Bull. Korean Math. Soc. **52** (2015), No. 2, pp. 513–524
http://dx.doi.org/10.4134/BKMS.2015.52.2.513

ON THE EQUATION $\phi(5^m - 1) = 5^n - 1$

Bernadette Faye, Florian Luca, and Amadou Tall

Abstract. Here, we show that the title equation has no positive integer solutions (m, n), where ϕ is the Euler function.

1. Introduction

Let $\phi(m)$ be the Euler function of the positive integer m. Problem 10626 from the *American Mathematical Monthly* [6] asks to find all positive integer solutions (m, n) of the Diophantine equation

$$\phi(5^m - 1) = 5^n - 1. \tag{1}$$

To our knowledge, no solution was ever received to this problem. Here, we prove the following result.

Theorem 1. *Equation* (1) *has no positive integer solution* (m, n)*.*

Results in this spirit appear in [3], [4], [6], [7], [8], [9], [10], [11].

2. The proof of Theorem 1

For the proof, we make explicit the arguments from [9] together with some specific features which we deduce from the factorizations of $5^k - 1$ for small values of k. Write

$$5^m - 1 = 2^{\alpha} p_1^{\alpha_1} \cdots p_r^{\alpha_r}. \tag{2}$$

Thus,

$$\phi(5^m - 1) = 2^{\alpha-1} p_1^{\alpha_1 - 1}(p_1 - 1) \cdots p_r^{\alpha_r - 1}(p_r - 1). \tag{3}$$

We achieve the proof of Theorem 1 as a sequence of lemmas. The first one is known but we give a proof of it for the convenience of the reader.

Lemma 2. *In equation* (1)*, m and n are not coprime.*

Received January 30, 2014; Revised May 20, 2014.
2010 *Mathematics Subject Classification.* 11D61.
Key words and phrases. Diophantine equations, applications of primitive divisors, applications of sieve methods.

©2015 Korean Mathematical Society

Proof. Suppose that $\gcd(m,n)=1$. Assume first that n is odd. Then $\mathrm{ord}_2(5^n-1)=2$, where for a prime p and a nonzero integer k we write $\mathrm{ord}_p(k)$ for the exact exponent of p in the factorization of k. Applying the ord_2 function in both sides of (3) and comparing it with (1), we get

$$(\alpha-1)+r \le (\alpha-1)+\sum_{i=1}^{r}\mathrm{ord}_2(p_i-1)=2.$$

If m is even, then $\alpha\ge 3$, and the above inequality shows that $\alpha=3,\ r=0$, so $5^m-1=8$, false. Thus, m is odd, so $\alpha=2$ and $r=1$. If $\alpha_1\ge 2$, then

$$p_1^{\alpha_1-1}\mid \gcd(\phi(5^m-1),5^m-1)=\gcd(5^m-1,5^n-1)=5^{\gcd(m,n)}-1=4,$$

a contradiction. So, $\alpha_1=1,\ 5^m-1=4p_1$, and

$$5^n-1=2(p_1-1)=\frac{5^m-1}{2}-2=\frac{5^m-5}{2},$$

which is impossible. Thus, n is even and since $\gcd(m,n)=1$, it follows that m is odd so $\alpha=2$. Furthermore, a previous argument shows that in (2) we have $\alpha_1=\cdots=\alpha_r=1$. Since m is odd, we have that $5\cdot(5^{(m-1)/2})^2\equiv 1 \pmod{p_i}$, therefore $\left(\frac{5}{p_i}\right)=1$ for $i=1,\ldots,r$. Here, $\left(\frac{a}{p}\right)$ is the Legendre symbol of a with respect to the odd prime p. Hence, $p_i\equiv 1,4 \pmod 5$. If $p_i\equiv 1 \pmod 5$, it follows that $5\mid \phi(5^m-1)=5^n-1$, a contradiction. Hence, $p_i\equiv 4 \pmod 5$ for $i=1,\ldots,r$. Reducing now relation (2) modulo 5, we get

$$4\equiv 4^{1+r} \pmod 5, \quad \text{therefore} \quad r\equiv 0 \pmod 2.$$

Reducing now equation

$$2(p_1-1)\cdots(p_r-1)=\phi(5^m-1)=5^n-1$$

modulo 5, we get

$$2\cdot(3^{r/2})^2\equiv 4 \pmod 5, \quad \text{therefore} \quad \left(\frac{2}{5}\right)=1,$$

a contradiction. □

Lemma 3. *If (m,n) satisfies equation (1), then m is not a multiple of any number d such that $p\mid 5^d-1$ for some prime $p\equiv 1 \pmod 5$.*

Proof. This is clear, for if not, then $5\mid (p-1)\mid \phi(5^d-1)\mid \phi(5^m-1)=5^n-1$, false. □

Since 29423041 is a prime dividing $5^{32}-1$, it follows by Lemma 3 that $\mathrm{ord}_2(m)\le 4$. From the Cunnigham project tables [1], we deduced that if $q\le 512$ is an odd prime, then 5^q-1 has a prime factor $p\equiv 1 \pmod 5$ except for $q\in\{17,41,71,103,223,257\}$. So, if $q\mid m$ is odd, then

(4) $$q\in\mathcal{Q}:=\{17,41,71,103,223,257\}\cup\{q>512\}.$$

Throughout the rest of the paper, we put

$$k = m - n. \tag{5}$$

Note that $k \geq 2$ because m and n are not coprime by Lemma 2. The next lemma gives an upper bound on k.

Lemma 4. *The following inequality holds*:

$$\log\left(\log\left(\frac{5^k e}{3.6}\right)\right) < 20 \sum_{\substack{q|m \\ q>2}} \frac{\log\log q}{q}. \tag{6}$$

Proof. Write

$$m = 2^{\alpha_0} \prod_{i=1}^{s} q_i^{\alpha_i}, \quad q_i \text{ odd prime, } i = 1, \ldots, s.$$

Recall that $\alpha_0 \leq 4$. None of the values $m = 1, 2, 4, 8, 16$ satisfies equation (1) for some n, so $s \geq 1$. Then

$$5^k < \frac{5^m - 1}{5^n - 1} = \frac{5^m - 1}{\phi(5^m - 1)} = \prod_{p|5^m-1} \left(1 + \frac{1}{p-1}\right). \tag{7}$$

For each prime number $p \neq 5$, we write ℓ_p for the order of appearance of p in the Lucas sequence of general term $5^n - 1$. That is, ℓ_p is the order of 5 modulo p. Clearly, if $p \mid 5^m - 1$, then $\ell_p = d$ for some divisor d of m. Thus, we can rewrite inequality (7) as

$$5^k < \prod_{d|m} \prod_{\ell_p=d} \left(1 + \frac{1}{p-1}\right). \tag{8}$$

If $p \mid m$ and ℓ_p is a power of 2, then $\ell_p \mid 16$, therefore $p \mid 5^{16} - 1$. Hence,

$$p \in \mathcal{P} = \{2, 3, 13, 17, 313, 11489\}.$$

Thus,

$$\prod_{\ell_p|16} \left(1 + \frac{1}{p-1}\right) \leq \prod_{p\in P} \left(1 + \frac{1}{p-1}\right) < 3.5. \tag{9}$$

Inserting (9) into (8), we get

$$\frac{5^k}{3.5} < \prod_{\substack{d|m \\ P(d)>2}} \prod_{\ell_p=d} \left(1 + \frac{1}{p-1}\right), \tag{10}$$

where for a nonzero integer ℓ we write $P(\ell)$ for the largest prime factor of ℓ with the convention that $P(\pm 1) = 1$. We take logarithms in inequality (10) above and use the inequality $\log(1+x) < x$ valid for all real numbers x to get

$$\log\left(\frac{5^k}{3.5}\right) < \sum_{\substack{d\mid m\\ P(d)>2}} \sum_{\ell_p=d} \frac{1}{p-1}.$$

If $\ell_p = d$, then $p \equiv 1 \pmod d$. If $P(d) > 2$, then since $d \mid m$, we get that every odd prime factor of d is in $\mathcal{Q}$. In particular, it is at least 17. Thus, $p > 34$. Hence,

$$\log\left(\frac{5^k}{3.5}\right) < \sum_{\substack{d\mid m\\ P(d)>2}} \sum_{\ell_p=d} \frac{1}{p} + \sum_{p\ge 37} \frac{1}{p(p-1)} < \sum_{\substack{d\mid m\\ P(d)>2}} \sum_{\ell_p=d} \frac{1}{p} + 0.007.$$

We thus get that

$$\log\left(\frac{5^k}{3.6}\right) < \sum_{\substack{d\mid m\\ P(d)>2}} S_d, \tag{11}$$

where

$$S_d := \sum_{\ell_p=d} \frac{1}{p}. \tag{12}$$

We need to bound S_d. For this, we first take

$$\mathcal{P}_d = \{p : \ell_p = d\}.$$

Put $\omega_d := \#\mathcal{P}_d$. Since $p \equiv 1 \pmod d$ for all $p \in \mathcal{P}_d$, we have that

$$(d+1)^{\omega_d} \le \prod_{p\in\mathcal{P}_d} p < 5^d - 1 < 5^d, \quad \text{therefore} \quad \omega_d < \frac{d\log 5}{\log(d+1)}. \tag{13}$$

We use the Brun-Titchmarsh theorem in the version due to Montgomery and Vaughan [12] which asserts that

$$\pi(x; d, 1) < \frac{2x}{\phi(d)\log(x/d)} \quad \text{for all} \quad x > d \ge 2, \tag{14}$$

where $\pi(x; d, 1)$ stands for the number of primes $p \le x$ with $p \equiv 1 \pmod d$. Put $\mathcal{Q}_d := \{p < 4d : p \equiv 1 \pmod d\}$. Clearly, $\mathcal{Q}_d \subset \{d+1, 2d+1, 3d+1\}$ and since $d \mid m$ and $3 \notin \mathcal{Q}$, it follows that d is not a multiple of 3. In particular, one of $d+1$ and $2d+1$ is a multiple of 3, so that at most one of these two numbers can be a prime. We now split S_d as follows:

$$S_d \le \sum_{\substack{p<4d\\ p\equiv 1 \pmod d}} \frac{1}{p} + \sum_{\substack{4d\le p\le d^2\\ p\equiv 1 \pmod d\\ \ell_p=d}} \frac{1}{p} + \sum_{\substack{p>d^2\\ \ell_p=d}} \frac{1}{p} := T_1 + T_2 + T_3. \tag{15}$$

Clearly,

$$T_1 = \sum_{p \in \mathcal{Q}_d} \frac{1}{p}. \tag{16}$$

For S_2, we use estimate (14) and Abel's summation formula to get

$$\begin{aligned} T_2 &\le \frac{\pi(x; d, 1)}{x}\Big|_{x=4d}^{d^2} + \int_{4d}^{d^2} \frac{\pi(t; d, 1)}{t^2} dt \\ &\le \frac{2d^2}{d^2\phi(d)\log d} + \frac{2}{\phi(d)} \int_{4d}^{d^2} \frac{dt}{t \log(t/d)} \\ &\le \frac{2}{\phi(d)\log d} + \frac{2}{\phi(d)} \log\log(t/d)\Big|_{t=4d}^{d^2} \\ &= \frac{2\log\log d}{\phi(d)} + \frac{2}{\phi(d)}\left(\frac{1}{\log d} - \log\log 4\right). \end{aligned}$$

The expression $1/\log d - \log\log 4$ is negative for $d \ge 34$, so

$$T_2 < \frac{2\log\log d}{\phi(d)} \quad \text{for all} \quad d \ge 34. \tag{17}$$

Inequality (17) holds for $d = 17$ as well, since there

$$T_2 < S_{17} = \frac{1}{409} + \frac{1}{466344409} < 0.003 < 0.13 < \frac{2\log\log 17}{\phi(17)}.$$

Hence, inequality (17) holds for all divisors d of m with $P(d) > 2$.

As for T_3, we have by (13),

$$T_3 < \frac{\omega_d}{d^2} < \frac{\log 5}{d\log(d+1)}. \tag{18}$$

Hence, collecting (16), (17) and (18), we get that

$$S_d < \sum_{p \in \mathcal{Q}_d} \frac{1}{p} + \frac{2\log\log d}{\phi(d)} + \frac{\log 5}{d\log(d+1)}. \tag{19}$$

We now show that

$$S_d < \frac{3\log\log d}{\phi(d)}. \tag{20}$$

Since $\phi(d) < d$ and at most one of $d+1$ and $2d+1$ is prime, we get, via (19), that

$$\begin{aligned} S_d &< \frac{1}{d+1} + \frac{1}{3d+1} + \frac{2\log\log d}{\phi(d)} + \frac{\log 5}{d\log(d+1)} \\ &< \frac{1}{\phi(d)}\left(\frac{4}{3} + 2\log\log d + \frac{\log 5}{\log(d+1)}\right). \end{aligned}$$

So, in order to prove (20), it suffices that

$$\frac{4}{3}+\frac{\log 5}{\log(d+1)}<2\log\log d, \quad \text{which holds for all } \ d>200.$$

The only possible divisors d of m with $P(d)>2$ (so, whose odd prime factors are in $\mathcal{Q}$), and with $d\le 200$ are

$$R:=\{17,34,41,68,71,82,103,136,142,164\}. \tag{21}$$

We checked individually that for each of the values of d in R given by (21), inequality (20) holds.

Now we write $d=2^{\alpha_d}d_1$, where $\alpha_d\in\{0,1,2,3,4\}$ and d_1 is odd. Since $d_1\ge 17>2^{\alpha_d}$, we have that $d<d_1^2$. Hence, keeping d_1 fixed and summing over α_d, we have that

$$\begin{aligned}\sum_{\alpha_d=0}^{4} S_{2^{\alpha_d}d_1} &< 3\sum_{\alpha_d=0}^{4}\frac{\log(2\log d_1)}{\phi(d_1)}\left(1+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\right)\\ &< \frac{8.7\log(2\log d_1)}{\phi(d_1)}.\end{aligned} \tag{22}$$

Inserting inequalities (20) and (22) into (11), we get that

$$\log\left(\frac{5^k}{3.6}\right)<\sum_{\substack{d_1\mid m\\ d_1>1\\ d_1 \text{ odd}}}\frac{8.7\log(2\log d_1)}{\phi(d_1)}. \tag{23}$$

The function

$$a\mapsto 8.7\log(2\log a)$$

is sub–multiplicative when restricted to the set $\mathcal{A}=\{a\ge 17\}$. That is, the inequality

$$8.7\log(2\log(ab))\le 8.7\log(2\log a)\cdot 8.7\log(2\log b) \quad \text{holds if} \quad \min\{a,b\}\ge 17.$$

Indeed, to see why this is true, assume say that $a\le b$. Then $\log ab\le 2\log b$, so it is enough to show that

$$8.7\log 2+8.7\log(2\log b)\le 8.7\log(2\log a)\cdot 8.7\log(2\log b)$$

which is equivalent to

$$8.7\log(2\log b)\,(8.7\log(2\log a)-1)>8.7\log 2,$$

which is clear for $\min\{a,b\}\ge 17$. It thus follows that

$$\sum_{\substack{d_1\mid m\\ d_1>1\\ d_1 \text{ odd}}}\frac{8.7\log(2\log d_1)}{\phi(d_1)}<\prod_{q\mid m}\left(1+\sum_{i\ge 1}\frac{8.7\log(2\log q^i)}{\phi(q^i)}\right)-1.$$

Inserting the above inequality into (23), taking logarithms and using the fact that $\log(1+x) < x$ for all real numbers x, we get

$$\log\left(\log\left(\frac{5^k e}{3.6}\right)\right) < \sum_{q|m}\sum_{i\geq 1}\frac{8.7\log(2\log q^i)}{\phi(q^i)}. \tag{24}$$

Next we show that

$$\sum_{i\geq 1}\frac{8.7\log(2\log(q^i))}{\phi(q^i)} < \frac{20\log\log q}{q} \quad \text{for} \quad q \in \mathcal{Q}. \tag{25}$$

We check that it holds for $q = 17$. So, from now on, $q \geq 41$. Since

$$\log(2\log q^i) = \log(2i) + \log\log q < (1 + \log i) + \log\log q \leq i + \log\log q,$$

we have that

$$\begin{aligned}\sum_{i=1}^{\infty}\frac{\log(2\log(q^i))}{\phi(q^i)} &< \sum_{i\geq 1}\frac{i}{q^{i-1}(q-1)} + \log\log q\sum_{i\geq 1}\frac{1}{q^{i-1}(q-1)}\\ &= \frac{q^2}{(q-1)^3} + (\log\log q)\left(\frac{q}{(q-1)^2}\right)\\ &< (\log\log q)\left(\frac{q^2}{(q-1)^3} + \frac{q}{(q-1)^2}\right)\\ &= (\log\log q)\left(\frac{2q^2-q}{(q-1)^3}\right)\end{aligned}$$

because $\log\log q > 1$. Thus, it suffices that

$$8.7\left(\frac{2q^2-q}{(q-1)^3}\right) < \frac{20}{q}, \quad \text{which holds for} \quad q \geq 41.$$

Hence, (25) holds, therefore (24) implies

$$\log\left(\log\left(\frac{5^k e}{3.6}\right)\right) < 20\sum_{\substack{q|m\\ q>2}}\frac{\log\log q}{q}, \tag{26}$$

which is exactly (6). This finishes the proof of the lemma. □

Lemma 5. *If $q < 10^4$ and $q \mid m$, then $q \mid n$.*

Proof. This is clear for $q = 2$, since then $24 \mid 5^2 - 1 \mid 5^m - 1$, therefore $8 = \phi(24) \mid \phi(5^m - 1) = 5^n - 1$, so n is even. Let now q be odd. Look at the number

$$\frac{5^q - 1}{4} = r_1^{\beta_1}\cdots r_l^{\beta_l}. \tag{27}$$

Assume that $l \geq 2$. Since $r_i \equiv 1 \pmod q$ for $i = 1, \ldots, l$, we have that $q^2 \mid (r_1 - 1)\cdots(r_l - 1) \mid \phi(5^m - 1) = 5^n - 1$. Since $q \| 5^{q-1} - 1$ for all odd $q < 10^4$, we get that, $q \mid n$, as desired. So, it remains to show that $l \geq 2$ in

(27). We do this by contradiction. Suppose that $l = 1$. Since $r_1 \equiv 4 \pmod 5$, reducing equation (27) modulo 5 we get that

$$1 \equiv 4^{\beta_1} \pmod 5,$$

so β_1 is even. Hence,

$$\frac{5^n - 1}{5 - 1} = \square.$$

However, the equation

$$\frac{x^n - 1}{x - 1} = \square$$

for integers $x > 1$ and $n > 2$ has been solved by Ljunggren [5] who showed that the only possibilities are $(x, n) = (3, 5),\ (7, 4)$. This contradiction shows that $l \geq 2$ and finishes the proof of this lemma. $\square$

Remark. Apart from Ljunggren's result, the above proof was based on the computational fact that if $q < 10^4$ is an odd prime, then $q \| 5^{q-1} - 1$. In fact, the first prime failing this test is $q = 20771$.

Lemma 6. *We have* $k = 2$.

Proof. We split the odd prime factors p of m in two subsets

$$U = \{q \mid n\} \quad \text{and} \quad V = \{q \nmid n\}.$$

By Lemma 4, we have

$$\log\left(\log\left(\frac{5^k e}{3.6}\right)\right) \leq 20\left(\sum_{q \in U} \frac{\log\log q}{q} + \sum_{p \in V} \frac{\log\log q}{q}\right) := 20(T_1 + T_2). \tag{28}$$

We first bound T_2. By Lemma 5, if $q \in V$, then $q > 10^4$. In particular, $q > 512$. Let $t \geq 9$, and put $I_t = [2^t, 2^{t+1}) \cap V$. Suppose that $r_1, \ldots, r_u$ are all the members of I_t. By the Primitive Divisor Theorem (see [2]), $5^{dr_u} - 1$ has a primitive prime factor for all divisors d of $r_1 \cdots r_{u-1}$, and this prime is congruent to 1 modulo r_u. Since the number $r_1 \cdots r_{u-1}$ has 2^{u-1} divisors, we get that

$$2^{u-1} \leq \mathrm{ord}_{r_u}(\phi(5^m - 1)) = \mathrm{ord}_{r_u}(5^n - 1).$$

Since $r_u \nmid n$, we get that $\mathrm{ord}_{r_u}(5^n - 1) = \mathrm{ord}_{r_u}(5^{r_u - 1} - 1)$, so

$$2^{u-1} \leq \mathrm{ord}_{r_u}(5^{r_u-1} - 1) < \frac{\log 5^{r_u}}{\log r_u} = \frac{r_u \log 5}{\log r_u} < \frac{2^{t+1} \log 5}{(t+1)\log 2}.$$

The above inequality implies that $u \leq t - 1$, for if not, then $u \geq t$, and we would get that

$$2^{t-1} \leq \frac{2^{t+1} \log 5}{(t+1) \log 2}, \quad \text{or} \quad 4 \log 5 \geq (t+1) \log 2 \geq 10 \log 2,$$

a contradiction. This shows that $\#I_t \le t-1$ for all $t \ge 9$. Hence,

$$20T_2 \le \sum_{t\ge 9} \frac{20(t-1)\log\log 2^t}{2^t} < 1.4.$$

Hence, we get that

$$\log\left(\log\left(\frac{5^k e}{3.6}\right)\right) < 20 \sum_{\substack{q\mid \gcd(m,k)\\ q>2}} \frac{\log\log q}{q} + 1.4. \tag{29}$$

We use (29) to bound k by better and better bounds. We start with

$$\log\left(\log\left(\frac{5^k e}{3.6}\right)\right) < 20(\log\log k)\left(\sum_{\substack{q\mid k\\ q>2}} \frac{1}{q}\right) + 1.4,$$

which is implied by (29). Assume $k \ge 3$. We have

$$\sum_{q\mid k} \frac{1}{q} < \sum_{d\mid k} \frac{1}{d} = \frac{\sigma(k)}{k} < \frac{k}{\phi(k)} < 1.79\log\log k + \frac{2.5}{\log\log k},$$

where the last inequality above holds for all $k \ge 3$ (see inequality (3.41) in [13]). We thus get that

$$\log k < \log(k\log 5 + 1 - \log(3.6)) < 20 \times 1.79(\log\log k)^2 + 51.4,$$

which gives $\log k < 2163$. Since

$$\sum_{17\le q\le 2243} \log q > 2166 > \log k,$$

it follows that

$$T_1 = \sum_{\substack{q\mid \gcd(m,k)\\ q>2}} \frac{\log\log q}{q} < \sum_{17\le q\le 2243} \frac{\log\log q}{q} < 1.48.$$

Hence,

$$\log\left(\log\left(\frac{5^k e}{3.6}\right)\right) < 20 \times 1.48 + 1.4; \quad \text{so} \quad k < 2\times 10^{13}.$$

By (4), the first few possible odd prime factors of m are 17, 41, 71, 103, 223, 257 and all others are > 512. Since

$$17 \times 41 \times 71 \times 103 \times 223 \times 257 \times 512 > 10^{14} > k,$$

it follows that

$$T_1 \le \frac{\log\log 17}{17} + \frac{\log\log 41}{41} + \frac{\log\log 71}{71} + \frac{\log\log 103}{103} + \frac{\log\log 223}{223} + \frac{\log\log 257}{257} < 0.143.$$

Hence,

$$\log\left(\log\left(\frac{5^k e}{3.6}\right)\right) < 20 \times 0.143 + 1.4; \quad \text{so} \quad k \le 44.$$

If follows that $\gcd(k, m)$ can have at most one odd prime factor, so

$$T_1 \le \frac{\log\log 17}{17} < 0.07,$$

therefore

$$\log\left(\log\left(\frac{5^k e}{3.6}\right)\right) < 20 \times 0.07 + 1.4 = 2.8; \quad \text{so} \quad k \le 11.$$

Thus, in fact k has no odd prime factor, giving that $T_1 = 0$, so

$$\log\left(\log\left(\frac{5^k e}{3.6}\right)\right) < 1.4, \quad \text{therefore} \quad k \le 2.$$

Since by Lemma 2, m and n are not coprime, it follows that in fact $k \ge 2$, so $k = 2$. ☐

Lemma 7. *$k > 2$.*

Proof. Let q_1 be the smallest odd prime factor of m which exists for if not $m \mid 16$, which is not possible. Let $q_1, \ldots, q_s$ be all the prime factors of m. For each divisor d of $q_2 \cdots q_{s-1}$, the number $5^{dq_1} - 1$ has a primitive divisor which is congruent to 1 modulo q_1. Since there are 2^{s-1} divisors of $q_2 \cdots q_s$, we get that

$$2^{s-1} \le \mathrm{ord}_{q_1}(\phi(5^m - 1)) = 5^n - 1.$$

Since q_1 does not divide n (otherwise it would divide $k = 2$), we get that $\mathrm{ord}_{q_1}(5^n - 1) = \mathrm{ord}_{q_1}(5^{q_1 - 1} - 1)$, and

$$2^{s-1} \le \mathrm{ord}_{q_1}\left(5^{q_1-1} - 1\right) < \frac{\log 5^{q_1}}{\log q_1} = \frac{q_1 \log 5}{\log q_1} < q_1.$$

Hence,

$$s < 1 + \frac{\log q_1}{\log 2}.$$

Lemma 4 now shows that

$$\log\left(\log\left(\frac{5^2 e}{3.6}\right)\right) < 20 \sum_{\substack{q \mid m \\ q > 2}} \frac{\log\log q}{q} < \frac{20 s \log\log q_1}{q_1}$$
$$< 20\left(1 + \frac{\log q_1}{\log 2}\right)\frac{\log\log q_1}{q_1}.$$

This gives $q_1 < 300$, so by Lemma 5, we have $q_1 \mid k$, which finishes the proof of this lemma. ☐

Obviously, Lemmas 6 and 7 contradict each other, which completes the proof of the theorem.

Acknowledgements. We thank the referee for a careful reading of the manuscript and for pointing out some computational errors in a previous version of the manuscript. We also thank Sam Wagstaff for sending us an updated version of some of the tables in [1]. This paper was written during a visit of F. L. at the AIMS-Senegal in January of 2014. He thanks the people of this institution for their hospitality. During the preparation of this paper, F. L. was also supported in part by Project PAPIIT IN104512 (UNAM).

References

[1] J. Brillhart, D. H. Lehmer, J. L. Selfridge, B. Tuckerman, and S. S. Wagstaff, Jr., *Factorizations of* $b^n \pm 1$, $b = 2, 3, 5, 6, 7, 10, 11, 12$ *up to high powers*, American Mathematical Society, Providence, RI, 1988.

[2] R. D. Carmichael, *On the numerical factors of the arithmetic forms* $\alpha^n \pm \beta^n$, Ann. Math. (2) **15** (1913), no. 1-4, 30–70.

[3] J. Cilleruelo and F. Luca, *Repunit Lehmer numbers*, Proc. Edinb. Math. Soc. **54** (2011), no. 1, 55–65.

[4] M. T. Damir, B. Faye, F. Luca, and A. Tall, *Members of Lucas sequences whose Euler function is a power of* 2, Fibonacci Quart. **52** (2014), no. 1, 3–9.

[5] W. Ljunggren, *Some theorems on indeterminate equations of the form* $xn-1/x-1 = yq$, Norsk Mat. Tidsskr. **25** (1943), 17–20.

[6] F. Luca, *Problem 10626*, American Math. Monthly **104** (1997), 871.

[7] ———, *Equations involving arithmetic functions of Fibonacci and Lucas numbers*, Fibonacci Quart. **38** (2000), no. 1, 49–55.

[8] ———, *Euler indicators of binary recurrence sequences*, Collect. Math. **53** (2002), no. 2, 133–156.

[9] ———, *On the Euler function of repdigits*, Czechoslovak Math. J. **58** (2008), no. 1, 51–59.

[10] F. Luca and M. Mignotte, $\phi(F_{11}) = 88$, Divulg. Mat. **14** (2006), no. 2, 101–106.

[11] F. Luca and P. Stănică, *Equations with arithmetic functions of Pell numbers*, Bull. Soc. Mat. Roumanie, to appear.

[12] H. L. Montgomery and R. C. Vaughan, *The large sieve*, Mathematika **20** (1973), 119–134.

[13] J. B. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. **6** (1962), 64–94.

Bernadette Faye
AIMS-Sénégal
Km 2 route de Joal (Centre IRD Mbour)
BP: 64566 Dakar-Fann, Sénégal
E-mail address: bernadette@aims-senegal.org

Florian Luca
School of Mathematics
University of the Witwatersrand
P. O. Box Wits 2050, South Africa
E-mail address: florian.luca@wits.ac.za

Amadou Tall
AIMS-Sénégal
Km 2 route de Joal (Centre IRD Mbour)
BP: 64566 Dakar-Fann, Sénégal
E-mail address: amadou.tall@aims-senegal.org