (1) Let $a\in R$ be a non-zero-divisor. Then $(a)\supseteq(a^2)\supseteq\cdots\supseteq(a^n)\supseteq$ is a descending chain of w-ideals of R. By hypothesis there is an integer n such that $(a^n)=(a^{n+1})$. It follows directly that a is a unit.

(2) If $\mathfrak{m}_1,\mathfrak{m}_2,\ldots,\mathfrak{m}_n,\ldots$ are maximal w-ideals of R, then

$$\mathfrak{m}_1\supseteq(\mathfrak{m}_1\mathfrak{m}_2)_w\supseteq\cdots\supseteq(\mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_n)_w\supseteq\cdots$$

is a descending chain of w-ideals of R. Hence there is an integer n such that $(\mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_n)_w = (\mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_n\mathfrak{m}_{n+1})_w$. Hence $\mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_n \subseteq \mathfrak{m}_{n+1}$. It follows that $\mathfrak{m}_{n+1} = \mathfrak{m}_i$ for some i, $1 \leqslant i \leqslant n$. Hence R has only finitely many maximal w-ideals. □

Lemma 2.19 ([12, Corollary 3.22]). *Let R be a w-Noetherian ring. If I is an ideal of R with* $\operatorname{ann}(I) = 0$*, then I contains a non-zero-divisor of R. In particular, if $J \in \mathrm{GV}(R)$, then J contains a non-zero-divisor of R.*

Theorem 2.20. *A ring R is Artinian if and only if R satisfies the DCC on w-ideals.*

Proof. It is enough to show "if" part. To show that R is Artinian, we must prove that R is a DW ring. Let A be a w-ideal of R. From Lemma 2.18(2) we may assume that $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ are all maximal w-ideals of R. By Proposition 2.14, $R_{\mathfrak{m}_i}$ is Artinian, and hence $A_{\mathfrak{m}_i}$ is finitely generated. Take $\{a_{ij}\} \subseteq A$, for $j = 1, \ldots, m$, such that $\{\frac{a_{ij}}{1}\}$ is a generating set of $A_{\mathfrak{m}_i}$ over $R_{\mathfrak{m}_i}$, $i = 1, \ldots, n$. It is routine to verify that $A = (\{a_{ij}\})_w$. Therefore, A is of finite type, whence R is w-Noetherian. Let $J \in \mathrm{GV}(R)$. By Lemma 2.19, J has a non-zero-divisor. By Lemma 2.18(1), $J = R$. Hence R is a DW ring. □

From Theorem 2.20, it is no use to define w-Artinian rings to satisfy the DCC on w-ideals.

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