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MULTIDIMENSIONAL BSDES WITH UNIFORMLY CONTINUOUS GENERATORS AND GENERAL TIME INTERVALS

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Abstract. This paper is devoted to solving a multidimensional backward stochastic differential equation with a general time interval, where the generator is uniformly continuous in (y, z) non-uniformly with respect to t. By establishing some results on deterministic backward differential equations with general time intervals, and by virtue of Girsanov's theorem and convolution technique, we prove a new existence and uniqueness result for solutions of this kind of backward stochastic differential equations, which extends the results of [8] and [6] to the general time interval case.

1. Introduction

In this paper, we are concerned with the following multidimensional backward stochastic differential equation (BSDE for short in the remaining):

$$y_t = \xi + \int_t^T g(s, y_s, z_s)\,\mathrm{d}s - \int_t^T z_s\,\mathrm{d}B_s, \quad t \in [0, T], \tag{1}$$

where T satisfies $0 \le T \le +\infty$ called the terminal time; ξ is a k-dimensional random vector called the terminal condition; the random function $g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbf{R}^k \times \mathbf{R}^{k\times d} \mapsto \mathbf{R}^k$ is progressively measurable for each (y, z), called the generator of BSDE (1); and B is a d-dimensional Brownian motion. The solution $(y_t, z_t)_{t\in[0,T]}$ is a pair of adapted processes. The triple (ξ, T, g) is called the parameters of BSDE (1). We also denote by BSDE (ξ, T, g) the BSDE with the parameters (ξ, T, g). The nonlinear BSDEs were initially introduced by Pardoux and Peng [12]. They proved an existence and uniqueness result for solutions of multidimensional BSDEs under the assumptions that the generator g is Lipschitz continuous in (y, z) uniformly with respect to t, where

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the terminal time T is a finite constant. Since then, BSDEs have attracted more and more interesting and many applications on BSDEs have been found in mathematical finance, stochastic control, partial differential equations and so on (See [4] for details).

Many works including [1], [2], [6], [8], [9], [10], [11] and [13], see also the references therein, have weakened the Lipschitz condition on the generator g to extend the existence and uniqueness result obtained in [12]. In particular, by virtue of some results on deterministic backward differential equations (DBDEs for short in the remaining), Hamadène [8] proved the existence for solutions of multidimensional BSDEs when the generator g is uniformly continuous in (y,z). Furthermore, by establishing an estimate for a linear-growth function, Fan, Jiang, and Davison [6] obtained the uniqueness result under the same assumptions as those in [8]. It should be pointed out that all these works mentioned above only deal the BSDEs with finite time intervals.

Chen and Wang [3] first extended the terminal time to the general case and proved the existence and uniqueness for solutions of BSDEs under the assumptions that the generator g is Lipschitz continuous in (y,z) non-uniformly with respect to t, which improves the result of [12] to the infinite time interval case. Furthermore, [5] and [7] relaxed the Lipschitz condition of [3] and obtained two existence and uniqueness results for solutions of BSDEs with general time intervals, which generalizes the results of [11] and [10] respectively.

In this paper, by establishing some results on solutions of DBDEs with general time intervals and by virtue of Girsanov's theorem and convolution technique, we put forward and prove a general existence and uniqueness result for solutions of multidimensional BSDEs with general time intervals and uniformly continuous generators in (y,z) (see Theorem 7 in Section 3), which extends the results of [8] and [6] to the general time interval case. It should be mentioned that the uniform continuous assumptions for the generator are not necessarily uniform with respect to t in this result.

We would like to mention that some new troubles arise naturally when we change the terminal time of the BSDE and the DBDE from the finite case to the general case. For example, in the case of $T=+\infty$, the integration of a constant over $[0,T]$ is not finite any more, $\int_0^T u(t)\,\mathrm{d}t \le C\sup_{t\in[0,T]} u(t)$ may not hold any longer, and $\int_0^T v^2(s)\,\mathrm{d}s<+\infty$ can not imply $\int_0^T v(s)\,\mathrm{d}s<+\infty$. All these troubles are well overcome in this paper. Furthermore, although the whole idea of the proof for the existence and uniqueness of Theorem 7 originates from [8] and [6] respectively, some different arguments from those employed in [8] is used to prove the existence part of Theorem 7. More specifically, in the Step 1 of the proof for the existence part of Theorem 7, the proof of Lemma 12 is completely different from that of the corresponding result in [8], and we do not use the iteration technique used in [8] for solutions of BSDE (ξ,T,g^n) (see (19) in Section 4). In addition, the Step 3 of our proof for the existence part is also very different from that in [8]. As a result, the proof procedure is simplified at certain degree.

This paper is organized as follows. Section 2 introduces some usual notations and establishes some results on the solutions of DBDEs with general time intervals. Section 3 is devoted to stating the existence and uniqueness result on BSDEs — Theorem 7, and Section 4 gives the detailed proof of Theorem 7.

2. Notations and some results on DBDEs

First of all, let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space carrying a standard d-dimensional Brownian motion $(B_t)_{t\geq 0}$ and let $(\mathcal{F}_t)_{t\geq 0}$ be the natural σ-algebra filtration generated by $(B_t)_{t\geq 0}$. We assume that $\mathcal{F}_T = \mathcal{F}$ and $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous and complete. In this paper, the Euclidean norm of a vector $y \in \mathbf{R}^k$ will be defined by $|y|$, and for a $k\times d$ matrix z, we define $|z| = \sqrt{Tr(zz^*)}$, where and hereafter z^* represents the transpose of z. Let $\langle x, y\rangle$ represent the inner product of x, $y \in \mathbf{R}^k$.

Let $L^2(\Omega, \mathcal{F}_T, \mathbf{P}; \mathbf{R}^k)$ be the set of $\mathbf{R}^k$-valued and $\mathcal{F}_T$-measurable random variables ξ such that $\|\xi\|^2_{L^2} := \mathbf{E}[|\xi|^2] < +\infty$ and let $\mathcal{S}^2(0, T; \mathbf{R}^k)$ denote the set of $\mathbf{R}^k$-valued, adapted and continuous processes $(Y_t)_{t\in[0,T]}$ such that

$$\|Y\|_{\mathcal{S}^2} := \left(\mathbf{E}\left[\sup_{t\in[0,T]} |Y_t|^2\right]\right)^{1/2} < +\infty.$$

Moreover, let $\mathrm{M}^2(0, T; \mathbf{R}^{k\times d})$ denote the set of (equivalent classes of) $(\mathcal{F}_t)$-progressively measurable $\mathbf{R}^{k\times d}$-valued processes $(Z_t)_{t\in[0,T]}$ such that

$$\|Z\|_{\mathrm{M}^2} := \left(\mathbf{E}\left[\int_0^T |Z_t|^2 \,\mathrm{d}t\right]\right)^{1/2} < +\infty.$$

Obviously, $\mathcal{S}^2(0, T; \mathbf{R}^k)$ is a Banach space and $\mathrm{M}^2(0, T; \mathbf{R}^{k\times d})$ is a Hilbert space.

Finally, let $\mathbf{S}$ be the set of all non-decreasing continuous functions $\rho(\cdot) : \mathbf{R}^+ \mapsto \mathbf{R}^+$ with $\rho(0) = 0$ and $\rho(x) > 0$ for all $x > 0$, where and hereafter $\mathbf{R}^+ := [0, +\infty)$.

As mentioned above, we will deal only with the multidimensional BSDE which is an equation of type (1), where the terminal condition ξ is $\mathcal{F}_T$-measurable, the terminal time T satisfies $0 \leq T \leq +\infty$, and the generator g is $(\mathcal{F}_t)$-progressively measurable for each (y, z). In this paper, we use the following definition.

Definition 1. A pair of processes $(y_t, z_t)_{t\in[0,T]}$ taking values in $\mathbf{R}^k \times \mathbf{R}^{k\times d}$ is called a solution of BSDE (1), if $(y_t, z_t)_{t\in[0,T]}$ belongs to the space $\mathcal{S}^2(0, T; \mathbf{R}^k)$ $\times\mathrm{M}^2(0, T; \mathbf{R}^{k\times d})$ and $\mathrm{d}\mathbf{P}$ − a.s., BSDE (1) holds true for each $t \in [0, T]$.

The following Lemma 2 comes from [10], which will be used later.

Lemma 2. *Let $p \in \mathbf{N}$, $f(\cdot) : \mathbf{R}^p \mapsto \mathbf{R}$ be a continuous and linear-growth function, i.e., there exists a positive constant K such that $|f(x)| \le K(1+|x|)$ for all $x \in \mathbf{R}^p$. Then $f_n(x) = \inf_{y\in\mathbf{R}^p}\{f(y)+n|x-y|\}$, $x \in \mathbf{R}^p$, is well defined for $n \ge K$ and satisfies*

(i) *Linear growth: for each $x \in \mathbf{R}^p$, $|f_n(x)| \le K(1+|x|)$;*
(ii) *Monotonicity in n: for each $x \in \mathbf{R}^p$, $f_n(x)$ increases in n;*
(iii) *Lipschitz continuous: for each x_1, $x_2 \in \mathbf{R}^p$, we have $|f_n(x_1)-f_n(x_2)| \le n|x_1-x_2|$;*
(iv) *Strong convergence: if $x_n \to x$, then $f_n(x_n) \to f(x)$ as $n \to +\infty$.*

In the following, we will establish some propositions on DBDEs with general time intervals, which will play important roles in the proof of our main result.

Proposition 3. *Let $0 \le T \le +\infty$ and $f(t,y) : [0,T]\times\mathbf{R} \mapsto \mathbf{R}$ satisfy the following two assumptions:*

(B1) *there exists a function $u(\cdot) : \mathbf{R}^+ \mapsto \mathbf{R}^+$ with $\int_0^T u(t)\,\mathrm{d}t < +\infty$ such that for each y_1, $y_2 \in \mathbf{R}$ and $t \in [0,T]$,*
$$|f(t,y_1)-f(t,y_2)| \le u(t)|y_1-y_2|;$$
(B2) $\int_0^T |f(t,0)|\,\mathrm{d}t < +\infty$.

Then for each $\delta \in \mathbf{R}$ the following DBDE

$$y_t = \delta + \int_t^T f(s,y_s)\,\mathrm{d}s, \quad t \in [0,T], \tag{2}$$

has a unique continuous solution $(y_t)_{t\in[0,T]}$ such that $\sup_{t\in[0,T]}|y_t| < +\infty$.

Proof. For each $\beta(\cdot) : \mathbf{R}^+ \mapsto \mathbf{R}^+$ such that $\int_0^T \beta(t)\,\mathrm{d}t < +\infty$, let $\mathcal{H}_{\beta(\cdot)}$ denote the set of the continuous functions $(y_t)_{t\in[0,T]}$ such that

$$\|y\|_{\beta(\cdot)} := \left(\sup_{t\in[0,T]}\left[\mathrm{e}^{-\int_t^T \beta(r)\,\mathrm{d}r}|y_t|^2\right]\right)^{1/2} < +\infty.$$

It is easy to verify that $\mathcal{H}_{\beta(\cdot)}$ is a Banach space. Note that for any $y_\cdot \in \mathcal{H}_{\beta(\cdot)}$, in view of (B1) and (B2),

$$\begin{aligned}\int_0^T |f(t,y_t)|\,\mathrm{d}t &\le \int_0^T u(t)|y_t|\,\mathrm{d}t + \int_0^T |f(t,0)|\,\mathrm{d}t\\ &\le \int_0^T u(t)\,\mathrm{d}t\cdot\left[\mathrm{e}^{\int_0^T \beta(r)\,\mathrm{d}r}\|y\|_{\beta(\cdot)}\right] + \int_0^T |f(t,0)|\,\mathrm{d}t < +\infty.\end{aligned}$$

For any $y_\cdot \in \mathcal{H}_{\beta(\cdot)}$, define

$$Y_t := \delta + \int_t^T f(s,y_s)\,\mathrm{d}s, \quad t \in [0,T].$$

Then we have

$$\begin{aligned}\|Y\|_{\beta(\cdot)} &= \sqrt{\sup_{t\in[0,T]}\left[\mathrm{e}^{-\int_t^T \beta(r)\,\mathrm{d}r}|Y_t|^2\right]} \le \sup_{t\in[0,T]}|Y_t| \\ &\le |\delta| + \int_0^T |f(t,y_t)|\,\mathrm{d}t < +\infty.\end{aligned}$$

Thus, we have constructed a mapping $\Phi : \mathcal{H}_{\beta(\cdot)} \mapsto \mathcal{H}_{\beta(\cdot)}$ such that $\Phi(y_t) = Y_t$ for each $t \in [0,T]$. Next we prove that this mapping is strictly contractive when $\beta(\cdot)$ is chosen appropriately.

Take $y^1_\cdot,\ y^2_\cdot \in \mathcal{H}_{\beta(\cdot)}$ and assume that $\Phi(y^1_t) = Y^1_t$, $\Phi(y^2_t) = Y^2_t$ for each $t \in [0,T]$. Let us set $\hat{Y}_\cdot := Y^1_\cdot - Y^2_\cdot$, $\hat{y}_\cdot := y^1_\cdot - y^2_\cdot$. Then we have

$$\begin{aligned}\mathrm{d}\left[\mathrm{e}^{-\int_s^T \beta(r)\,\mathrm{d}r}|\hat{Y}_s|^2\right] &= \mathrm{e}^{-\int_s^T \beta(r)\,\mathrm{d}r}\big(\beta(s)|\hat{Y}_s|^2\,\mathrm{d}s + 2\hat{Y}_s\,\mathrm{d}\hat{Y}_s\big) \\ &= \mathrm{e}^{-\int_s^T \beta(r)\,\mathrm{d}r}\big(\beta(s)|\hat{Y}_s|^2\,\mathrm{d}s - 2\hat{Y}_s(f(s,y^1_s) - f(s,y^2_s))\,\mathrm{d}s\big),\end{aligned}$$

from which it follows that

$$\mathrm{e}^{-\int_t^T \beta(r)\,\mathrm{d}r}|\hat{Y}_t|^2 = \int_t^T \mathrm{e}^{-\int_s^T \beta(r)\,\mathrm{d}r}\left[2\hat{Y}_s\big(f(t,y^1_s) - f(t,y^2_s)\big) - \beta(s)|\hat{Y}_s|^2\right]\mathrm{d}s.$$

By (B1) and the inequality $2ab \le \lambda a^2 + b^2/\lambda$ ($\lambda > 0$) we get that

$$\begin{aligned}\mathrm{e}^{-\int_t^T \beta(r)\,\mathrm{d}r}|\hat{Y}_t|^2 &\le \int_t^T \mathrm{e}^{-\int_s^T \beta(r)\,\mathrm{d}r}\left[2\sqrt{u(s)}|\hat{Y}_s|\sqrt{u(s)}|\hat{y}_s| - \beta(s)|\hat{Y}_s|^2\right]\mathrm{d}s \\ &\le \int_t^T \mathrm{e}^{-\int_s^T \beta(r)\,\mathrm{d}r}\left[\big(\lambda u(s) - \beta(s)\big)|\hat{Y}_s|^2 + \frac{u(s)}{\lambda}|\hat{y}_s|^2\right]\mathrm{d}s,\end{aligned}$$

from which it follows that, with choosing $\lambda > 4\int_0^T u(s)\,\mathrm{d}s$ and $\beta(s) = \lambda u(s)$,

$$\begin{aligned}\|\hat{Y}\|^2_{\beta(\cdot)} &\le \frac{1}{\lambda}\int_0^T \mathrm{e}^{-\int_s^T \beta(r)\,\mathrm{d}r}u(s)|\hat{y}_s|^2\,\mathrm{d}s \\ &\le \frac{1}{\lambda}\sup_{t\in[0,T]}\left[\mathrm{e}^{-\int_t^T \beta(r)\,\mathrm{d}r}|\hat{y}_t|^2\right]\int_0^T u(s)\,\mathrm{d}s < \frac{1}{4}\|\hat{y}\|^2_{\beta(\cdot)}.\end{aligned}$$

So $\|\hat{Y}\|_{\beta(\cdot)} < \frac{1}{2}\|\hat{y}\|_{\beta(\cdot)}$, which implies that Φ is a contractive mapping from $\mathcal{H}_{\beta(\cdot)}$ to $\mathcal{H}_{\beta(\cdot)}$. Then the conclusion follows from the fixed point theorem immediately. $\square$

From the proof of Proposition 3, we can directly obtain the following Proposition 4.

Proposition 4. *Assume* $0 \le T \le +\infty$, f *satisfies* (B1) *and* (B2), $(y_t)_{t\in[0,T]}$ *is the unique continuous solution of DBDE* (2) *such that* $\sup_{t\in[0,T]}|y_t| < +\infty$,

C is an arbitrary constant and y_t^n is defined recursively as follows, for each $n \in \mathbf{N}$ and $\delta \in \mathbf{R}$,

$$y_t^1 = C; \quad y_t^{n+1} = \delta + \int_t^T f(s, y_s^n)\, \mathrm{d}s, \quad t \in [0, T].$$

Then $y_t^n \to y_t$ as $n \to +\infty$ for each $t \in [0, T]$.

Proposition 5 illustrates the comparison theorem for solutions of DBDE with general time intervals.

Proposition 5. *Let $0 \le T \le +\infty$, f and f' satisfy* (B1)–(B2), *$(y_t)_{t\in[0,T]}$ and $(y'_t)_{t\in[0,T]}$ with $\sup_{t\in[0,T]} \big(|y_t| + |y'_t|\big) < +\infty$ satisfy respectively DBDE* (2) *and the following DBDE, for some $\delta' \in \mathbf{R}$,*

$$y'_t = \delta' + \int_t^T f'(s, y'_s)\, \mathrm{d}s, \quad t \in [0, T].$$

Assume that $f(t, y'_t) \ge f'(t, y'_t)$ for each $t \in [0, T]$. Then, we have

(i) (*Comparison theorem*) *if $\delta \ge \delta'$, then $y_t \ge y'_t$ for each $t \in [0, T]$;*

(ii) (*Strict comparison theorem*) *if $\delta > \delta'$, then $y_t > y'_t$ for each $t \in [0, T]$.*

Proof. For each $t \in [0, T]$, let us set

$$a(t) := \begin{cases} \dfrac{f(t, y_t) - f(t, y'_t)}{y_t - y'_t}, & y_t \ne y'_t; \\ 0, & y_t = y'_t, \end{cases}$$

$b(t) := f(t, y'_t) - f'(t, y'_t)$ and $\hat{y}_t := y_t - y'_t$. From (B1) and (B2) we can deduce that $\int_0^T |a(s)|\, \mathrm{d}s \le \int_0^T u(s)\, \mathrm{d}s < +\infty$ and $\int_0^T b(s)\, \mathrm{d}s < +\infty$. Then we have

$$\begin{aligned} \hat{y}_t &= \delta - \delta' + \int_t^T \big(f(s, y_s) - f(s, y'_s) + f(s, y'_s) - f'(s, y'_s)\big)\, \mathrm{d}s \\ &= \delta - \delta' + \int_t^T a(s)\hat{y}_s\, \mathrm{d}s + \int_t^T b(s)\, \mathrm{d}s, \quad t \in [0, T]. \end{aligned}$$

As a result, in view of the conditions that $\delta \ge \delta'$ and $b(t) \ge 0$ for each $t \in [0, T]$,

$$\hat{y}_t = \mathrm{e}^{\int_t^T a(s)\, \mathrm{d}s} \left[\delta - \delta' + \int_t^T b(s) \mathrm{e}^{\int_s^T a(r)\, \mathrm{d}r}\, \mathrm{d}s\right] \ge 0, \quad t \in [0, T].$$

Furthermore, if $\delta > \delta'$, then $\hat{y}_t > 0$ for each $t \in [0, T]$. The proof is then completed. □

Proposition 6. *Let $0 \le T \le +\infty$, $u(\cdot)$ be defined in* (B1) *and $\varphi(\cdot) : \mathbf{R}^+ \mapsto \mathbf{R}^+$ be a continuous function such that $\varphi(x) \le ax + b$ for all $x \in \mathbf{R}^+$, where a and b are two given nonnegative constants. Then for each $\delta \in \mathbf{R}^+$, the following DBDE*

$$y_t^\delta = \delta + \int_t^T u(s)\varphi(y_s^\delta)\, \mathrm{d}s, \quad t \in [0, T], \tag{3}$$

has a solution $(y_t^\delta)_{t\in[0,T]}$ *such that* $\sup_{t\in[0,T]}|y_t^\delta|<+\infty$. *In addition,*

(i) *if* $\delta>0$ *and* $\varphi(x)>0$ *for all* $x>0$*, then DBDE* (3) *has a unique solution*;

(ii) *if* $\delta=0$ *and* $\varphi(\cdot)\in\mathbf{S}$ *with* $\int_{0+}\varphi^{-1}(x)\,\mathrm{d}x=+\infty$*, then DBDE* (3) *has a unique solution* $y_t\equiv 0$.

Proof. Let $\varphi_n(x):=\inf_{y\in\mathbf{R}}\{\overline{\varphi}(y)+n|x-y|\}$ with $\overline{\varphi}(y)=\varphi(|y|)$ for $y\in\mathbf{R}$. Then it follows from Lemma 2 and Proposition 3 that $\varphi_n(x)$ is well defined on $\mathbf{R}$ for each $n\ge a$, and for each $n\ge a$, the following two DBDEs

$$y_t^{n,\delta}=\delta+\int_t^T u(s)\varphi_n(y_s^{n,\delta})\,\mathrm{d}s,\quad t\in[0,T] \tag{4}$$

and

$$\overline{y}_t^\delta=\delta+\int_t^T\big(au(s)\overline{y}_s^\delta+bu(s)\big)\,\mathrm{d}s,\quad t\in[0,T]$$

have, respectively, unique solutions $y_t^{n,\delta}$ and $\overline{y}_t^\delta$ with $\sup_{t\in[0,T]}|y_t^{n,\delta}|<+\infty$ and $\sup_{t\in[0,T]}|\overline{y}_t^\delta|<+\infty$. Clearly, $y_t^{n,\delta}\ge 0$ and $\overline{y}_t^\delta\ge 0$ for each $t\in[0,T]$ and $n\ge a$. By Proposition 5 and the fact that $\varphi_n\le\varphi_{n+1}$, we have $y^{n,\delta}\le y^{n+1,\delta}\le\overline{y}^\delta$. Therefore, for each $t\in[0,T]$, the limit of the sequence $\{y_t^{n,\delta}\}_{n=a}^{+\infty}$ must exist, we denote it by y_t^δ. In view of (i) and (iv) in Lemma 2, using Lebesgue's dominated convergence theorem we can obtain that

$$\lim_{n\to+\infty}\int_0^T u(s)\varphi_n(y_s^{n,\delta})\,\mathrm{d}s=\int_0^T u(s)\overline{\varphi}(y_s^\delta)\,\mathrm{d}s=\int_0^T u(s)\varphi(y_s^\delta)\,\mathrm{d}s.$$

Thus, by passing to the limit in both sides of DBDE (4) we deduce that

$$y_t^\delta=\delta+\int_t^T u(s)\varphi(y_s^\delta)\,\mathrm{d}s,\quad t\in[0,T],$$

which means that y_t^δ is a solution of DBDE (3).

Let us now suppose that $\delta>0$ and $\varphi(x)>0$ for all $x>0$. For each $z\ge\delta$, set $G(z):=\int_z^1\varphi^{-1}(x)\,\mathrm{d}x$. It is easy to see that $-\infty=G(+\infty)<G(z_1)<G(z_2)<G(\delta)$ for each $z_1>z_2>\delta$. Then the inverse function of $G(z)$ must exist, we denote it by $G^{-1}(u)$ for $u\le G(\delta)$. Let y_t^δ be a solution of DBDE (3). It is obvious that $y_t^\delta\ge\delta$ and $\mathrm{d}G(y_t^\delta)=u(t)\,\mathrm{d}t$. Hence, $G(y_T^\delta)-G(y_t^\delta)=\int_t^T u(s)\,\mathrm{d}s$, which implies that $y_t^\delta=G^{-1}\big(G(\delta)-\int_t^T u(s)\,\mathrm{d}s\big)$ for each $t\in[0,T]$. The proof of (i) is then completed.

Finally, we prove that (ii) is also right. Assume that $\delta=0$ and $\varphi(\cdot)\in\mathbf{S}$ with $\int_{0+}\varphi^{-1}(x)\,\mathrm{d}x=+\infty$. For each $z>0$, set $H(z):=\int_z^1\varphi^{-1}(x)\,\mathrm{d}x$. It is clear that $-\infty=H(+\infty)<H(z_1)<H(z_2)<H(0)=+\infty$ for each $z_1>z_2>0$. Then the inverse function of $H(z)$ must exist, we denote it by $H^{-1}(u)$ for each $u\in\mathbf{R}$. Now let y_t^0 be a continuous solution of DBDE (3). Then $y_t^0\ge 0$ and

$\mathrm{d}H(y_t^0) = u(t)\,\mathrm{d}t$. Hence, for each $0 \le t \le t_1 < T$,

$$y_t^0 = H^{-1}\left(H(y_{t_1}^0) - \int_t^{t_1} u(s)\,\mathrm{d}s\right).$$

Furthermore, noticing that $H(y_{t_1}^0) \to H(y_T^0) = H(0) = +\infty$ as $t_1 \to T$ and $H^{-1}(+\infty) = 0$, we know that $y_t^0 = 0$ for each $t \le T$. Therefore, DBDE (3) has a unique solution $y_t \equiv 0$. □

3. Main result

In this section, we will state the main result of this paper. Let us first introduce the following assumptions with respect to the generator g of BSDE (1), where $0 \le T \le +\infty$.

- (H1) g is uniformly continuous in y non-uniformly with respect to t, i.e., there exist a deterministic function $u(\cdot) : [0,T] \mapsto \mathbf{R}^+$ with $\int_0^T u(t)\,\mathrm{d}t < +\infty$ and a linear-growth function $\rho(\cdot) \in \mathbf{S}$ such that $\mathrm{d}\mathbf{P} \times \mathrm{d}t -$ a.e., for each $y_1,\ y_2 \in \mathbf{R}^k$ and $z \in \mathbf{R}^{k\times d}$,
$$|g(\omega,t,y_1,z) - g(\omega,t,y_2,z)| \le u(t)\rho(|y_1 - y_2|);$$
Furthermore, we assume that $\int_{0^+} \rho^{-1}(u)\,\mathrm{d}u = +\infty$;
- (H2) g is uniformly continuous in z non-uniformly with respect to t, i.e., there exist a deterministic function $v(\cdot) : [0,T] \mapsto \mathbf{R}^+$ with $\int_0^T \big(v(t) + v^2(t)\big)\,\mathrm{d}t < +\infty$ and a linear-growth function $\phi(\cdot) \in \mathbf{S}$ such that $\mathrm{d}\mathbf{P} \times \mathrm{d}t -$ a.e., for each $y \in \mathbf{R}^k$ and $z_1,\ z_2 \in \mathbf{R}^{k\times d}$,
$$|g(\omega,t,y,z_1) - g(\omega,t,y,z_2)| \le v(t)\phi(|z_1 - z_2|);$$
- (H3) For any $i = 1,\dots,k$, $g_i(\omega,t,y,z)$, the ith component of g, depends only on the ith row of z;
- (H4) $\mathbf{E}\left[\left(\int_0^T |g(\omega,t,0,0)|\,\mathrm{d}t\right)^2\right] < +\infty$.

In the sequel, we denote the linear-growth constant for $\rho(\cdot)$ and $\phi(\cdot)$ in (H1) and (H2) by $A > 0$, i.e., $\rho(x) \le A(1+x)$ and $\phi(x) \le A(1+x)$ for all $x \in \mathbf{R}^+$. In the remaining of this paper, we put an i at upper left of $y \in \mathbf{R}^k$, $z \in \mathbf{R}^{k\times d}$ to represent the ith component of y and the ith row of z, like ${}^i y$ and ${}^i z$.

The main result of this paper is the following Theorem 7, whose proof will be given in the next section.

Theorem 7. *Assume that $0 \le T \le +\infty$ and g satisfies* (H1)–(H4). *Then for each $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbf{P}; \mathbf{R}^k)$, BSDE* (1) *has a unique solution.*

Remark 8. In the corresponding assumptions in [8] and [6] the $u(t)$, $v(t)$ appearing in (H1) and (H2) are bounded by a constant $c > 0$, and T is a finite real number. However, in our framework the $u(t)$, $v(t)$ may be unbounded. In addition, Theorem 7 also considers the case of $T = +\infty$. Therefore, Theorem 7 generalizes the corresponding results in [8] and [6].

Example 9. Let $0 \le T \le +\infty$, and for each $i = 1, \dots, k$ and $(\omega, t, y, z) \in \Omega \times [0, T] \times \mathbf{R}^k \times \mathbf{R}^{k\times d}$, define the generator $g = (g_1, \dots, g_k)$ by

$$g_i(\omega, t, y, z) = f_1(t)\big(h(|y|) + 1\big) + f_2(t)\sqrt{|{}^i z|} + |B_t(\omega)|,$$

where

$$\begin{aligned} f_1(t) &= \frac{1}{\sqrt{t}}\mathbf{1}_{0<t<\delta} + \frac{1}{\sqrt{1+t^2}}\mathbf{1}_{\delta\le t\le T}, \\ f_2(t) &= \frac{1}{\sqrt[4]{t}}\mathbf{1}_{0<t<\delta} + \frac{1}{(1+t)^2}\mathbf{1}_{\delta\le t\le T}, \\ h(x) &= x\ln\frac{1}{x}\mathbf{1}_{0\le x\le\delta} + [h'(\delta-)(x-\delta) + h(\delta)]\mathbf{1}_{x>\delta}, \end{aligned}$$

with δ small enough. Since $h(0) = 0$ and h is concave and increasing, we have $h(x_1 + x_2) \le h(x_1) + h(x_2)$ for all $x_1,\ x_2 \in \mathbf{R}^+$, which implies that $|h(x_1) - h(x_2)| \le h(|x_1 - x_2|)$. Thus, note that

$$\int_{0^+} \frac{1}{x\ln\frac{1}{x}}\,\mathrm{d}x = +\infty.$$

We know that the generator g satisfies assumptions (H1)–(H4) with $u(t) = f_1(t)$, $v(t) = f_2(t)$. It then follows from Theorem 7 that for each $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbf{P}; \mathbf{R}^k)$, BSDE (1) has a unique solution $(y_t, z_t)_{t\in[0,T]}$.

It should be mentioned that the above conclusion can not be obtained by the result of [8], [6] and other existing results.

4. Proof of the main result

This section will give the proof of our main result — Theorem 7. Before starting the proof, let us first introduce the following Lemma 10, which comes from Lemma 1.1 and Theorem 1.2 in [3]. Note that Lemma 10 remains valid in the multidimensional case since their arguments are done via a standard contraction combined with a priori estimates without using the comparison theorem. The following assumption will be used in Lemma 10, where we suppose $0 \le T \le +\infty$:

(A1) There exist two deterministic functions $u(\cdot), v(\cdot) : [0, T] \mapsto \mathbf{R}^+$ with $\int_0^T \big(u(t) + v^2(t)\big)\,\mathrm{d}t < +\infty$ such that $\mathrm{d}\mathbf{P} \times \mathrm{d}t$ − a.e., for each y_1, $y_2 \in \mathbf{R}^k$ and $z_1,\ z_2 \in \mathbf{R}^{k\times d}$,

$$|g(t, y_1, z_1) - g(t, y_2, z_2)| \le u(t)|y_1 - y_2| + v(t)|z_1 - z_2|.$$

Lemma 10 (Theorem 1.2 in [3]). *Assume that* $0 \le T \le +\infty$ *and* g *satisfies* (A1) *and* (H4). *Then for each* $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbf{P}; \mathbf{R}^k)$, *BSDE* (1) *has a unique solution* $(y_t, z_t)_{t\in[0,T]}$.

4.1. Proof of the uniqueness part of Theorem 7

The idea of the proof of this part is partly motivated by [6]. Let $(y_t^1, z_t^1)_{t\in[0,T]}$ and $(y_t^2, z_t^2)_{t\in[0,T]}$ be two solutions of BSDE (1). Then we have the following Lemma 11.

Lemma 11. *The process $(y_t^1 - y_t^2)_{t\in[0,T]}$ is uniformly bounded, i.e., there exists a positive constant $C_1 > 0$ such that*

$$\mathrm{d}\mathbf{P} \times \mathrm{d}t - \text{a.e.}, \quad |y_t^1 - y_t^2| \le C_1. \tag{5}$$

Moreover, for each $n \in \mathbf{N}$, $i = 1, 2, \ldots, k$ and $0 \le r \le t \le T$, we have

$$\mathbf{E}^{n,i}\left[|{}^i y_t^1 - {}^i y_t^2| \middle| \mathcal{F}_r\right] \le a_n + \int_t^T \mathbf{E}^{n,i}\left[u(s)\rho(|y_s^1 - y_s^2|)\middle|\mathcal{F}_r\right] \mathrm{d}s, \tag{6}$$

where

$$a_n = \phi\left(\frac{2A}{n + 2A}\right) \int_0^T v(s)\,\mathrm{d}s,$$

and $\mathbf{E}^{n,i}[X|\mathcal{F}_t]$ represents the conditional expectation of random variable X with respect to $\mathcal{F}_t$ under a probability measure $\mathbf{P}^{n,i}$ on $(\Omega, \mathcal{F})$, which depends on n and i, and which is absolutely continuous with respect to $\mathbf{P}$.

Proof. Using Itô's formula to $|y_t^1 - y_t^2|^2$ we arrive, for each $t \in [0, T]$, at

$$\begin{aligned} |y_t^1 - y_t^2|^2 + \int_t^T |z_s^1 - z_s^2|^2\,\mathrm{d}s &= 2\int_t^T \langle y_s^1 - y_s^2, g(s, y_s^1, z_s^1) - g(s, y_s^2, z_s^2)\rangle\,\mathrm{d}s \\ &\quad - 2\int_t^T \langle y_s^1 - y_s^2, (z_s^1 - z_s^2)\,\mathrm{d}B_s\rangle. \end{aligned} \tag{7}$$

The inner product term including g can be enlarged by (H1)–(H2) and the basic inequality $2ab \le 2a^2 + b^2/2$ as follows:

$$\begin{aligned} &2\langle y_s^1 - y_s^2, g(s, y_s^1, z_s^1) - g(s, y_s^2, z_s^2)\rangle \\ &\le 2|y_s^1 - y_s^2||g(s, y_s^1, z_s^1) - g(s, y_s^2, z_s^1) + g(s, y_s^2, z_s^1) - g(s, y_s^2, z_s^2)| \\ &\le 2|y_s^1 - y_s^2|\big(Au(s)|y_s^1 - y_s^2| + Av(s)|z_s^1 - z_s^2| + Au(s) + Av(s)\big) \\ &\le B(s)|y_s^1 - y_s^2|^2 + \frac{1}{2}|z_s^1 - z_s^2|^2 + Au(s) + Av(s), \end{aligned}$$

where $B(s) = 2Au(s) + 2A^2v^2(s) + Au(s) + Av(s)$. Putting the previous inequality into (7) we can obtain that for each $t \in [0, T]$,

$$|y_t^1 - y_t^2|^2 \le \int_t^T B(s)|y_s^1 - y_s^2|^2\,\mathrm{d}s - 2\int_t^T \langle y_s^1 - y_s^2, (z_s^1 - z_s^2)\,\mathrm{d}B_s\rangle + C,$$

where $C=\int_0^T A(u(s)+v(s))\,\mathrm{d}s$. Note that both $(y_t^1,z_t^1)_{t\in[0,T]}$ and $(y_t^2,z_t^2)_{t\in[0,T]}$ belong to the process space $\mathcal{S}^2(0,T;\mathbf{R}^k)\times \mathrm{M}^2(0,T;\mathbf{R}^{k\times d})$. By the Burkholder-Davis-Gundy (BDG for short in the remaining) inequality and Hölder's inequality we have that there exists a positive constant $K'>0$ such that

$$\begin{aligned}
&\mathbf{E}\left[\sup_{t\in[0,T]}\left|\int_0^t \langle y_s^1-y_s^2,(z_s^1-z_s^2)\,\mathrm{d}B_s\right|\right]\\
&\le K'\mathbf{E}\left[\sqrt{\int_0^T |y_s^1-y_s^2|^2|z_s^1-z_s^2|^2\,\mathrm{d}s}\right]\\
&\le K'\sqrt{\mathbf{E}\left[\sup_{t\in[0,T]}|y_t^1-y_t^2|^2\right]}\sqrt{\mathbf{E}\left[\int_0^T |z_s^1-z_s^2|^2\,\mathrm{d}s\right]}<+\infty,
\end{aligned}$$

which implies that $(\int_0^t\langle y_s^1-y_s^2,(z_s^1-z_s^2)\,\mathrm{d}B_s\rangle)_{t\in[0,T]}$ is an $(\mathcal{F}_t,\mathbf{P})$-martingale. Then we have that for each $0\le r\le t\le T$,

$$\mathbf{E}\left[|y_t^1-y_t^2|^2\middle|\mathcal{F}_r\right]\le\int_t^T B(s)\mathbf{E}\left[|y_s^1-y_s^2|^2\middle|\mathcal{F}_r\right]\,\mathrm{d}s+C.$$

By Lemma 4 in [7] we have

$$\mathbf{E}\left[|y_t^1-y_t^2|^2\middle|\mathcal{F}_r\right]\le C\mathrm{e}^{\int_0^T B(s)\,\mathrm{d}s}:=(C_1)^2,$$

which yields (5) after taking $r=t$.

In the sequel, by (H3) we have that for each $t\in[0,T]$,

$${}^iy_t^1-{}^iy_t^2=\int_t^T\left(g_i(s,y_s^1,{}^iz_s^1)-g_i(s,y_s^2,{}^iz_s^2)\right)\mathrm{d}s-\int_t^T({}^iz_s^1-{}^iz_s^2)\,\mathrm{d}B_s.$$

Then, (H3) and Tanaka's formula lead to that, for each $t\in[0,T]$,

$$\begin{aligned}
|{}^iy_t^1-{}^iy_t^2|\le&\int_t^T \mathrm{sgn}({}^iy_s^1-{}^iy_s^2)\left(g_i(s,y_s^1,{}^iz_s^1)-g_i(s,y_s^2,{}^iz_s^2)\right)\mathrm{d}s\\
&-\int_t^T\mathrm{sgn}({}^iy_s^1-{}^iy_s^2)({}^iz_s^1-{}^iz_s^2)\,\mathrm{d}B_s.
\end{aligned}\tag{8}$$

Furthermore, it follows from (H1) and (H2) that

$$|g_i(s,y_s^1,{}^iz_s^1)-g_i(s,y_s^2,{}^iz_s^2)|\le u(s)\rho(|y_s^1-y_s^2|)+v(s)\phi(|{}^iz_s^1-{}^iz_s^2|).\tag{9}$$

Recalling that $\phi(\cdot)$ is a non-decreasing function from $\mathbf{R}^+$ to itself with at most linear-growth. From [6] we know that for each $n\in\mathbf{N}$ and $x\in\mathbf{R}^+$,

$$\phi(x)\le(n+2A)x+\phi\left(\frac{2A}{n+2A}\right).\tag{10}$$

Thus, combining (8)–(10) we get that for each $n\in\mathbf{N}$,

$$|{}^iy_t^1-{}^iy_t^2|\le\phi\left(\frac{2A}{n+2A}\right)\int_0^T v(s)\,\mathrm{d}s$$

$$+\int_t^T \big(u(s)\rho(|y_s^1-y_s^2|)+(n+2A)v(s)|{}^iz_s^1-{}^iz_s^2|\big)\,\mathrm{d}s$$
$$-\int_t^T \operatorname{sgn}({}^iy_s^1-{}^iy_s^2)({}^iz_s^1-{}^iz_s^2)\,\mathrm{d}B_s,\quad t\in[0,T].$$

Now for each $t\in[0,T]$, let

$$e_t^{n,i}:=(n+2A)\frac{\operatorname{sgn}({}^iy_t^1-{}^iy_t^2)({}^iz_t^1-{}^iz_t^2)^*}{|{}^iz_t^1-{}^iz_t^2|}\mathbf{1}_{|{}^iz_t^1-{}^iz_t^2|\neq 0}.$$

Then, $(e_t^{n,i})_{t\in[0,T]}$ is a $\mathbf{R}^d$-valued, bounded and $(\mathcal{F}_t)$-adapted process. It follows from Girsanov's theorem that $B_t^{n,i}=B_t-\int_0^t e_s^{n,i}v(s)\,\mathrm{d}s$, $t\in[0,T]$, is a d-dimensional Brownian motion under the probability $\mathbf{P}^{n,i}$ on $(\Omega,\mathcal{F})$ defined by

$$\frac{\mathrm{d}\mathbf{P}^{n,i}}{\mathrm{d}\mathbf{P}}=\exp\left\{\int_0^T v(s)(e_s^{n,i})^*\,\mathrm{d}B_s-\frac{1}{2}\int_0^T v^2(s)|e_s^{n,i}|^2\,\mathrm{d}s\right\}.$$

Thus, for each $n\in\mathbf{N}$ and $t\in[0,T]$,

$$\begin{aligned}|{}^iy_t^1-{}^iy_t^2|\leq{}&\phi\left(\frac{2A}{n+2A}\right)\int_0^T v(s)\,\mathrm{d}s+\int_t^T u(s)\rho(|y_s^1-y_s^2|)\,\mathrm{d}s\\&-\int_t^T \operatorname{sgn}({}^iy_s^1-{}^iy_s^2)({}^iz_s^1-{}^iz_s^2)\,\mathrm{d}B_s^{n,i}.\end{aligned}\tag{11}$$

Moreover, the process $\big(\int_0^t \operatorname{sgn}({}^iy_s^1-{}^iy_s^2)({}^iz_s^1-{}^iz_s^2)\,\mathrm{d}B_s^{n,i}\big)_{t\in[0,T]}$ is an $(\mathcal{F}_t,\mathbf{P}^{n,i})$-martingale. In fact, let $\mathbf{E}^{n,i}[X]$ represent the expectation of the random variable X under $\mathbf{P}^{n,i}$. By the BDG inequality and Hölder's inequality we know that there exists a positive constant $K''>0$ such that for each $n\in\mathbf{N}$,

$$\begin{aligned}&\mathbf{E}^{n,i}\left[\sup_{t\in[0,T]}\left|\int_0^t \operatorname{sgn}({}^iy_s^1-{}^iy_s^2)({}^iz_s^1-{}^iz_s^2)\,\mathrm{d}B_s^{n,i}\right|\right]\\&\leq K''\mathbf{E}^{n,i}\left[\sqrt{\int_0^T |{}^iz_s^1-{}^iz_s^2|^2\,\mathrm{d}s}\right]\\&\leq K''\sqrt{\mathbf{E}\left[\left(\frac{\mathrm{d}\mathbf{P}^{n,i}}{\mathrm{d}\mathbf{P}}\right)^2\right]}\sqrt{\mathbf{E}\left[\int_0^T |{}^iz_s^1-{}^iz_s^2|^2\,\mathrm{d}s\right]}<+\infty.\end{aligned}$$

Thus, for each $n\in\mathbf{N}$ and $0\leq r\leq t\leq T$, by taking the condition expectation with respect to $\mathcal{F}_r$ under $\mathbf{P}^{n,i}$ in both sides of (11), we can get the desired result (6). The proof of Lemma 11 is complete. □

In the sequel, we can prove the uniqueness part of Theorem 7. First, let $\overline{\rho}(y)=\rho(|y|)$ for each $y\in\mathbf{R}$, and for each $n\in\mathbf{N}$, define $\rho_n(\cdot):\mathbf{R}\mapsto\mathbf{R}^+$ by

$$\rho_n(x)=\sup_{y\in\mathbf{R}}\{\overline{\rho}(y)-n|x-y|\}.$$

It follows from Lemma 2 that ρ_n is well defined for $n \geq A$, Lipschitz continuous, non-increasing in n and converges to $\overline{\rho}$. Then, for each $n \geq A$, by Proposition 3 we can let f_t^n be the unique solution of the following DBDE

$$f_t^n = a_n + \int_t^T [u(s)\rho_n(k \cdot f_s^n)]\,\mathrm{d}s, \quad t \in [0,T]. \tag{12}$$

Noticing that ρ_n and a_n are both decreasing in n, we have $0 \leq f_t^{n+1} \leq f_t^n$ for each $n \geq A$ by Proposition 5, which implies that the sequence $\{f_t^n\}_{n=1}^{+\infty}$ converges point wisely to a function f_t. Thus, by sending $n \to +\infty$ in (12), it follows from Lemma 2 and the Lebesgue dominated convergence theorem that

$$f_t = \int_t^T \big(u(s)\overline{\rho}(k \cdot f_s)\big)\,\mathrm{d}s = \int_t^T \big(u(s)\rho(k \cdot f_s)\big)\,\mathrm{d}s, \quad t \in [0,T].$$

Recalling that $\rho(\cdot) \in \mathbf{S}$ and $\int_{0^+} \rho^{-1}(u)\,\mathrm{d}u = +\infty$, Proposition 6 yields that $f_t \equiv 0$.

Now, for each $n \geq A$, $j \geq 1$ and $t \in [0,T]$, let $f_t^{n,j}$ be the function defined recursively as follows:

$$f_t^{n,1} := C_1; \qquad f_t^{n,j+1} := a_n + \int_t^T \big(u(s)\rho_n(k \cdot f_s^{n,j})\big)\,\mathrm{d}s, \tag{13}$$

where C_1 is defined in (5). Noticing that ρ_n is Lipschitz continuous, by Proposition 4 we know that $f_t^{n,j}$ converges point wisely to f_t^n as $j \to +\infty$ for each $t \in [0,T]$ and $n \geq A$.

On the other hand, it is easy to check by induction that for each $n \geq A$, $j \geq 1$ and $i = 1, \ldots, k$,

$$|{}^i y_t^1 - {}^i y_t^2| \leq f_t^{n,j} \leq f_0^{n,j}, \quad t \in [0,T]. \tag{14}$$

Indeed, (14) holds true for $j = 1$ due to (5). Suppose (14) holds true for $j \geq 1$. Then, for each $t \in [0,T]$,

$$u(t)\rho(|y_t^1 - y_t^2|) \leq u(t)\rho(k \cdot f_t^{n,j}) \leq u(t)\rho_n(k \cdot f_t^{n,j}).$$

In view of (6) with $r = t$ as well as (13), we can deduce that for each $n \geq A$ and $i = 1, 2, \ldots, k$,

$$|{}^i y_t^1 - {}^i y_t^2| \leq f_t^{n,j+1} \leq f_0^{n,j+1}, \quad t \in [0,T],$$

which is the desired result.

Finally, by sending first $j \to +\infty$ and then $n \to +\infty$ in (14), we obtain that $\sup_{t\in[0,T]} |{}^i y_t^1 - {}^i y_t^2| = 0$ for each $i = 1, 2, \ldots, k$. That is, the solution of BSDE (1) is unique. The proof of the uniqueness part is then completed.

4.2. Proof of the existence part of Theorem 7

The idea of the proof of this part is enlightened by [8]. But some different arguments are used, and then the proof procedure is simplified at certain degree.

Assume that the generator g satisfies (H1)–(H4) and $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbf{P}; \mathbf{R}^k)$. Without loss of generality, we assume that $u(t)$ and $v(t)$ in (H1) and (H2) are both strictly positive functions. Otherwise, we can use $u(t)+\mathrm{e}^{-t}$ and $v(t)+\mathrm{e}^{-t}$ instead of them respectively.

We have the following Lemma 12.

Lemma 12. *Let g satisfy* (H1)–(H3)*, and assume that $u(t)>0$ and $v(t)>0$ for each $t\in[0,T]$. Then there exists a generator sequence $\{g^n\}_{n=1}^{+\infty}$ such that*

(i) *For each $n\in\mathbf{N}$, $g^n(t,y,z)$ is a mapping from $\Omega\times[0,T]\times\mathbf{R}^k\times\mathbf{R}^{k\times d}$ into $\mathbf{R}^k$ and is $(\mathcal{F}_t)$-progressively measurable. Moreover, we have* $\mathrm{d}\mathbf{P}\times\mathrm{d}t-$a.e.*, for each $y\in\mathbf{R}^k$ and $z\in\mathbf{R}^{k\times d}$,*

$$|g^n(t,y,z)|\le|g(t,0,0)|+kAu(t)(1+|y|)+kAv(t)(1+|z|);$$

(ii) *For each $n\in\mathbf{N}$, $g^n(t,y,z)$ satisfies* (H3)*, and* $\mathrm{d}\mathbf{P}\times\mathrm{d}t-$a.e.*, for each $y_1,\ y_2\in\mathbf{R}^k$ and $z_1,\ z_2\in\mathbf{R}^{k\times d}$, we have*

$$|g^n(t,y_1,z_1)-g^n(t,y_2,z_2)|\le ku(t)\rho(|y_1-y_2|)+kv(t)\phi(|z_1-z_2|),$$
$$|g^n(t,y_1,z_1)-g^n(t,y_2,z_2)|\le k(n+A)\big(u(t)|y_1-y_2|+v(t)|z_1-z_2|\big);$$

(iii) *For each $n\in\mathbf{N}$, there exists a non-increasing deterministic functions sequence $b_n(\cdot):[0,T]\mapsto\mathbf{R}^+$ with $\int_0^T b_n(t)\,\mathrm{d}t\to 0$ as $n\to+\infty$ such that* $\mathrm{d}\mathbf{P}\times\mathrm{d}t-$a.e.*, for each $y\in\mathbf{R}^k$ and $z\in\mathbf{R}^{k\times d}$,*

$$|g^n(t,y,z)-g(t,y,z)|\le kb_n(t).$$

Proof. For each $i=1,\dots,k$, by (H1)–(H3) we deduce that $\mathrm{d}\mathbf{P}\times\mathrm{d}t-$a.e., for each $n\in\mathbf{N}$, $y,\ p\in\mathbf{R}^k$ and $z,\ q\in\mathbf{R}^{k\times d}$,

$$\begin{aligned}&g_i(t,p,{}^iq)+(n+A)\big(u(t)|p-y|+v(t)|{}^iq-{}^iz|\big)\\ \ge\ &g_i(t,0,0)-\big(Au(t)(1+|p|)+Av(t)(1+|{}^iq|)\big)\\ &\ +Au(t)(|p|-|y|)+Av(t)(|{}^iq|-|{}^iz|)\\ \ge\ &g_i(t,0,0)-Au(t)(1+|y|)-Av(t)(1+|{}^iz|).\end{aligned}\tag{15}$$

Thus, for each $n\in\mathbf{N}$, $i=1,\dots,k$, $y\in\mathbf{R}^k$ and $z\in\mathbf{R}^{k\times d}$, we can define the following $(\mathcal{F}_t)$-progressively measurable function:

(16)
$$g_i^n(t,y,z)=\inf_{(p,q)\in\mathbf{R}^k\times\mathbf{R}^{k\times d}}\left\{g_i(t,p,{}^iq)+(n+A)\big(u(t)|p-y|+v(t)|{}^iq-{}^iz|\big)\right\},$$

and it depends only on the ith row of z. Obviously, $g_i^n(t,y,{}^iz)\le g_i(t,y,{}^iz)$, and it follows from (15) that

$$g_i^n(t,y,{}^iz)\ge-|g_i(t,0,0)|-Au(t)(1+|y|)-Av(t)(1+|{}^iz|).$$

Hence, for each $n\in\mathbf{N}$, g_i^n is well defined and (i) holds true with setting $g^n:=(g_1^n,g_2^n,\dots,g_k^n)$.

Furthermore, it follows from (16) that

$$g_i^n(t,y,{}^iz)=\inf_{(\overline{p},\overline{q})\in\mathbf{R}^k\times\mathbf{R}^{k\times d}}\left\{g_i(t,y-\overline{p},{}^iz-{}^i\overline{q})+(n+A)\big(u(t)|\overline{p}|+v(t)|{}^i\overline{q}|\big)\right\}.$$

Thus, in view of (H1)–(H2) and the following basic inequality

$$\left|\inf_{x\in D} f_1(x) - \inf_{x\in D} f_2(x)\right| \le \sup_{x\in D} |f_1(x) - f_2(x)|, \tag{17}$$

we have, $\mathrm{d}\mathbf{P}\times \mathrm{d}t-$a.e., for each $n\in\mathbf{N}$, $i=1,\dots,k$, y_1, $y_2\in\mathbf{R}^k$ and z_1, $z_2\in\mathbf{R}^{k\times d}$,

$$\begin{aligned}
&|g_i^n(t,y_1,{}^iz_1)-g_i^n(t,y_2,{}^iz_2)|\\
\le&\sup_{(\overline{p},\overline{q})\in\mathbf{R}^k\times\mathbf{R}^{k\times d}}|g_i(t,y_1-\overline{p},{}^iz_1-{}^i\overline{q})-g_i(t,y_2-\overline{p},{}^iz_2-{}^i\overline{q})|\\
\le&\, u(t)\rho(|y_1-y_2|)+v(t)\phi(|{}^iz_1-{}^iz_2|),
\end{aligned}$$

which means that (H1)–(H2) hold true for g_i^n.

In the sequel, it follows from (16) and (17) that, $\mathrm{d}\mathbf{P}\times \mathrm{d}t-$a.e., for each $n\in\mathbf{N}$, $i=1,\dots,k$, y_1, $y_2\in\mathbf{R}^k$ and z_1, $z_2\in\mathbf{R}^{k\times d}$,

$$\begin{aligned}
&|g_i^n(t,y_1,{}^iz_1)-g_i^n(t,y_2,{}^iz_2)|\\
\le&\sup_{(p,q)\in\mathbf{R}^k\times\mathbf{R}^{k\times d}}|(n+A)(u(t)|p-y_1|+v(t)|{}^iq-{}^iz_1|)\\
&\qquad\qquad-(n+A)(u(t)|p-y_2|+v(t)|{}^iq-{}^iz_2|)|\\
\le&\,(n+A)\big(u(t)|y_1-y_2|+v(t)|{}^iz_1-{}^iz_2|\big).
\end{aligned}$$

Hence, (A1) is right for g_i^n.

Finally, for each $n\in\mathbf{N}$, $i=1,\dots,k$, $y\in\mathbf{R}^k$, $z\in\mathbf{R}^{k\times d}$ and $t\in[0,T]$, let

$$\mathbf{H}_{n,i,t}(y,z):=\left\{(p,q)\in\mathbf{R}^k\times\mathbf{R}^{k\times d}: u(t)|p-y|+v(t)|{}^iq-{}^iz|>\tfrac{2A}{n}\big(u(t)+v(t)\big)\right\},$$

then

$$\mathbf{H}^c_{n,i,t}(y,z)=\left\{(p,q)\in\mathbf{R}^k\times\mathbf{R}^{k\times d}: u(t)|p-y|+v(t)|{}^iq-{}^iz|\le\tfrac{2A}{n}\big(u(t)+v(t)\big)\right\}.$$

For each $n\in\mathbf{N}$, $i=1,\dots,k$, $y\in\mathbf{R}^k$, $z\in\mathbf{R}^{k\times d}$, $t\in[0,T]$ and $(p,q)\in\mathbf{H}_{n,i,t}(y,z)$, it follows from (H1)–(H2) that

$$\begin{aligned}
&g_i(t,p,{}^iq)+(n+A)\big(u(t)|p-y|+v(t)|{}^iq-{}^iz|\big)\\
\ge&\, g_i(t,y,{}^iz)+n\big(u(t)|p-y|+v(t)|{}^iq-{}^iz|\big)-A\big(u(t)+v(t)\big)\\
>&\, g_i(t,y,{}^iz)+A\big(u(t)+v(t)\big),\quad \mathrm{d}\mathbf{P}-\text{a.s.}.
\end{aligned}$$

Then, since $g_i^n(t,y,{}^iz)\le g_i(t,y,{}^iz)$, we have that $\mathrm{d}\mathbf{P}\times \mathrm{d}t-$a.e., for each $n\in\mathbf{N}$, $i=1,\dots,k$, $y\in\mathbf{R}^k$ and $z\in\mathbf{R}^{k\times d}$,

$$g_i^n(t,y,{}^iz)=\inf_{(p,q)\in\mathbf{H}^c_{n,i,t}(y,z)}\left\{g_i(t,p,{}^iq)+(n+A)\big(u(t)|p-y|+v(t)|{}^iq-{}^iz|\big)\right\}. \tag{18}$$

In the sequel, (H1)–(H3) and (18) yield that, $\mathrm{d}\mathbf{P}\times \mathrm{d}t-$a.e., for each $n\in\mathbf{N}$, $i=1,\dots,k$, $y\in\mathbf{R}^k$ and $z\in\mathbf{R}^{k\times d}$,

$$g_i^n(t,y,{}^iz)\ge\inf_{(p,q)\in\mathbf{H}^c_{n,i,t}(y,z)}\{g_i(t,y,{}^iz)-u(t)\rho(|p-y|)-v(t)\phi(|{}^iq-{}^iz|)\}$$

$$\geq g_i(t, y, {}^i z) - b_n(t),$$

where

$$b_n(t) = u(t)\rho\left(\frac{2A}{n} \cdot \frac{u(t)+v(t)}{u(t)}\right) + v(t)\phi\left(\frac{2A}{n} \cdot \frac{u(t)+v(t)}{v(t)}\right).$$

Thus, $\mathrm{d}\mathbf{P} \times \mathrm{d}t -$ a.e., for each $n \in \mathbf{N}$, $i = 1, \ldots, k$, $y \in \mathbf{R}^k$ and $z \in \mathbf{R}^{k\times d}$,

$$0 \leq g_i(t, y, {}^i z) - g_i^n(t, y, {}^i z) \leq b_n(t).$$

It is clear that $b_n(t) \downarrow 0$ as $n \to +\infty$ for each $t \in [0, T]$. Since $\rho(\cdot)$ and $\phi(\cdot)$ are at most linear-growth, we have

$$\begin{aligned} b_n(t) &\leq Au(t) + \frac{2A^2}{n}\big(u(t)+v(t)\big) + Av(t) + \frac{2A^2}{n}\big(u(t)+v(t)\big) \\ &\leq (A + 4A^2)\big(u(t)+v(t)\big), \end{aligned}$$

from which and the Lebesgue dominated convergence theorem it follows that $\int_0^T b_n(t)\,\mathrm{d}t \to 0$ as $n \to +\infty$.

Thus, the sequence $g^n := (g_1^n, g_2^n, \ldots, g_k^n)$ is just the one we desire. The proof is complete. □

Now, we can give the proof of the existence part of Theorem 7. First, it follows from (i)–(ii) of Lemma 12 and (H4) that for each $n \in \mathbf{N}$, g^n satisfies (A1) and (H4). Then it follows from Lemma 10 that for each $n \in \mathbf{N}$ and $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbf{P}; \mathbf{R}^k)$, the following BSDE

$$y_t^n = \xi + \int_t^T g^n(s, y_s^n, z_s^n)\,\mathrm{d}s - \int_t^T z_s^n\,\mathrm{d}B_s, \quad t \in [0, T], \tag{19}$$

has a unique solution $(y_t^n, z_t^n)_{t\in[0,T]}$. The following proof will be split into three steps.

Step 1. In this step we show that $\{(y_t^n)_{t\in[0,T]}\}_{n=1}^{+\infty}$ is a Cauchy sequence in $\mathcal{S}^2(0, T; \mathbf{R}^k)$.

For each n, $m \in \mathbf{N}$, let $(y_t^n, z_t^n)_{t\in[0,T]}$ and $(y_t^m, z_t^m)_{t\in[0,T]}$ be, respectively, solutions of BSDE (ξ, T, g^n) and BSDE (ξ, T, g^m). Using Itô's formula to $|y_t^n - y_t^m|^2$ we arrive, for each $t \in [0, T]$, at

$$\begin{aligned} &|y_t^n - y_t^m|^2 + \int_t^T |z_s^n - z_s^m|^2\,\mathrm{d}s \\ &= 2\int_t^T \langle y_s^n - y_s^m, g^n(s, y_s^n, z_s^n) - g^m(s, y_s^m, z_s^m)\rangle\,\mathrm{d}s \\ &\quad - 2\int_t^T \langle y_s^n - y_s^m, (z_s^n - z_s^m)\,\mathrm{d}B_s\rangle. \end{aligned} \tag{20}$$

It follows from (ii)–(iii) in Lemma 12 and the basic inequality $2ab \leq 2a^2 + b^2/2$ that, with adding and subtracting the term $g^n(s, y_s^m, z_s^m)$,

$$2\langle y_s^n - y_s^m, g^n(s, y_s^n, z_s^n) - g^m(s, y_s^m, z_s^m)\rangle$$

$$
\begin{aligned}
&\le 2|y_s^n - y_s^m|\big(kAu(s)|y_s^n - y_s^m| + kAv(s)|z_s^n - z_s^m| \\
&\quad + kAu(s) + kAv(s) + \tau_{n,m}(s)\big) \\
&\le \big(D(s) + \tau_{n,m}(s)\big)|y_s^n - y_s^m|^2 + \frac{1}{2}|z_s^n - z_s^m|^2 + kAu(s) + kAv(s) + \tau_{n,m}(s),
\end{aligned}
$$

where

$$\tau_{n,m}(s) = k\big(b_n(s)+b_m(s)\big), \quad D(s) = 2kAu(s)+2k^2A^2v^2(s)+kAu(s)+kAv(s).$$

Putting the previous inequality into (20) and taking the conditional expectation with respect to $\mathcal{F}_r$ yield that, for each $0 \le r \le t \le T$ and $n, m \in \mathbf{N}$,

$$\mathbf{E}\left[|y_t^n - y_t^m|^2\middle|\mathcal{F}_r\right] \le \int_t^T \big(D(s) + \tau_{n,m}(s)\big)\mathbf{E}\left[|y_s^n - y_s^m|^2\middle|\mathcal{F}_r\right] \mathrm{d}s + C_{n,m},$$

where $C_{n,m} = \int_0^T \big(kAu(s) + kAv(s) + \tau_{n,m}(s)\big)\,\mathrm{d}s$. It follows from Lemma 4 in [7] that

$$
\begin{aligned}
\mathbf{E}\left[|y_t^n - y_t^m|^2\middle|\mathcal{F}_r\right] &\le C_{n,m}\mathrm{e}^{\int_0^T (D(s)+\tau_{n,m}(s))\,\mathrm{d}s} \\
&\le C_{1,1}\mathrm{e}^{\int_0^T (D(s)+\tau_{1,1}(s))\,\mathrm{d}s} := (C_2)^2.
\end{aligned}
$$

After taking $r = t$ in the previous inequality, we have that for each $n, m \in \mathbf{N}$, $\mathrm{d}\mathbf{P} \times \mathrm{d}t -$ a.e., $|y_t^n - y_t^m| \le C_2$.

Furthermore, it follows from (ii) in Lemma 12, (H3) and Tanaka's formula that for each $t \in [0,T]$,

$$
\begin{aligned}
|{}^iy_t^n - {}^iy_t^m| \le &\int_t^T \mathrm{sgn}({}^iy_s^n - {}^iy_s^m)\big(g_i^n(s, y_s^n, {}^iz_s^n) - g_i^m(s, y_s^m, {}^iz_s^m)\big)\,\mathrm{d}s \\
&- \int_t^T \mathrm{sgn}({}^iy_s^n - {}^iy_s^m)({}^iz_s^n - {}^iz_s^m)\,\mathrm{d}B_s. \qquad (21)
\end{aligned}
$$

It follows from (ii)–(iii) in Lemma 12 that, by adding and subtracting the term $g_i^n(s, y_s^m, {}^iz_s^m)$,

$$
\begin{aligned}
&|g_i^n(s, y_s^n, {}^iz_s^n) - g_i^m(s, y_s^m, {}^iz_s^m)| \\
&\le ku(s)\rho(|y_s^n - y_s^m|) + kv(s)\phi(|{}^iz_s^n - {}^iz_s^m|) + \tau_{n,m}(s). \qquad (22)
\end{aligned}
$$

Combining (21)–(22) with (10) we get that for each $n, m, q \in \mathbf{N}$ and $t \in [0,T]$,

$$
\begin{aligned}
|{}^iy_t^n - {}^iy_t^m| \le C_{n,m,q} &+ k\int_t^T \big(u(s)\rho(|y_s^n - y_s^m|) + (q+2A)v(s)|{}^iz_s^n - {}^iz_s^m|\big)\,\mathrm{d}s \\
&- \int_t^T \mathrm{sgn}({}^iy_s^n - {}^iy_s^m)({}^iz_s^n - {}^iz_s^m)\,\mathrm{d}B_s, \qquad (23)
\end{aligned}
$$

where

$$C_{n,m,q} := k\phi\left(\frac{2A}{q+2A}\right)\int_0^T v(s)\,\mathrm{d}s + \int_0^T \tau_{n,m}(s)\,\mathrm{d}s.$$

In the sequel, by virtue of Girsanov's theorem, in the same way as in the proof of Lemma 11 we can deduce from (23) that for each n, m, $q \in \mathbf{N}$, $i = 1, \ldots, k$, and $0 \le r \le t \le T$,

$$\mathbf{E}^{n,m,q,i}\left[|{}^i y_t^n - {}^i y_t^m|\big|\mathcal{F}_r\right] \le C_{n,m,q} + k\int_t^T \mathbf{E}^{n,m,q,i}\left[u(s)\rho(|y_s^n - y_s^m|)\big|\mathcal{F}_r\right] \mathrm{d}s,$$

where $\mathbf{E}^{n,m,q,i}[X|\mathcal{F}_t]$ represents the conditional expectation of random variable X with respect to $\mathcal{F}_t$ under a probability measure $\mathbf{P}^{n,m,q,i}$ on $(\Omega, \mathcal{F})$, which depends on n, m, q and i, and which is absolutely continuous with respect to $\mathbf{P}$.

Finally, note that $C_{n,m,q}$ tends non-increasingly to 0 as n, m, $q \to +\infty$. The same argument as in the proof of the uniqueness part of Theorem 7 yields that for each $i = 1, \ldots, k$,

$$\lim_{n,m\to+\infty} \mathbf{E}\left[\sup_{t\in[0,T]} |{}^i y_t^n - {}^i y_t^m|^2\right] = 0,$$

which means that $\{(y_t^n)_{t\in[0,T]}\}_{n=1}^{+\infty}$ is a Cauchy sequence in $\mathcal{S}^2(0,T;\mathbf{R}^k)$. We denote the limit by $(y_t)_{t\in[0,T]}$.

Step 2. In this step we show that $\{(z_t^n)_{t\in[0,T]}\}_{n=1}^{+\infty}$ is a Cauchy sequence in $\mathrm{M}^2(0,T;\mathbf{R}^{k\times d})$.

Using Itô's formula for $|y_t^n|^2$ defined in BSDE (19), we can obtain that

$$|y_t^n|^2 + \int_t^T |z_s^n|^2 \,\mathrm{d}s = |\xi|^2 + 2\int_t^T \langle y_s^n, g^n(s, y_s^n, z_s^n)\rangle \,\mathrm{d}s - 2\int_t^T \langle y_s^n, z_s^n \,\mathrm{d}B_s\rangle.$$

Let $G_n(\omega) := \sup_{t\in[0,T]} |y_t^n|$. It follows from the convergence of $\{(y_t^n)_{t\in[0,T]}\}_{n=1}^{+\infty}$ in $\mathcal{S}^2(0,T;\mathbf{R}^k)$ that $\sup_{n\in\mathbf{N}} \mathbf{E}\left[G_n^2(\omega)\right] < +\infty$. In view of (i) in Lemma 12 we have that for each $t \in [0,T]$,

$$\begin{aligned}
&|y_t^n|^2 + \int_t^T |z_s^n|^2 \,\mathrm{d}s \\
&\le |\xi|^2 - 2\int_t^T \langle y_s^n, z_s^n \,\mathrm{d}B_s\rangle \\
&\quad + 2G_n(\omega)\int_t^T \big(|g(s,0,0)| + kAu(s)(1+|y_s^n|) + kAv(s)(1+|z_s^n|)\big)\,\mathrm{d}s.
\end{aligned}$$

It follows from the BDG inequality that $(\int_0^t \langle y_t^n, z_t^n \,\mathrm{d}B_s\rangle)_{t\in[0,T]}$ is an $(\mathcal{F}_t, \mathbf{P})$-martingale. By the inequalities $2ab \le a^2 + b^2$, $2ab \le \lambda a^2 + b^2/\lambda$ ($\lambda := 2k^2A^2\int_0^T v^2(s)\,\mathrm{d}s$) and Hölder's inequality we deduce that for each $n \in \mathbf{N}$,

$$\mathbf{E}\left[\int_0^T |z_s^n|^2 \,\mathrm{d}s\right]$$

$$
\begin{aligned}
&\leq \mathbf{E}\left[|\xi|^2\right] + \left[1 + kA + \lambda + 2kA\int_0^T u(s)\,\mathrm{d}s\right]\sup_{n\in\mathbf{N}}\mathbf{E}\left[G_n^2(\omega)\right] \\
&\quad + \mathbf{E}\left[\left(\int_0^T |g(s,0,0)|\,\mathrm{d}s\right)^2\right] + kA\left(\int_0^T \big(u(s)+v(s)\big)\,\mathrm{d}s\right)^2 \\
&\quad + \frac{k^2A^2}{\lambda}\mathbf{E}\left[\left(\int_0^T v(s)|z_s^n|\,\mathrm{d}s\right)^2\right] \\
&\leq C_3 + \frac{1}{2}\mathbf{E}\left[\int_0^T |z_s^n|^2\,\mathrm{d}s\right],
\end{aligned}
$$

from which it follows that

$$
\sup_n \mathbf{E}\left[\int_0^T |z_s^n|^2\,\mathrm{d}s\right] \leq 2C_3 < +\infty, \tag{24}
$$

where C_3 is a positive constant and independent of n.

On the other hand, by taking expectation in both sides of (20), we have that for each n, $m \in \mathbf{N}$,

(25)

$$
\mathbf{E}\left[\int_0^T |z_s^n - z_s^m|^2\,\mathrm{d}s\right] \leq 2\mathbf{E}\left[\int_0^T \langle y_s^n - y_s^m, g^n(s,y_s^n,z_s^n) - g^m(s,y_s^m,z_s^m)\rangle\,\mathrm{d}s\right].
$$

It follows from (i) in Lemma 12 that

$$
\begin{aligned}
&2\langle y_s^n - y_s^m, g^n(s,y_s^n,z_s^n) - g^m(s,y_s^m,z_s^m)\rangle \\
&\leq 4k|y_s^n - y_s^m|\big[|g(s,0,0)| + Au(s)\big(G_n(\omega)+G_m(\omega)\big) \\
&\quad + Av(s)(|z_s^n| + |z_s^m|) + A(u(s)+v(s))\big].
\end{aligned}
$$

Putting the previous inequality into (25) and using Hölder's inequality and (24) yields that

$$
\begin{aligned}
&\mathbf{E}\left[\int_0^T |z_s^n - z_s^m|^2\,\mathrm{d}s\right] \\
&\leq 16k\sqrt{\mathbf{E}\left[\sup_{t\in[0,T]}|y_t^n - y_t^m|^2\right]}\sqrt{\mathbf{E}\left[\left(\int_0^T \left[|g(s,0,0)| + A\big(u(s)+v(s)\big)\right]\,\mathrm{d}s\right)^2\right]} \\
&\quad + 32kA\int_0^T u(s)\,\mathrm{d}s\sqrt{\sup_{n\in\mathbf{N}}\mathbf{E}\left[G_n^2(\omega)\right]}\sqrt{\mathbf{E}\left[\sup_{t\in[0,T]}|y_t^n - y_t^m|^2\right]} \\
&\quad + 32kA\sqrt{2C_3\int_0^T v^2(s)\,\mathrm{d}s}\sqrt{\mathbf{E}\left[\sup_{t\in[0,T]}|y_t^n - y_t^m|^2\right]}.
\end{aligned}
$$

Since $\sup_{n\in\mathbf{N}}\mathbf{E}[G_n^2(\omega)]<+\infty$, $\{(y_t^n)_{t\in[0,T]}\}_{n=1}^{+\infty}$ converges in $\mathcal{S}^2(0,T;\mathbf{R}^k)$ and

$$\begin{aligned}&\mathbf{E}\left[\left(\int_0^T\big[|g(s,0,0)|+A\big(u(s)+v(s)\big)\big]\,\mathrm{d}s\right)^2\right]\\&\le 2\mathbf{E}\left[\left(\int_0^T|g(s,0,0)|\,\mathrm{d}s\right)^2\right]+2\left(\int_0^T A\big(u(s)+v(s)\big)\,\mathrm{d}s\right)^2<+\infty,\end{aligned}$$

we can deduce that,

$$\lim_{n,m\to+\infty}\mathbf{E}\left[\int_0^T|z_s^n-z_s^m|^2\,\mathrm{d}s\right]=0.$$

Therefore, $\{(z_t^n)_{t\in[0,T]}\}_{n=1}^{+\infty}$ is a Cauchy sequence in $\mathrm{M}^2(0,T;\mathbf{R}^{k\times d})$. We denote by $(z_t)_{t\in[0,T]}$ the limit.

Step 3. This step will show that the process $(y_t,z_t)_{t\in[0,T]}$ is a solution of BSDE (1).

Now, we have known that for each fixed $t\in[0,T]$, the sequence $\{y_t^n\}_{n=1}^{+\infty}$ and $\{\int_t^T z_s^n\,\mathrm{d}B_s\}_{n=1}^{+\infty}$ converge in $L^2(\Omega,\mathcal{F}_T,\mathbf{P};\mathbf{R}^k)$ toward to y_t and $\int_t^T z_s\,\mathrm{d}B_s$ respectively. Next let us check the limit of $g^n(s,y_s^n,z_s^n)$ in BSDE (19). First, for each $t\in[0,T]$, we have

$$\begin{aligned}&\mathbf{E}\left[\left(\int_t^T|g^n(s,y_s^n,z_s^n)-g(s,y_s,z_s)|\,\mathrm{d}s\right)^2\right]\\&\le 2\mathbf{E}\left[\left(\int_0^T|g^n(s,y_s^n,z_s^n)-g(s,y_s^n,z_s^n)|\,\mathrm{d}s\right)^2\right]\\&\quad+2\mathbf{E}\left[\left(\int_0^T|g(s,y_s^n,z_s^n)-g(s,y_s,z_s)|\,\mathrm{d}s\right)^2\right].\end{aligned}\tag{26}$$

It follows from (iii) in Lemma 12 that the first term on the right side of (26) converges to 0 as $n\to+\infty$. Furthermore, by (H1)–(H2) and (10) we have that for each n, $m\in\mathbf{N}$,

$$\begin{aligned}&\mathbf{E}\left[\left(\int_0^T|g(s,y_s^n,z_s^n)-g(s,y_s,z_s)|\,\mathrm{d}s\right)^2\right]\\&\le 2(m+2A)^2\mathbf{E}\left[\left(\int_0^T\big(u(s)|y_s^n-y_s|+v(s)|z_s^n-z_s|\big)\,\mathrm{d}s\right)^2\right]\end{aligned}\tag{27}$$

$$+ 2 \left(\int_0^T \left[u(s)\rho\left(\frac{2A}{m+2A}\right) + v(s)\phi\left(\frac{2A}{m+2A}\right)\right] \mathrm{d}s \right)^2 .$$

Note that the second term on the right side of (27) converges to 0 as $m \to +\infty$. On the other hand, it follows from Hölder's inequality that

$$\begin{aligned} &\mathbf{E}\left[\left(\int_0^T \big(u(s)|y_s^n - y_s| + v(s)|z_s^n - z_s|\big)\,\mathrm{d}s\right)^2\right] \\ &\le 2\left(\int_0^T u(s)\,\mathrm{d}s\right)^2 \mathbf{E}\left[\sup_{t\in[0,T]} |y_t^n - y_t|^2\right] \\ &\quad + 2\int_0^T v^2(s)\,\mathrm{d}s\mathbf{E}\left[\int_0^T |z_s^n - z_s|^2\,\mathrm{d}s\right]. \end{aligned}$$

Thus, by virtue of the fact that $\{(y_t^n, z_t^n)_{t\in[0,T]}\}_{n=1}^{+\infty}$ is a Cauchy sequence in $\mathcal{S}^2(0,T;\mathbf{R}^k) \times \mathrm{M}^2(0,T;\mathbf{R}^{k\times d})$, taking $n \to +\infty$ and then $m \to +\infty$ in (27) and taking $n \to +\infty$ in (26) yield that for each $t \in [0,T]$,

$$\lim_{n\to+\infty} \mathbf{E}\left[\left|\int_t^T g^n(s, y_s^n, z_s^n)\,\mathrm{d}s - \int_t^T g(s, y_s, z_s)\,\mathrm{d}s\right|^2\right] = 0.$$

Subsequently, noticing that $(y_t)_{t\in[0,T]}$ is a continuous process, by passing to the limit in BSDE (19) we deduce that $\mathrm{d}\mathbf{P} -$ a.s.,

$$y_t = \xi + \int_t^T g(s, y_s, z_s)\,\mathrm{d}s - \int_t^T z_s\,\mathrm{d}B_s, \quad t \in [0,T],$$

which means that $(y_t, z_t)_{t\in[0,T]} \in \mathcal{S}^2(0,T;\mathbf{R}^k) \times \mathrm{M}^2(0,T;\mathbf{R}^{k\times d})$ is a solution of BSDE (1).

Remark 13. Note that if there exists a constant $K > 0$ such that the function $\phi(\cdot)$ appearing in (H2) satisfies $\phi(x) \le Kx$ for all $x \in \mathbf{R}^+$, then the condition $\int_0^T \big(v(t) + v^2(t)\big)\,\mathrm{d}t < +\infty$ in Theorem 7 can be weakened to $\int_0^T v^2(t)\,\mathrm{d}t < +\infty$ as in Lemma 10.

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