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UPPERS TO ZERO IN POLYNOMIAL RINGS WHICH ARE MAXIMAL IDEALS

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Abstract. Let D be an integrally closed domain with quotient field K, X be an indeterminate over D, $f = a_0 + a_1X + \cdots + a_nX^n \in D[X]$ be irreducible in $K[X]$, and $Q_f = fK[X] \cap D[X]$. In this paper, we show that Q_f is a maximal ideal of $D[X]$ if and only if $(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0}) \subseteq P$ for all nonzero prime ideals P of D; in this case, $Q_f = \frac{1}{a_0}fD[X]$. As a corollary, we have that if D is a Krull domain, then D has infinitely many height-one prime ideals if and only if each maximal ideal of $D[X]$ has height ≥ 2.

1. Introduction

Let D be an integral domain with quotient field K, $\bar{D}$ the integral closure of D in K, X an indeterminate over D, and $D[X]$ the polynomial ring over D. We say that a nonzero prime ideal Q of $D[X]$ is an *upper to zero* in $D[X]$ if $Q \cap D = (0)$; so each upper to zero in $D[X]$ has height-one. Clearly, if Q is an upper to zero in $D[X]$, then $Q = fK[X] \cap D[X]$ for some $f \in D[X]$ which is irreducible in $K[X]$.

Let $X_1, \ldots, X_n$ be indeterminates over D. It is known that the intersection of the nonzero prime ideals of D is zero if and only if $M \cap D \neq (0)$ for all maximal ideals M of $D[X_1, \ldots, X_n]$ [11, Theorem 14.10]. This result was first proved by Artin-Tate [2] in the Noetherian case and then by Nagata [11] in the general case. In [10], May used this result to give an elementary proof of the Nullstellensatz: if F is an algebraically closed field, then an ideal M of $F[X_1, \ldots, X_n]$ is maximal if and only if $M = (X_1 - a_1, \ldots, X_n - a_n)$ for $a_1, \ldots, a_n \in F$. Let A be the intersection of the nonzero prime ideals of D. It is clear by [11, Theorem 14.10] that $A \neq (0)$ if and only if there is a maximal ideal M of $D[X]$ with $M \cap D = (0)$. In this case, if we let $f = 1 + aX$ for $0 \neq a \in A$, then $Q_f := fK[X] \cap D[X]$ is a maximal ideal of $D[X]$ with $Q_f \cap D = (0)$ [9, Proof of Theorem 24] and $Q_f = fD[X]$. In this paper, we completely characterize uppers to zero in $D[X]$ that are maximal ideals.

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Let $f = a_0 + a_1X + \cdots + a_nX^n \in D[X]$ be such that f is irreducible in $K[X]$ and $Q_f = fK[X] \cap D[X]$. We show that $\sqrt{fD[X]}$ is a maximal ideal of $D[X]$ if and only if a_0 is a unit in D and $(a_1, \ldots, a_n) \subseteq P$ for all nonzero prime ideals P of D. Then we use this result to show that if D is integrally closed, then Q_f is a maximal ideal if and only if $(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0}) \subseteq P$ for all nonzero prime ideals P of D; in this case, $Q_f = \frac{1}{a_0}fD[X]$. Also, we give an example of a non-integrally closed integral domain D with $g \in D[X]$ such that $gK[X] \cap D[X]$ is a maximal ideal, but not a principal ideal. Finally, we show that if D is a Krull domain, then D has infinitely many height-one prime ideals if and only if each maximal ideal of $D[X]$ has height ≥ 2.

Let I be a nonzero fractional ideal of D. Then $I^{-1} = \{x \in K \mid xI \subseteq D\}$, $I_v = (I^{-1})^{-1}$, and $I_t = \cup\{J_v \mid J \subseteq I$ is a nonzero finitely generated ideal$\}$. We say that I is a *t-ideal* if $I_t = I$. A t-ideal of D is a *maximal t-ideal* if it is maximal among proper integral t-ideals. Let t-Max(D) denote the set of maximal t-ideals of D. It is well known that a maximal t-ideal is a prime ideal; each integral t-ideal is contained in a maximal t-ideal; $D = \cap_{P \in t\text{-Max}(D)} D_P$; and each prime ideal minimal over a t-ideal is a t-ideal (and hence each height-one prime ideal is a t-ideal and t-Max$(D) \neq \emptyset$ if D is not a field). A nonzero ideal I of D is said to be *t-invertible* if $(II^{-1})_t = D$; equivalently, $II^{-1} \nsubseteq P$ for all $P \in t$-Max(D).

2. Main results

Let D be an integral domain with quotient field K, X be an indeterminate over D, and $D[X]$ be the polynomial ring over D.

Lemma 1. *Let $f = a_0 + a_1X + \cdots + a_nX^n \in D[X]$ and I an ideal of D. Then $fD[X] + I[X] = D[X]$ if and only if $a_0D + I = D$ and $(a_1, \ldots, a_n)^m \subseteq I$ for some integer $m \geq 1$.*

Proof. Note that $fD[X] + I[X] = D[X]$ if and only if $D[X]/I[X] \cong (D/I)[X] = (\overline{f})$; $a_0D + I = D$ if and only if $\overline{a_0}$ is a unit in D/I; and $\overline{a_1}, \ldots, \overline{a_n}$ are nilpotent in D/I if and only if $a_i^{m_i} \in I$ for some integer $m_i \geq 1$, if and only if $(a_1, \ldots, a_n)^m \subseteq I$ for some integer $m \geq 1$. Thus the result follows directly from [3, Exercise 2, page 10] that $g = b_0 + b_1X + \cdots + b_mX^m \in (D/I)[X]$ is a unit if and only if b_0 is a unit in D/I and $b_1, \ldots, b_m$ are nilpotent in D/I. $\square$

Let $f = a_0 + a_1X + \cdots + a_nX^n \in D[X]$. We denote by $c_D(f)$ (or simply $c(f)$) the ideal of D generated by the coefficients of f, i.e., $c_D(f) = (a_0, a_1, \ldots, a_n)$. It is well known that if $g, h \in D[X]$, then there exists a positive integer m such that $c(g)^{m+1}c(h) = c(g)^m c(gh)$ [5, Theorem 28.1]. Using this result, we can easily show that $fK[X] \cap D[X] = fD[X]$ if and only if $c(f)^{-1} = D$ (see, for example, [1, Lemma 2.1(1)]). Also, it is well known that if D is integrally closed, then $gK[X] \cap D[X] = gc(g)^{-1}[X]$ for all $0 \neq g \in D[X]$ [5, Corollary 34.9].

Lemma 2. *Let $f = a_0 + a_1X + \cdots + a_nX^n \in D[X]$ be irreducible in $K[X]$. Then $\sqrt{fD[X]}$ is a maximal ideal of $D[X]$ if and only if a_0 is a unit in D and $(a_1, \ldots, a_n) \subseteq P$ for all nonzero prime ideals P of D. In this case, $\sqrt{fD[X]} = fD[X]$.*

Proof. ($\Rightarrow$) Suppose that $\sqrt{fD[X]}$ is a maximal ideal of $D[X]$, and let P be a nonzero prime ideal of D; then $P \not\subseteq \sqrt{fD[X]}$, and hence $fD[X]+P[X] = D[X]$. Thus by Lemma 1, $a_0 \notin P$ and $(a_1, \ldots, a_n) \subseteq P$. In particular, since P is arbitrary, a_0 must be a unit in D.

($\Leftarrow$) First, note that $fK[X] \cap D[X] = fD[X]$ since a_0 is a unit in D. Let Q be a prime ideal of $D[X]$ with $f \in Q$. If $Q \cap D \neq (0)$, then $(a_1, \ldots, a_n) \subseteq Q \cap D$ and $fD[X] + (Q \cap D)[X] \subseteq Q$. Note that $a_0D + Q \cap D = D$ because a_0 is a unit; so by Lemma 1, $D[X] = fD[X] + (Q \cap D)[X] \subseteq Q$, a contradiction. Hence $Q \cap D = (0)$, and thus $Q = fK[X] \cap D[X]$ and Q is the unique prime ideal of $D[X]$ that contains f. Therefore $\sqrt{fD[X]} = Q = fD[X]$ is a maximal ideal. □

We are now ready to prove the main result of this paper.

Theorem 3. *Let D be an integrally closed domain, $f = a_0+a_1X+\cdots+a_nX^n \in D[X]$ be such that f is irreducible in $K[X]$, and $Q_f = fK[X] \cap D[X]$. Then Q_f is a maximal ideal of $D[X]$ if and only if $(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0}) \subseteq P$ for all nonzero prime ideals P of D. In this case, $Q_f = \frac{1}{a_0}fD[X]$.*

Proof. ($\Rightarrow$) Since Q_f is maximal, Q_f is a maximal t-ideal, and hence Q_f is t-invertible [7, Theorem 1.4]. Note that $Q_f = fc_D(f)^{-1}[X]$ and $(ID[X])_t = I_tD[X]$ for all nonzero fractional ideals I of D [6, Proposition 3.4]; hence $c_D(f)^{-1}$ is t-invertible. Let P be a maximal t-ideal of D. Then $(Q_f)_{D\setminus P}$ is a maximal ideal of $D_P[X]$. Note that $c_D(f)c_D(f)^{-1} \not\subseteq P$; so $c_D(f)D_P$ is invertible. Hence $c_D(f)D_P = (c_D(f)D_P)_t = (c_D(f)_tD_P)_t \supseteq c_D(f)_tD_P \supseteq c_D(f)D_P$ [8, Lemma 3.4(3)] and $c_D(f)D_P = a_iD_P$ for some i [5, Proposition 7.4(2)], and thus $c_D(f)_tD_P = a_iD_P$. Note also that $(Q_f)_{D\setminus P} = fK[X] \cap D_P[X] = \frac{1}{a_i}fD_P[X]$. Hence by Lemma 2, $\frac{a_0}{a_i}$ is a unit in D_P; so $a_iD_P = a_0D_P$. Thus $c_D(f)_t = \cap_{P\in t\text{-Max}(D)}c_D(f)_tD_P = \cap_{P\in t\text{-Max}(D)}a_0D_P = a_0D$ [8, Proposition 2.8(3)]. This implies that $Q_f = fK[X] \cap D[X] = \frac{1}{a_0}fD[X]$. Again, by Lemma 2, $(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0}) \subseteq P$ for all nonzero prime ideals P of D.

($\Leftarrow$) Let $h = \frac{1}{a_0}f = 1 + \frac{a_1}{a_0}X + \cdots + \frac{a_n}{a_0}X^n$. Then by Lemma 2, $Q_f = hK[X] \cap D[X] = hD[X]$ is a maximal ideal of $D[X]$. □

We next give an example which shows that Theorem 3 does not hold for a non-integrally closed domain.

Example 4. Let D be a one-dimensional quasi-local domain with maximal ideal P such that $\bar{D}$ is quasi-local with maximal ideal Q and $P\bar{D} \subsetneq Q$. (For example, let F be a field, t an indeterminate over F, and $D = F[\![t^2, t^3]\!]$ be a subring of the power series ring $F[\![t]\!]$. Then (t^2, t^3) is a maximal ideal of

D, $\bar{D} = F[\![t]\!]$ is a local PID with maximal ideal $t\bar{D}$, and $(t^2, t^3)\bar{D} \subsetneq t\bar{D}$.) Choose $u = \frac{b}{a} \in Q \setminus P\bar{D}$, where $a, b \in D$, and let $f = a + bX \in D[X]$. Then $fK[X] \cap D[X]$ is a maximal ideal of $D[X]$, but not a principal ideal.

Proof. Obviously, $fK[X] \cap \bar{D}[X] = fc_{\bar{D}}(f)^{-1}[X] = \frac{1}{a}f\bar{D}[X]$. Hence $fK[X] \cap \bar{D}[X]$ is a maximal ideal by Theorem 3, and thus $fK[X] \cap D[X]$ is a maximal ideal of $D[X]$ because $\bar{D}[X]$ is integral over $D[X]$. Next, assume that $fK[X] \cap D[X]$ is principal. Then there exists an $h \in D[X]$ such that $fK[X] \cap D[X] = hD[X]$. Note that $hK[X] = fK[X]$; so $f = \alpha h$ for some $\alpha \in K$ and $c_D(h) = D$. Hence $(a, b) = c_D(f) = \alpha D$, and thus (a, b) is invertible. But, in this case, either $(a, b) = aD$ or $(a, b) = bD$ because D is quasi-local. If $(a, b) = aD$, then $\frac{b}{a} \in D \subseteq \bar{D}$, and hence $\frac{b}{a} \in Q \cap D = P \subseteq P\bar{D}$, a contradiction. Assume $(a, b) = bD$. Then $\frac{a}{b} \in D \subseteq \bar{D}$, and hence $1 = \frac{b}{a} \cdot \frac{a}{b} \in Q$, a contradiction. Thus $fK[X] \cap D[X]$ is not principal. □

Corollary 5. *Let $f = a_0 + a_1X + \cdots + a_nX^n \in D[X]$ be such that f is irreducible in $K[X]$. Then the following statements are equivalent.*

(1) *$Q_f := fK[X] \cap D[X]$ is a maximal ideal of $D[X]$.*
(2) *$M_f := fK[X] \cap \bar{D}[X]$ is a maximal ideal of $\bar{D}[X]$.*
(3) *$(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0}) \subseteq P$ for all nonzero prime ideals P of $\bar{D}$.*

In particular, if a_0 is a unit in D, then Q_f is maximal if and only if $(a_1, \ldots, a_n) \subseteq P$ for all nonzero prime ideals P of D.

Proof. (1) ⇔ (2) This follows from [9, Theorem 44] because $M_f \cap D[X] = Q_f$ and $\bar{D}[X]$ is integral over $D[X]$. (2) ⇔ (3) Theorem 3. The "in particular" part follows because $\bar{D}$ is integral over D. □

Corollary 6. *Let R be a subring of K containing D, $f \in D[X]$ be irreducible in $K[X]$, and $Q_f = fK[X] \cap D[X]$. If Q_f is a maximal ideal, then $fK[X] \cap R[X]$ is a maximal ideal of $R[X]$.*

Proof. Let $\bar{R}$ be the integral closure of R in K. Then $\bar{D} \subseteq \bar{R} \subseteq K$, and hence each $a \in \bar{D}$ that is contained in all nonzero prime ideals of $\bar{D}$ is contained in all nonzero prime ideals of $\bar{R}$. Thus, by Corollary 5, $fK[X] \cap R[X]$ is a maximal ideal of $R[X]$. □

An integral domain D is called a *G-domain* if $K = D[\frac{1}{c}]$ for some $0 \neq c \in D$. It is clear that D is a G-domain if and only if the intersection of the nonzero prime ideals of D is nonzero [4, Lemma 3].

Corollary 7. *Let A be the intersection of the nonzero prime ideals of D.*

(1) ([9, Proof of Theorem 24]) *If $f = 1 + aX$ for $0 \neq a \in A$, then $Q_f = fK[X] \cap D[X]$ is a maximal ideal of $D[X]$ and $Q_f = fD[X]$.*
(2) ([9, Theorem 24]) *D is a G-domain (i.e., $A \neq (0)$) if and only if there exists a maximal ideal M of $D[X]$ which satisfies $M \cap D = (0)$.*
(3) (cf. [11, Theorem 14.10]) *$A = (0)$ if and only if $M \cap D \neq (0)$ for all maximal ideals M of $D[X]$.*

Proof. Note that $A \subseteq P$ for all nonzero prime ideals P of $\bar{D}$; so if we let B be the intersection of the nonzero prime ideals of $\bar{D}$, then $A \neq (0)$ if and only if $B \neq (0)$.

(1) It is clear that f is irreducible in $K[X]$ and $Q_f = fD[X]$. Thus, the result is an immediate consequence of Corollary 5.

(2) If D is a G-domain, then $K = D[\frac{1}{c}]$ for some $0 \neq c \in D$. Clearly, $c \in A$. So if we set $f = 1 + cX$ and $Q_f = fK[X] \cap D[X]$, then Q_f is a maximal ideal of $D[X]$ by (1) and $Q_f \cap D = (0)$. For the converse, note that $B \neq (0)$ by Corollary 5. Thus $A \neq (0)$.

(3) This follows from (2). □

Let $X^1(D)$ be the set of height-one prime ideals of D. Clearly, if each nonzero prime ideal of D contains a height-one prime ideal, then $\cap_{P \in X^1(D)} P$ is equal to the intersection of the nonzero prime ideals of D. So if D is a Krull domain (resp., principal ideal domain (PID)), then the intersection of the nonzero prime ideals of D is zero if and only if $|X^1(D)| = \infty$. Also, it is well known and easy to prove that D is a Krull domain with $|X^1(D)| < \infty$ if and only if D is a semilocal PID [4, Theorem 1]. Thus by Corollary 7(3), we have:

Corollary 8. *A Krull domain D has infinitely many height-one prime ideals if and only if each maximal ideal of $D[X]$ has height ≥ 2.*

Corollary 9 ([12, Theorem 2])**.** *A PID D has infinitely many non-associate prime elements if and only if each maximal ideal of $D[X]$ has height* 2*.*

Proof. This follows because the (Krull) dimension of $D[X]$ over a PID D is 2. □

We end this paper with a concrete example of uppers to zero in $D[X]$ that are maximal ideals. This also shows that the converse of Corollary 6 does not hold.

Example 10. Let $\mathbb{Z}$ be the ring of integers, $\mathbb{Q}$ be the field of rational numbers, $f = a_0 + a_1X + \cdots + a_nX^n \in \mathbb{Z}[X]$ with $\gcd(a_0, a_1, \ldots, a_n) = 1$, $\Delta = \{P \in \mathrm{Spec}(\mathbb{Z}) \mid a_i \in P$ for $i = 1, \ldots, n$ and $a_0 \notin P\}$, $S = \mathbb{Z} \setminus \cup_{P \in \Delta} P$, and $D = \mathbb{Z}_S$. Note that $f\mathbb{Q}[X] \cap \mathbb{Z}[X] = f\mathbb{Z}[X]$ and $f\mathbb{Q}[X] \cap D[X] = fD[X]$. Hence if f is irreducible in $\mathbb{Q}[X]$, then $fD[X]$ is a maximal ideal by Lemma 2, but $f\mathbb{Z}[X]$ is not a maximal ideal of $\mathbb{Z}[X]$ by Corollary 9. (In fact, if R is a ring such that $\mathbb{Z} \subseteq R \subseteq \mathbb{Q}$ and $fR[X]$ is maximal, then $D \subseteq R$.) For example, let $f = 10 + 15X + 45X^3 + 3X^4$. Then (i) $5 \mid 10$, $5 \mid 15$, and $5 \mid 45$, (ii) $5 \nmid 3$ and $5^2 \nmid 10$, and hence f is irreducible in $\mathbb{Q}[X]$ by Eisenstein's Criterion. Thus if we set $D = \mathbb{Z}_{3\mathbb{Z}}$, then $fD[X]$ is a maximal ideal of $D[X]$, while $f\mathbb{Z}[X]$ is not a maximal ideal of $\mathbb{Z}[X]$.

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