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SOME UNIFORM GEOMETRICAL PROPERTIES IN BANACH SPACES

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Abstract. In this paper, we investigate relationships among property (k-β), weak property (β_k), k-nearly uniformly convexity and property (A_k).

1. Introduction

Let $(X, \|\cdot\|)$ be a real Banach space and X^* the dual space of X. By B_X and S_X, we denote the closed unit ball of X and the unit sphere of X, respectively. For a sequence (x_n) in X, we let $\operatorname{sep}(x_n) = \inf\{\|x_n - x_m\| : n \neq m\}$. Denote by $\mathbb{N}$ and $\mathbb{R}$ the set of natural numbers and real numbers, respectively. For $A,\ B \subset \mathbb{N}$, we write $A < B$ if $\max A < \min B$. For a Banach space X with a basis (e_n) and $x,\ y \in X$, we write $x < y$ if supp $x <$ supp y, where supp $x = \{i \in \mathbb{N} : a_i \neq 0,\ x = \sum_{i=1}^{\infty} a_i e_i\}$.

A Banach space is said to be reflexive if the natural embedding $\eta : X \to X^{**}$ is onto. A Banach space X is said to have Banach-Saks property (BS) if any bounded sequence in the space admits a subsequence whose arithmetic means converge in norm. In similar way, we say that a Banach space X has weak Banach-Saks property (w-BS) if any weakly convergent sequence in the space admits a subsequence whose arithmetic means converge in norm. Since any weakly convergent sequence is norm bounded, it follows that Banach-Saks property implies weak Banach-Saks property. We note that weak Banach-Saks property and Banach-Saks property coincide in the reflexive Banach space. A Banach space X is said to be uniformly convex (UC) if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $x, y \in B_X$ and $\|x - y\| \geq \epsilon$, $\frac{1}{2}\|x + y\| \leq 1 - \delta$.

S. Kakutani [2] showed that unform convexity implies Banach-Saks property. T. Nishiura and D. Waterman [6] proved that Banach-Saks property implies reflexivity in Banach spaces.

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A Banach space is said to have property (A_k) if it is reflexive and there exists a number δ, $0 < \delta < 1$, such that for a weakly null sequence (x_n), there exist $n_1 < n_2 < \cdots < n_k$ with $\left\| \frac{1}{k} \sum_{i=1}^{k} x_{n_i} \right\| \leq 1 - \delta$. We say that X has the property (A_∞) if it has the property (A_k) for some $k \in \mathbb{N}$.

Let $x_1, x_2, \ldots, x_{k+1} \in X$. The k-dimensional volume enclosed by $x_1, x_2, \ldots,$ x_{k+1} is given by

$$V(x_1, x_2, \ldots, x_{k+1}) = \sup \left\{ \begin{vmatrix} 1 & \cdots & 1 \\ f_1(x_1) & \cdots & f_1(x_{k+1}) \\ \vdots & & \vdots \\ f_k(x_1) & \cdots & f_k(x_{k+1}) \end{vmatrix} \; f_i \in B_{X^*}, \; i = 1, 2, \ldots, k \right\}.$$

A Banach space X is said to be k-uniformly rotund (k-UR), $k \geq 1$, if for each $\epsilon > 0$ there exists a $\delta > 0$ such that for $x_i \in B_X$ $(i = 1, 2, \ldots, k+1)$ and $V(x_1, x_2, \ldots, x_{k+1}) \geq \epsilon$, $\frac{1}{k+1} \left\| \sum_{i=1}^{k+1} x_i \right\| \leq 1 - \delta$.

J. R. Partington [7] introduced the notion of property (A_k) and show the following strict implications:

$$(UC) \Rightarrow (A_2) \Rightarrow (A_3) \Rightarrow \cdots \Rightarrow (A_\infty) \Rightarrow (BS)$$

A Banach space X is said to have the property (k-β) if for each $\epsilon > 0$, there exists $\delta > 0$ such that for any $x \in B_X$ and any sequence $(x_n) \subset B_X$ with $\operatorname{sep}(x_n) > \epsilon$ there exist $n_i \in \mathbb{N}$, $i = 1, 2, \ldots, k$ with $n_1 < n_2 < \cdots < n_k$ such that

$$\left\| \frac{1}{k+1} \left(x + \sum_{i=1}^{k} x_{n_i} \right) \right\| \leq 1 - \delta.$$

A Banach space X is said to be k-nearly uniformly convex (k-NUC) if for each $\epsilon > 0$, there exists $\delta > 0$ such that for any sequence $(x_n) \subset B_X$ with $\operatorname{sep}(x_n) > \epsilon$ there exist $n_i \in \mathbb{N}$, $i = 1, 2, \ldots, k$ with $n_1 < n_2 < \cdots < n_k$ such that

$$\left\| \frac{1}{k} \left(\sum_{i=1}^{k} x_{n_i} \right) \right\| \leq 1 - \delta.$$

D. Kutzarova [3] introduced the notion of property (k-β) and k-nearly uniformly convexity and show the following strict implications:

$$\begin{array}{ccccccccccc} (UC) & \Rightarrow & (\beta_1) & \Rightarrow & 2\text{-}NUC & \Rightarrow & (\beta_2) & \Rightarrow & 3\text{-}NUC & \Rightarrow \cdots \Rightarrow & (BS) \\ & & & & \Downarrow & & & & \Downarrow & & \\ & & & & A_2 & & & & A_3 & \cdots & \end{array}$$

A Banach space X has the weak property (β_k) if it is reflexive and there exists $\delta > 0$ such that for any $x \in B_X$ and any weakly null sequence $(x_n) \subset B_X$ there exist $n_i \in \mathbb{N},\ i = 1, 2, \ldots, k$ with $n_1 < n_2 < \cdots < n_k$ such that

$$\left\| \frac{1}{k+1} \left(x + \sum_{i=1}^{k} x_{n_i} \right) \right\| \leq 1 - \delta.$$

We say that X has the weak property (β_∞) if it has the weak property (β_k) for some $k \in \mathbb{N}$. K. G. Cho and C. S. Lee [1] introduced the notion of weak property (β_k) and show the following strict implications:

$$(UC) \Rightarrow (w\text{-}\beta_1) \Rightarrow (w\text{-}\beta_2) \Rightarrow \cdots \Rightarrow (w\text{-}\beta_\infty) \Rightarrow (BS)$$

In this paper, we show that the property $(k - \beta)$ implies the weak property (β_k) but the converse does not hold.

2. Main parts

We begin with the proposition:

Proposition 2.1. *Let X be a Banach space.*

1. *If X has the weak property (β_k), then it has the property (A_{k+1}) for all $k \geq 1$.*
2. *If X has the property (A_k), then it has the weak property (β_k) for all $k \geq 2$.*

Proof. (1) Choose a corresponding $\delta > 0$ according to the definition of weak property (β_k). Let (x_n) be a weakly null sequence in B_X. Take $x = x_1 = x_{n_1}$, then for $(x_n)_{n \geq 2}$ there exist $2 \leq n_2 < n_3 < \cdots < n_{k+1}$ such that

$$\frac{1}{k+1} \left\| \sum_{i=1}^{k+1} x_{n_i} \right\| = \frac{1}{k+1} \left\| x + \sum_{i=2}^{k+1} x_{n_i} \right\| \leq 1 - \delta.$$

This means that X has the property (A_{k+1}).

(2) Choose a corresponding $\delta > 0$ according to the definition of property (A_k). Let $x \in B_X$ and (x_n) be a weakly null sequence in B_X. Since X has the property (A_k), there exist $n_1 < n_2 < \cdots < n_k$ such that

$$\frac{1}{k} \left\| \sum_{i=1}^{k} x_{n_i} \right\| \leq 1 - \delta.$$

Then

$$\frac{1}{k+1} \left\| x + \sum_{i=1}^{k} x_{n_i} \right\| \leq \frac{1}{k+1} \left(1 + k(1 - \delta)\right) = 1 - \frac{k\delta}{k+1}$$

Taking $\delta_1 = \frac{k\delta}{k+1}$, we get the result. □

We want to show the converses of Proposition 2.1 do not hold. We need lemma, example and proposition. The following lemma and example can be found in [4] and [7], respectively.

Lemma 2.2. *l_1-direct sum of finitely many Banach spaces with k-NUC has k-NUC.*

Example 1. For $x = (a_n) \in l_2$, we define a norm $\|x\|_{(k)}$ by

$$\|x\|_{(k)} = \left[\sup_{n_1 < n_2 < \cdots < n_k} \left(\sum_{i=1}^{k} |a_{n_i}| \right)^2 + \sum_{n \neq n_1, n_2, \ldots, n_k} |a_n|^2 \right]^{\frac{1}{2}}.$$

Then $\|x\|_2 \leq \|x\|_{(k)} \leq \sqrt{k}\|x\|_2$. Let $X_k = \left(l_2, \|\cdot\|_{(k)}\right)$. Then X_k has the property (A_{k+1}) but not the property (A_k) [7].

Proposition 2.3. *The property $(k\text{-}\beta)$ implies the weak property β_k.*

Proof. Let $\delta\left(\frac{1}{2}\right)$ be chosen according to the definition of $(k\text{-}\beta)$ for $\epsilon = \frac{1}{2}$. Put $\delta = \min\left\{\frac{k}{2(k+1)}, \delta\left(\frac{1}{2}\right)\right\}$. Let $x \in B_X$ and (x_n) be a weakly null sequence in B_X. If there exist $n_1 < n_2 < \cdots < n_k$ such that $\|x_{n_i}\| \leq \frac{1}{2}$, then $\left\|\frac{1}{k+1}\left(x + \sum_{i=1}^{k} x_{n_i}\right)\right\| \leq \frac{1}{k+1}\left(1 + \frac{k}{2}\right) = 1 - \frac{k}{2(k+1)} \leq 1 - \delta$. We now consider the case that $\|x_n\| > \frac{1}{2}$ except finite n less than k. Assume $\|x_n\| > \frac{1}{2}$ for all $n \in \mathbb{N}$. We note that $\liminf_{n\to\infty} \|x_n - x_m\| \geq \|x_m\| > \frac{1}{2}$ for every m. Put $n_1 = 1$. Since $\liminf_{n\to\infty} \|x_n - x_1\| \geq \|x_1\| > \frac{1}{2}$, there exists n_2 with $n_2 > n_1$ such that $\|x_{n_1} - x_{n_2}\| > \frac{1}{2}$. Having chosen $x_{n_1}, \ldots, x_{n_l}$ with $\|x_{n_i} - x_{n_j}\| > \frac{1}{2}$ whenever $i \neq j,\ 1 \leq i,\ j \leq l$. Since $\liminf_{n\to\infty} \|x_n - x_{n_i}\| \geq \|x_{n_i}\| > \frac{1}{2}$ for $i = 1, 2, \ldots, l$, there exists n_{l+1} with $n_{l+1} > n_l$ such that $\|x_{n_i} - x_{n_{l+1}}\| > \frac{1}{2}$ for $i = 1, 2, \ldots, l$.

We get a subsequence (x_{n_i}) of (x_n) with $\text{sep}(x_{n_i}) \geq \frac{1}{2}$. Then there exist $x_{n_{i_1}}, \ldots, x_{n_{i_k}}$ such that

$$\frac{1}{k+1}\left\|x + \sum_{j=1}^{k} x_{n_{i_j}}\right\| \leq \delta\left(\frac{1}{2}\right) \leq 1 - \delta.$$

This completes the proof. □

We are ready to show that the converses of Proposition 2.1 do not hold. The techniques are same with [3].

Proposition 2.4. (1) *There exists a Banach space with the property (A_{k+1}) which does not have the weak property (β_k) for each $k \geq 1$.*

(2) *There exists a Banach space with the weak property (β_k) which does not have the property (A_k) for each $k \geq 2$.*

Proof. (1) Since l_2 is uniformly convex, it is 2-NUC. By Lemma 2.2, $(l_2 + \mathbb{R})_{l_1}$ is 2-NUC. Since 2-NUC implies the property (A_2) [3], $(l_2 + \mathbb{R})_{l_1}$ has the property (A_2). Let (e_n) be the usual unit bases of l_2. Then $((e_n, 0))$ is weakly null in $(l_2 + \mathbb{R})_{l_1}$. But for $(0,1) \in (l_2 + \mathbb{R})_{l_1}$, $\|(0,1) + (e_n, 0)\|_{(l_2+\mathbb{R})_{l_1}} = \|(e_n, 1)\|_{(l_2+\mathbb{R})_{l_1}} = 2$. This means that $(l_2 + \mathbb{R})_{l_1}$ does not have the weak property (β_1).

For $k \geq 2$, X_k is k-UR [5]. Since k-UR implies (k-β) [3], X_k is (k-β). Since (k-β) implies $(k+1)$-NUC [3], X_k is $(k+1)$-NUC. By Lemma 2.2, $(X_k+\mathbb{R})_{l_1}$ is $(k+1)$-NUC. Since $(k+1)$-NUC implies the property A_{k+1} [3], $(X_k+\mathbb{R})_{l_1}$ has the property A_{k+1}.

We note that X_k does not have the property A_k [7]. Then for all $\delta > 0$, there exists weakly null sequence $(x_n(\delta))$ in B_{X_k} such that $\frac{1}{k}\left\|\sum_{i=1}^k x_{n_i}(\delta)\right\|_{(k)} > 1-\delta$ for all $n_1 < n_2 < \cdots < n_k$. Let $y = (0,1)$, $y_n = (x_n(\delta), 0)$. Then y, $y_n \in B_{(X_k+\mathbb{R})_{l_1}}$, (y_n) is weakly null in $(X_k+\mathbb{R})_{l_1}$ and

$$\begin{aligned}\frac{1}{k+1}\left\|y+\sum_{i=1}^k y_{n_i}\right\|_{(X_k+\mathbb{R})_{l_1}} &= \frac{1}{k+1}\left(1+\left\|\sum_{i=1}^k x_{n_i}(\delta)\right\|_{(k)}\right) \\ &\geq \frac{1}{k+1}\left(1+k(1-\delta)\right) = 1-\frac{k}{k+1}\delta\end{aligned}$$

for all $n_1 < n_2 < \cdots < n_k$. This means that $(X_k+\mathbb{R})_{l_1}$ does not have the weak property (β_k).

(2) X_k does not have the property A_k [7] and is k-UR [5]. Since k-UR implies (k-β) [3], X_k is (k-β). By Proposition 2.3, X_k has the weak property β_k. This completes the proof. □

We finally investigate the converse of Proposition 2.3.

Example 2. Let $X = \left(\prod_{p\geq 2} l_p\right)_2$ and $(e_n^{(p)})$ be a usual unit basis for l_p. For $n = 0,1,2,\ldots$ and $1 \leq p \leq n+1$, let $f_{\frac{n(n+1)}{2}+p} = (0,\ldots,0,e_{n-p+2}^{(p+1)},0,\ldots)$ with all zero except pth coordinate $e_{n-p+2}^{(p+1)}$. Then (f_i) is a basis for $X = \left(\prod_{p\geq 2} l_p\right)_2$.

We need a lemma.

Lemma 2.5. *Let $x, y \in B_{\left(\prod_{p\geq 2} l_p\right)_2}$ with $x < y$. Then $\|x+y\| \leq \sqrt{2}$.*

Proof. Let $x = \sum_{i=1}^n a_i f_i$ and $y = \sum_{i=n+1}^\infty a_i f_i$. For $p \geq 2$, let

$$A_p = \left\{ i \in \text{supp } x : i = \frac{m(m+1)}{2} + p \ \text{ for } m \geq 0 \right\}$$

and

$$B_p = \left\{ i \in \text{supp } y : i = \frac{m(m+1)}{2} + p \ \text{ for } m \geq 0 \right\}.$$

Then for all $p \geq 2$, we know $A_p < B_p$ and

$$\|x+y\|^2 = \sum_{p=2}^\infty \left\| \sum_{i\in A_p} a_i e_i^{(p)} + \sum_{i\in B_p} a_i e_i^{(p)} \right\|_p^2$$

$$= \sum_{p=2}^{\infty}\left(\sum_{i\in A_p\cup B_p}|a_i|^p\right)^{\frac{2}{p}}$$
$$\leq \sum_{p=2}^{\infty}\left(\sum_{i\in A_p}|a_i|^p\right)^{\frac{2}{p}} + \sum_{p=2}^{\infty}\left(\sum_{i\in B_p}|a_i|^p\right)^{\frac{2}{p}}$$
$$= \|x\|^2 + \|y\|^2 = 2.$$

This completes the proof. □

Using the above lemma, we get the following.

Proposition 2.6. *There exists a Banach space with the weak property* (β_1) *which is not k-NUC for all* $k \geq 2$.

Proof. Let $X = \left(\prod_{p\geq 2} l_p\right)_2$. Let $x = \sum_{i=1}^{\infty} a_i f_i$, $x_n = \sum_{i=1}^{\infty} b_i^{(n)} f_i \in B_X$ and (x_n) be a weakly null sequence in B_X. Then there exists $N \in \mathbb{N}$ such that $\left\|\sum_{i=N+1}^{\infty} a_i f_i\right\| \leq \frac{1}{4}$. Since (x_n) be a weakly null sequence in B_X, there exists $n_0 \in \mathbb{N}$ such that $\left\|\sum_{i=1}^{N} b_i^{(n_0)} f_i\right\| \leq \frac{1}{4}$. Then by Lemma 2.5,

$$\|x + x_{n_0}\| \leq \left\|\sum_{i\leq N} a_i f_i + \sum_{i\geq N+1} b_i^{(n_0)} f_i\right\| + \left\|\sum_{i=N+1}^{\infty} a_i f_i\right\| + \left\|\sum_{i=1}^{N} b_i^{(n_0)} f_i\right\|$$
$$\leq \sqrt{2} + \frac{1}{2} = 2\left(1 - \left(\frac{3}{4} - \frac{\sqrt{2}}{2}\right)\right).$$

Taking $\delta = \frac{3}{4} - \frac{\sqrt{2}}{2}$, $X = \left(\prod_{p\geq 2} l_p\right)_2$ has the weak property (β_1). On the other hand, X fails to have an equivalent NUC norm [3]. We get the result. □

The above proposition also implies that there exists a space which satisfies the weak property (β_1) but does not satisfy the property $(k\text{-}\beta)$ for all k. By [1], [3], [7], Propositions 2.1, 2.3, 2.4 and 2.6, we finally get the following strict implications:

$$\begin{array}{ccccccccccccc} (UC) & \Rightarrow & (\beta_1) & \Rightarrow & 2\text{-}NUC & \Rightarrow & (\beta_2) & \Rightarrow & 3\text{-}NUC & \Rightarrow & \cdots & \Rightarrow & (BS) \\ & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \cdots & & \\ (UC) & \Rightarrow & (w\text{-}\beta_1) & \Rightarrow & (A_2) & \Rightarrow & (w\text{-}\beta_2) & \Rightarrow & (A_3) & \Rightarrow & \cdots & \Rightarrow & (BS) \end{array}$$

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References

[1] K. G. Cho and C. S. Lee, *Weak property* (β_k), Korean J. Math. **20** (2012), no. 4, 415–422.
[2] S. Kakutani, *Weak convergence in uniformly convex spaces*, Tôhoku Math. J. **45** (1938), 347–354.
[3] D. Kutzarova, *k-beta and k-Nearly Uniformly Convex Banach spaces*, J. Math. Anal. Appl. **162** (1991), no. 2, 322–338.
[4] D. Kutzarova and T. Landes, *NUC and related properties of direct sums*, Bolletino U.M.I.(7)8-A (1994), 45–54.
[5] B.-L. Lin and X.-Y. Yu, *On the k-uniform rotund and the fully convex Banach spaces*, J. Math. Anal. Appl. **110** (1985), no. 2, 407–410.
[6] T. Nishiura and D. Waterman, *Reflexivity and summability*, Studia Math. **23** (1963), 53–57.
[7] J. R. Partington, *On the Banach-Saks property*, Math. Proc. Cambridge Philos. Soc. **82** (1977), no. 3, 369–374.

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