Bull. Korean Math. Soc. **52** (2015), No. 2, pp. 541–548
http://dx.doi.org/10.4134/BKMS.2015.52.2.541

A NOTE ON w-NOETHERIAN RINGS

Shiqi Xing and Fanggui Wang

Abstract. Let R be a commutative ring. An R-module M is called a w-Noetherian module if every submodule of M is of w-finite type. R is called a w-Noetherian ring if R as an R-module is a w-Noetherian module. In this paper, we present an exact version of the Eakin-Nagata Theorem on w-Noetherian rings. To do this, we prove the Formanek Theorem for w-Noetherian rings. Further, we point out by an example that the condition (†) in the Chung-Ha-Kim version of the Eakin-Nagata Theorem on SM domains is essential.

1. Introduction

Throughout the paper, all the rings are commutative rings with $1 \neq 0$.

Let $R \subseteq T$ be an extension of rings. If T is a finitely generated R-module, it is well-known that if R is Noetherian, then so is T. In 1968, P. M. Eakin ([4]) and M. Nagata ([7]) independently proved the converse: If T is Noetherian, then R is also Noetherian. This theorem is usually called the Eakin-Nagata Theorem in commutative algebra. After the notion of strong Mori domains (SM domains for short) was introduced by F. G. Wang and R. L. McCasland (see [11] and [12]), many classical theorems on Noetherian rings have been generalized to SM domains, for example, Hilbert Basis Theorem ([12, Theorem 1.13]), Principal Ideal Theorem ([12, Corollary 1.11]), Krull-Akizuki Theorem ([12, Theorem 3.4]), Matijevic Theorem ([9, Theorem 1.5]), Mori-Nagata Theorem ([2, Theorem 3.1]), Matlis Theorem and Cartan-Eilenberg-Bass Theorem on injective modules ([6, Proposition 2.6] & [6, Theorem 2.9]). It is natural to ask how to present the Eakin-Nagata Theorem on SM domains. Let $R \subseteq T$ be an extension of domains and let T as R-module be a w-finite type w-module. Recently, Chung, Ha and Kim proved in [3] that when $R \subseteq T$ satisfies the condition (†) (i.e., if N is a co-semi-divisorial R-module, then $\mathrm{Hom}_R(T, N)$ is a co-semi-divisorial T-module), R is also an SM domain if T is an SM domain. Naturally, we ask whether the statement on the Eakin-Nagata Theorem for

Received March 6, 2014; Revised May 8, 2014.
2010 *Mathematics Subject Classification.* 13B02, 13A15, 13E99.
Key words and phrases. w-moudle, w-finite type, w-Noetherian module, w-Noetherian ring.

SM domains by Chung-Ha-Kim is exact and whether the condition (†) can be deleted.

It is worthy noting that the notions of w-modules and SM domains have been generalized to an arbitrary commutative ring, see [13, 15]. So it is also natural to ask what is the exact statement on the Eakin-Nagata Theorem for w-Noetherian rings. Let M as R-module be finitely generated and faithful. In his paper [5] Formanek proved that if M satisfies the ascending chain condition for submodules of M of the form of IM, where I is an ideal of R, then R is a Noetherian ring. In this paper, we first prove the Formanek Theorem for w-Noetherian rings. As a corollary, we obtain the exact form of the Eakin-Nagata Theorem for w-Noetherian rings.

To see the essence of the condition (†) in the Chung-Ha-Kim version on the Eakin-Nagata Theorem, we give some equivalent characterizations on it, for commutative rings. We also post an example for which if the condition (†) is deleted, that T is an SM domains does not imply that R is an SM domain.

Now, we recall some material of w-modules. Following [15], an ideal J of R is called a GV-ideal, denoted by $J \in \mathrm{GV}(R)$, if J is finitely generated and the natural homomorphism $\phi : J \to \mathrm{Hom}_R(J, R)$ is an isomorphism. An R-module M is called GV-*torsion-free* if $Jx = 0$ with $J \in \mathrm{GV}(R)$ and $x \in M$ implies $x = 0$. M is called GV-*torsion* if there exists $J \in \mathrm{GV}(R)$ such that $Jx = 0$ for any $x \in M$. GV-torsion-free and GV-torsion mean co-semi-divisrial and w-null respectively in [3]. For a GV-torsion-free module M, set

$$M_w = \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in \mathrm{GV}(R)\},$$

which is called the w-*envelope* of M, where $E(M)$ is the injective hull of M. If $M = M_w$, then M is called a w-*module* (over R). In particular, if A is an ideal of R with $A = A_w$, then A is called a w-ideal of R. Let $R \subseteq T$ be an extension of rings. As in [14], T is called w-linked over R if T as R-module is a w-module. When $R \subseteq T$ is an extension of integral domains, the w-linked extension is said to be t-linked in a lot of literature.

Let $f : A \to B$ be an R-homomorphism. Then, as in [10], f is called a w-*epimorphism* (resp., w-*monomorphism* and w-*isomorphism*) if $f_P : A_P \to B_P$ is an epimorphism (resp., a monomorphism and an isomorphism) for any maximal w-ideal P of R. A sequence of modules and homomorphisms $A \to B \to C$ is called w-*exact sequence* if the sequence $A_P \xrightarrow{f_P} B_P \xrightarrow{g_P} C_P$ is exact for any maximal w-ideal P of R. An R-module M is said to be of w-*finite type* if there exists a w-exact sequence $F \to M \to 0$, where F is finitely generated free; equivalently, there is a finitely generated submodule N of M such that M/N is GV-torsion. And M is called a w-*Notherian module* if every submodule of M is of w-finite type. In particular, if R as R-module is a w-Noetherian module, then R is called a w-*Notherian ring.* If M is a w-module, then M is w-Noetherian if and only if M satisfies the ascending chain condition for w-submodules of M. Certainly, if R is a domain, then a torsion-free w-modules M is w-Notherian module if and only if M is an SM module; R is w-Noetherian if and only if R is an SM domain. For unexplained terminologies and notations, we refer to [10], [14] and [15].

2. The main results

We start by the following observation for w-Noetherian rings.

Theorem 2.1. *Let M be a* GV*-torsion-free w-Noetherian module of w-finite type. Set $I = \text{ann}(M)$. Then R/I as an R-module is w-Notherian. In particular, if M is faithful, then R is a w-Notherian ring.*

Proof. Since M is of w-finite type, there is a finitely generated submodule $N = Rx_1 + \cdots + Rx_2$ such that M/N is GV-torsion. Define $f : R \to M^n$ by $f(r) = (rx_1, \ldots, rx_n)$ for $r \in R$. Then $\ker(f) = \text{ann}(N)$.

Now we prove $\text{ann}(M) = \text{ann}(N)$. To do this, we show $(\text{ann}(M))_P = \text{ann}(M_P)$ for all maximal w-ideal P of R. In fact, let $r \in \text{ann}(M)$. Then $rM = 0$, whence $\frac{r}{1}M_P = 0$. Thus we have $(\text{ann}(M))_P \subseteq \text{ann}(M_P)$. On the other hand, if $r \in R$ and $s \notin P$ with $\frac{r}{s}M_P = 0$, then we have $\frac{r}{s}N_P = 0$. Since N is finitely generated, we have $s_1rN = 0$ for some $s_1 \notin P$. For any $x \in M$, take a GV-ideal J with $Jx \subseteq N$. Then $Js_1rx = 0$. Because M is GV-torsion-free, we have $s_1rx = 0$. Hence $s_1r \in \text{ann}(M)$, and therefore $\text{ann}(M_P) \subseteq (\text{ann}(M))_P$. Thus we get $(\text{ann}(M))_P = \text{ann}(M_P)$.

Since M is GV-torsion-free and $M_w = N_w$, $M_P = N_P$ by [15, Corollary 3.10]. Hence $(\text{ann}(M))_P = \text{ann}(M_P) = \text{ann}(N_P) = (\text{ann}(N))_P$. Noting that $\text{ann}(M)$ and $\text{ann}(N)$ are w-ideals, we have $\text{ann}(M) = \text{ann}(N)$ by [15, Corollary 3.10]. Hence the induced map $\overline{f} : R/I \to M^n$ is a monomorphism. So R/I is a w-Noetherian R-module by [15, Proposition 4.5]. □

Before we prove the Formanek Theorem for w-Noetherian rings, we need the following lemma.

Lemma 2.2. *Let M be a* GV*-torsion-free R-module. Then M is w-Noetherian module if and only if M_w is a w-Noetherian module.*

Proof. Note that the inclusion map $M \to M_w$ is a w-isomorphism. Apply [10, Proposition 3.5]. □

Now, we can prove the Formanek theorem for w-Noetherian rings.

Theorem 2.3. *Let M be a faithful w-module of w-finite type. Then M has ACC of submodules of M of the form $(IM)_w$ if and only if R is a w-Noetherian ring, where I is an ideal of R.*

Proof. Suppose R is a w-Noetherian ring. Certainly M is a w-Noetherian module by [10, Theorem 3.6] since M is of w-finite type. Hence M has ACC on submodules of M of the form $(IM)_w$.

For the converse, by Theorem 2.1, it is sufficient to show that M is w-Noetherian. If not, set

$$\Omega = \{(IM)_w \mid I \text{ is an ideal of } R \text{ and } M/(IM)_w \text{ is not } w\text{-Noetherian}\}.$$

By hypothesis Ω has a maximal element $(BM)_w$. Set

$$S = \{A \mid A \text{ is an ideal of } R \text{ with } (AM)_w = (BM)_w\}.$$

Let $\{A_i\}$ be a chain in S. Then $A = \bigcup_i A_i$ is an ideal of R. It is clear that $(BM)_w = (A_iM)_w \subseteq (AM)_w$. On the other hand, if $y \in (AM)_w$, then there exists $J \in \mathrm{GV}(R)$ such that $Jy \subseteq AM$. Write $J = (b_1, \ldots, b_n)$. Then, for each i, there exists some A_{k_i} such that $b_iy \subseteq A_{k_i}M$. Hence, there exists some A_k such that $Jy \subseteq A_kM$. Therefore $y \in (A_kM)_w = (BM)_w$. Thus $(AM)_w = (BM)_w$ and hence A is an upper bound of the chain $\{A_i\}$. It follows that S has a maximal element in Ω, say C.

If $I \supsetneq C$, then $(IM)_w \neq (CM)_w = (BM)_w$. So $(IM)_w \supsetneq (BM)_w$, which implies that $M/(IM)_w$ is a w-Noetherian module. Note that $M/(IM)_w$ is w-Notherian if and only if $(M/(IM)_w)_w$ is w-Noetherian by Lemma 2.2. By replacing $(M/(CM)_w)_w$ by M, we can assume that M is not w-Noetherian but $M/(IM)_w$ is w-Noetherian for any nonzero ideal I of R.

Set $S' = \{N \mid N \text{ is a } w\text{-submodule of } M \text{ and } M/N \text{ is faithful}\}$. Since M is faithful, $0 \in S'$, and hence S' is not empty. Let $\{N_i\}$ be a chain in S' and put $N = \bigcup N_i$. Then N is a w-submodule of M by [15, Proposition 2.6]. We conclude that M/N is faithful. In fact, since M is of w-finite type, there exists a finitely generated submodule F such that $F_w = M$. Write $F = Rx_1 + \cdots + Rx_n$. If $\mathrm{ann}(M/N) \neq 0$, take $0 \neq a \in \mathrm{ann}(M/N)$. Then $ax_i \in N$. Hence there exists some N_k such that $ax_i \in N_k$ for each i. Thus $aF \subseteq N_k$, and hence $aM = aF_w \subseteq (aF)_w \subseteq (N_k)_w = N_k$. Consequently, $a \in \mathrm{ann}(M/N_k) = 0$, a contradiction. Hence M/N is faithful and N is the upper bound of $\{N_i\}$. By Zorn's Lemma, S' has a maximal element, say E. Since M is of w-finite type, it follows that M/E is also a w-finite type GV-torsion-free module by [10, Proposition 1.3] and [15, Theorem 2.7]. Now we prove that M/E is a w-Noetherian module. In this case we obtain that R is a w-Noetherian ring by Theorem 2.1.

Assume by contradiction that M/E is not w-Noetherian. Then there exists a non-finite type w-submodule N of M by [15, Proposition 4.2]. By replacing $(M/E)_w$ by M we can assume that (a) M is not a w-Noetherian module; (b) $M/(IM)_w$ is a w-Noetherian module for any non-zero ideal I of R; (c) M/N is not faithful for each non-zero w-submodule N of M.

Since M is not w-Noetherian, there is a non-finite type w-submodule N of M. By (c), take $0 \neq a \in R$ with $aM \subseteq N$. We conclude that aM is of w-finite type. (Note that we do not have $(aM)_w = aM_w$ for commutative rings in general.) In fact, if $x \in M$, then $Jx \subseteq F$ for some $J \in \mathrm{GV}(R)$. Hence $Jax \subseteq aF$. So $ax \subseteq (aF)_w$. Thus $aM \subseteq (aF)_w$, whence $(aM)_w = (aF)_w$. It follows that $(aM)_w$ is of w-finite type. Also, since $M/(aM)_w$ is a w-Noetherian module by (b), it follows that $N/(aM)_w$ is of w-finite type. Hence we see from the exact sequence $0 \to (aM)_w \to N \to N/(aM)_w \to 0$ that N is of w-finite type by [10, Proposition 1.3], a contradiction. □

Let $R \subseteq T$ be a w-linked extension of rings. For any ideal A of T, denote by A_W the w-envelope of A as a T-module, which is different from the w-envelope A_w of A as an R-module. If $A_w = A$, then we say that A is a w_R-ideal of T. T is said to be a w_R-Noetherian ring if T has the ascending chain condition of w_R-ideals of T. When $R \subseteq T$ is a w-linked extension of integral domains, then w_R is a finite character star-operation on T. Now we record the following easy facts and omit their proofs.

Lemma 2.4. *Let $R \subseteq T$ be a w-linked extension of rings. Then the following statements hold.*

(i) *For any ideal A of T, $A_w \subseteq A_W$.*
(ii) *If A is a w-ideal of T, then A is a w_R-ideal of T.*
(ii) *If A is a w_R-ideal of T, then $A \bigcap R$ is a w-ideal of R.*
(iv) *For any proper w_R-ideal A of T, there is a maximal w_R-ideal P with $A \subseteq P$. Therefore, T has certainly a maximal w_R-ideal.*
(v) *If T is a w_R-Noetherian ring, then T is w-Noetherian.*

Now, we can present the exact version of the Eakin-Nagata Theorem for w-Noetherian rings by making use of the Formanek Theorem for w-Noetherian rings.

Theorem 2.5. *Let $R \subseteq T$ be a w-linked extension of rings in which T as an R-module is of w-finite type. Then R is a w-Noetherian ring if and only if T is a w_R-Noetherian ring.*

Proof. Suppose R is w-Noetherian. Since T is a w-finite type w-module over R, T is a w-Noetherian R-module by [13, Lemma 3.5]. Hence T is a w_R-Notherian ring. Conversely, suppose T is a w_R-Noetherian ring and I is an ideal of R. Then $(IT)_w$ is a w_R-ideal of T. Note that T is certainly a faithful R-module. Since T has ACC for w_R-ideals of T of the form $(IT)_w$, it follows that R is w-Noetherian by Theorem 2.3. □

Corollary 2.6. *Let $R \subseteq T$ be a w-linked extension of rings in which T as an R-module is of w-finite type. If R is a w-Noetherian ring, then T is a w-Noetherian ring.*

Recall from [3] the Chung-Ha-Kim version of Eakin-Nagata Theorem for SM domains:

Let $R \subseteq T$ be an w-linked extension of integral domains in which T is a w-finite type R-module. Assume that $R \subseteq T$ satisfies (†). If T is an SM domain, then R is also an SM domain.

To explain the condition (†) in [3], the other condition is posted as follows:

(♯): For each prime ideal Q of T with $Q \bigcap R \neq 0$, $(Q \bigcap R)_t \subsetneq D$ implies $Q_t \subsetneq T$.

It was proved that the conditions (†) and (♯) are equivalent in [3] when $R \subseteq T$ is a w-linked extension of integral domains and T is of w-finite type over R. To clarify what these conditions mean for extensions of commutative rings, we give the following theorem.

Theorem 2.7. *Let $R \subseteq T$ be a w-linked extension of rings. Then the following statements is equivalent*:

(i) *Each maximal w_R-ideal of T is a maximal w-ideal of T.*

(ii) *If $J \in \mathrm{GV}(T)$, then there exists $J' \in \mathrm{GV}(R)$ such that $J' \subseteq J$.*

(iii) *Each w_R-ideal of T is a w-ideal of T.*

If R and T are integral domains, the conditions above are equivalent to the following condition:

(iv) *For each prime ideal Q of T, $(Q \bigcap R)_t \subsetneq R$ implies $Q_t \subsetneq T$, that is, the condition ($\sharp$) holds.*

Also, if T is of w-finite type over R, the conditions above are equivalent to the following condition:

(v) *If N is a* GV*-torsion-free R-module, then $\mathrm{Hom}_R(T, N)$ is a* GV*-torsion-free T-module, that is, the condition (†) holds.*

Proof. (i)⇒(ii). Let $J \in \mathrm{GV}(T)$. If $J_w \neq T$, then there exists a maximal w_R-ideal Q of T such that $J_w \subseteq Q$ by Lemma 2.4. Also, it means that Q is a maximal w-ideal of T by hypothesis. Since $J \subseteq J_w \subseteq Q$ and $J_W = T$, it means that $Q = T$, a contradiction. Hence, $J_w = T$, and hence there is $J' \in \mathrm{GV}(R)$ such that $J' = J' \cdot 1 \subseteq J$.

(ii)⇒(iii). Suppose A is a w_R-ideal of T. If $Jx \subseteq A$ with $J \in \mathrm{GV}(T)$ and $x \in T$, take $J' \in \mathrm{GV}(R)$ such that $J' \subseteq J$. Hence $J'x \subseteq A$. Since A is a w_R-ideal of T, it implies that $x \in A_w = A$. Hence A is a w-ideal of T.

(iii)⇒(i). It is trivial.

Now we assume that $R \subseteq T$ is a w-linked extension of integral domains.

(i)⇒(iv). Let Q be a prime ideal of T such that $(Q \bigcap R)_t \neq R$. Then there exists a maximal t-ideal P of R such that $Q \bigcap R \subseteq (Q \bigcap R)_t \subseteq P \neq R$. Note that P is also a maximal w-ideal of R by [1, Cororally 2.17] and $Q \bigcap R$ is a w-ideal of R by [12, Proposition 1.1]. Hence Q is a w_R-ideal of T by [14, Theorem 3.7]. It follows that there exists a maximal w_R-ideal M of T such that $Q \subseteq M$. Since M is a maximal w-ideal of T by hypothesis, M is also a maximal t-ideal of T. Hence $Q_t \subseteq M_t = M \neq T$.

(iv)⇒(i). Let Q be a maximal w_R-ideal of T. Then $Q \bigcap R$ is a w-ideal of R by Lemma 2.4. Hence there exists a maximal w-ideal P of R such that $Q \bigcap R \subseteq P$. Noting that P is a maximal t-ideal of R by [1, Corollary 2.17], we have $(Q \bigcap R)_t \subseteq P \neq R$. It means that $Q_t \neq T$ by hypothesis. Hence there exists a maximal t-ideal Q' of T such that $Q \subseteq Q_t \subseteq Q'$. Noting that Q' is a w-ideal of T and hence a w_R-ideal of T, we have $Q = Q'$ by the maximality of Q. It follows that Q is a maximal w-ideal of T.

(vi)⇔(v). See [3, Proposition 2.7]. □

Corollary 2.8. *Let $R \subseteq T$ be a w-linked extension of rings in which T is of w-finite type over R. Assume that T satisfies one of the former three equivalent* *conditions in Theorem 2.7. Then R is a w-Noetherian ring if and only if T is a w-Noetherian ring.*

Corollary 2.9. *Let $R \subseteq T$ be a w-linked extension of integral domains in which T is of w-finite type over R. Assume that T satisfies one of the equivalent conditions in Theorem 2.7. Then R is an SM domain if and only if T is an SM domain.*

Let $R \subseteq T$ be a w-linked extension of domains in which T as an R-module is of w-finite type. Now we will give an example to show that the Chung-Ha-Kim version of Eakin-Nagata Theorem does not hold if the condition (†) is deleted. That is when T is an SM domain, R does not have to be an SM domain.

Let

$$\begin{array}{ccc} R & \longrightarrow & T \\ \downarrow & & \downarrow \\ D & \longrightarrow & F \end{array}$$

be a commutative diagram of rings and homomorphisms. If R, D and T are domains, F is a field, and R is a proper subring of T, then the commutative diagram is called a Milnor square. Thus there is a maximal ideal M of T such that $T/M \cong F$. In particular, if $R = D + M$, then this Milnor square is called a $D+M$ construction. For a Milnor square $RDTF$, we usually write $F = T/M$ and regard that M is a common ideal of T and R with $D = R/M$.

Lemma 2.10. *Let $RDTF$ be a Milnor square. If D and F are fields and $[F : D] < \infty$, then T as an R-module is finitely generated.*

Proof. Let $\pi : T \to T/M = F$ be the natural map. Since $n := [F : D] < \infty$, there are elements $x_1 = 1, x_2, \dots, x_n \in T$ such that $\pi(x_1), \pi(x_2), \dots, \pi(x_n)$ is a basis of F over D. Set $A = Rx_1 + \cdots + Rx_n$. Then A is a fractional ideal of R and $A \subseteq T$. Since $1 \in A$, it means that $AT = T$. So $M = MT = MAT = MA \subseteq A$.

Let $x \in T$. Then $\pi(x) = \pi(r_1)\pi(x_1) + \cdots + \pi(r_n)\pi(x_n)$ for $r_i \in R$, $i = 1, 2, \dots, n$. Hence $x - (r_1x_1 + \cdots + r_nx_n) \in M \subseteq A$. Thus we have $x \in A$. Therefore $A = T$, that is, T as an R-module is finitely generated. □

Example 2.11. Let $D \subseteq F$ be an extension of fields with $[F : D] < \infty$. Let $T = F[X_1, \dots, X_n, \dots]$ be the polynomial ring over F with countably infinitely many indeterminates. Construct the Milnor square as follows:

$$\begin{array}{ccc} R & \longrightarrow & T \\ \downarrow & & \downarrow \\ D & \longrightarrow & F \end{array}$$

Then T is w-linked over R and T is finitely generated over R by Lemma 2.10. By [8, Theorem 4.7] T is an SM domain. Let $M = \sum_{i=1}^{\infty} TX_i$. Then M is a maximal ideal of T and T_M is not Noetherian. By [8, Proposition 3.7], R is not an SM domain. Note that M is a maximal w_R-ideal of T, but not a maximal w-ideal of T.

Acknowledgements. The authors would like to thank the referee for a careful reading and relevant comments. This work is supported by National Natural Science Foundation of China(Grant No. 11171240).

References

[1] D. D. Anderson and S. J. Cook, *Two star operations and their induced lattices*, Comm. Algebra **29** (2000), no. 5, 2461–2475.

[2] G. W. Chang and M. Zafrullah, *The w-integral closure of integral domains*, J. Algebra **295** (2006), no. 1, 195–210.

[3] W. Chung, J. Ha, and H. Kim, *Some remarks on strong Mori domains*, Houston J. Math. **38** (2012), no. 4, 1051–1059.

[4] P. M. Eakin, *The converse to a well known theorem on Noetherian rings*, Math. Ann. **177** (1968), 278–282.

[5] E. Formanek, *Die stätze von Bertini für lokale Ringe*, Math. Ann. **229** (1977), 97–111.

[6] H. Kim, E. S. Kim, and Y. S. Park, *Injective modules over strong Mori domains*, Houston J. Math. **34** (2008), no. 2, 349–360.

[7] M. Nagata, *A type of subrings of a Noetherian ring*, J. Math. Kyoto Univ. **8** (1968), 465–467.

[8] M. H. Park, *Groups rings and semigroup rings over strong Mori domains*, J. Pure Appl. Algebra **163** (2001), no. 3, 301–318.

[9] ———, *On overrings of Strong Mori domains*, J. Pure Appl. Algebra **172** (2002), no. 1, 79–85.

[10] F. G. Wang, *Finitely presented type modules and w-coherent rings*, J. Sichuan Normal Univ. **33** (2010), 1–9.

[11] F. G. Wang, and R. L. McCasland, *On w-modules over strong Mori domains*, Comm. Algebra **25** (1997), no. 4, 1285–1306.

[12] ———, *On strong Mori domains*, J. Pure Appl. Algebra **135** (1999), no. 2, 155–165.

[13] F. G. Wang, J. Zhang, *Injective modules over w-Noetherian rings*, Acta math. Sinica (Chin. Ser.) **53** (2010), no. 6, 1119–1130.

[14] L. Xi, F. G. Wang, and Y. Tian, *On w-linked overrings*, J. Math. Res. Exposition **31** (2011), 337–346.

[15] H. Y. Yin, F. G. Wang, X. S. Zhu, and Y. H. Chen, *w-modules over commutative rings*, J. Korean Math. Soc. **48** (2011), no. 1, 146–151.

Shiqi Xing
College of Mathematics and Software Science
Sichuan Normal University
Chengdu 610068, P. R. China
E-mail address: sqxing@yeah.net

Fanggui Wang
College of Mathematics and Software Science
Sichuan Normal University
Chengdu 610068, P. R. China
E-mail address: wangfg2004@163.com