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SIGNED A-POLYNOMIALS OF GRAPHS AND POINCARÉ POLYNOMIALS OF REAL TORIC MANIFOLDS

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Abstract. Choi and Park introduced an invariant of a finite simple graph, called *signed a-number*, arising from computing certain topological invariants of some specific kinds of real toric manifolds. They also found the signed a-numbers of path graphs, cycle graphs, complete graphs, and star graphs.

We introduce a *signed a-polynomial* which is a generalization of the signed a-number and gives *a-, b-, and c-numbers*. The signed a-polynomial of a graph G is related to the Poincaré polynomial $P_{M(G)}(z)$, which is the generating function for the Betti numbers of the real toric manifold $M(G)$. We give the generating functions for the signed a-polynomials of not only path graphs, cycle graphs, complete graphs, and star graphs, but also complete bipartite graphs and complete multipartite graphs. As a consequence, we find the Euler characteristic number and the Betti numbers of the real toric manifold $M(G)$ for complete multipartite graphs G.

1. Introduction

A *signed a-number* of a finite simple graph G is a graph invariant introduced by Choi and Park [3], denote by $\mathrm{sa}(G)$, as follows:

- $\mathrm{sa}(\emptyset) = 1$.
- $\mathrm{sa}(G)$ is the product of signed a-numbers of connected components of G.
- $\mathrm{sa}(G) = 0$ if G is a connected graph on odd number of vertices.
- If G is connected with even number of vertices, then $\mathrm{sa}(G)$ is given by the negative of the sum of signed a-numbers of all induced subgraphs G' of G except itself G.

Let the *a-number* $\mathrm{a}(G)$ be the absolute value of the signed a-number of G, the *b-number* $\mathrm{b}(G)$ the sum of signed a-numbers induced subgraphs of G, and the *c-numbers* $\mathrm{c}_i(G)$ the sum of a-numbers of induced subgraphs of G with i vertices.

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These numbers arise from computing certain *topological invariants* of some specific kinds of real toric manifolds which are important objects in toric topology. For a finite simple graph G, a *building set* $B(G)$ is consisting of connected induced subgraphs of G and a *nestohedron* $P_{\mathcal{B}(G)}$ is defined as the Minkowski sum of simplices

$$P_{\mathcal{B}(G)} = \sum_{I \in B(G)} \Delta_I,$$

which is called a *graph associahedron*. Since every nestohedron is a *Delzant polytope* [5, Proposition 7.10], the real toric manifold $M(G)$ can be defined as the set of real points in the toric manifold, which is associated to the normal fan of the graph associahedron $P_{\mathcal{B}(G)}$ as Delzant polytope. For further information, see [2, 5, 6].

Recently, Choi and Park [3, Theorem 1.1] showed that the Euler characteristic $\chi(M(G))$ of $M(G)$ is equal to $\mathrm{b}(G)$ and the i-th rational Betti number $\beta_i(M(G))$ of $M(G)$ is equal to $\mathrm{c}_{2i}(G)$. We remark that $\mathrm{c}_{2i}(G)$ is the same with $\mathrm{a}_i(G)$ in [3]. They also computed these numbers of path graphs P_{2n}, cycle graphs C_{2n}, complete graphs K_{2n}, and star graphs $K_{1,2n-1}$.

In this paper, we introduce a *signed a-polynomial* which is a generalization of the signed a-number and gives a-, b-, and c-numbers. The signed a-polynomial of a graph G is related to the Poincaré polynomial $P_{M(G)}(z)$, which is the generating function for the Betti numbers of the real toric manifold $M(G)$. The relation will be shown in equation (7). We give the signed a-polynomials of not only path graphs, cycle graphs, complete graphs, and star graphs, but also complete bipartite graphs $K_{p,q}$ and complete multipartite graphs $K_{p_1,\dots,p_m}$. As a consequence, we find $\chi(M(G))$ and $\beta_i(M(G))$ for $G = K_{p,q}$ and $G = K_{p_1,\dots,p_m}$.

2. Preliminaries

From now on, we assume that a graph is finite, undirected, and simple. We rewrite a formal definition of a signed a-number $\mathrm{sa}(G)$ of a graph $G = (V, E)$ in the previous section as

$$\mathrm{sa}(G) = \begin{cases} 1 & \text{if } G \text{ is the empty graph,} \\ 0 & \text{if } G \text{ is connected and } |V| \text{ is odd,} \\ -\displaystyle\sum_{V' \subsetneq V} \mathrm{sa}(G|_{V'}) & \text{if } G \text{ is connected and } |V| \text{ is even} \geq 2, \\ \displaystyle\prod_{G' \in \mathrm{comp}(G)} \mathrm{sa}(G') & \text{if } G \text{ is disconnected,} \end{cases}$$

where $G|_{V'}$ is the induced subgraph of G by a vertex subset V' and $\mathrm{comp}(G)$ is the set of connected components of G. From the above definition, it is easy to check that $\mathrm{sa}(G) = 0$ for every graph G with at least one connected component on odd number of vertices; and $\sum_{V' \subseteq V} \mathrm{sa}(G|_{V'}) = 0$ for every nonempty graph

G on V with every connected component on even number of vertices. Thus, we find a simpler equivalent definition of a signed a-number as follows.

Definition 1. A *signed a-number* $\mathrm{sa}(G)$ of a graph $G = (V, E)$ is defined by

(1)
$$\mathrm{sa}(G) = \begin{cases} 1 & \text{if } G \text{ is the empty graph,} \\ 0 & \text{if } G \text{ has a connected component on odd number} \\ & \text{of vertices,} \\ -\sum_{V' \subsetneq V} \mathrm{sa}(G|_{V'}) & \text{otherwise.} \end{cases}$$

Consequently, we define a-, b-, and c-numbers of a graph with the signed a-numbers.

Definition 2. The *a-, b-,* and *c-numbers* of a graph G, denoted by $\mathrm{a}(G)$, $\mathrm{b}(G)$, and $\mathrm{c}_i(G)$, are defined by

$$\mathrm{a}(G) = (-1)^{|V|/2}\,\mathrm{sa}(G), \tag{2}$$

$$\mathrm{b}(G) = \sum_{V' \subseteq V} \mathrm{sa}(G|_{V'}), \tag{3}$$

$$\mathrm{c}_i(G) = \sum_{\substack{V' \subseteq V \\ |V'|=i}} \mathrm{a}(G|_{V'}) = (-1)^{i/2} \sum_{\substack{V' \subset V \\ |V'|=i}} \mathrm{sa}(G|_{V'}). \tag{4}$$

By definition, for any graph G, it holds that $\mathrm{c}_i(G) = 0$ if i is odd, and $\mathrm{c}_n(G) = \mathrm{a}(G)$ if n is the number of vertices of G. From a topological viewpoint [3, Remark 2.2], it is obvious that $\mathrm{a}(G)$ and $\mathrm{c}_i(G)$ are nonnegative integers.

3. On signed a-polynomials

Now, we introduce a generalization of a-, b-, and c-numbers of graphs.

Definition 3 (Signed a-polynomial)**.** The *signed a-polynomial* $\mathrm{sa}(G; t)$ of a graph G is defined by

$$\mathrm{sa}(G; t) = \sum_{V' \subseteq V(G)} \mathrm{sa}(G|_{V'})\; t^{|V \setminus V'|}, \tag{5}$$

where $V(G)$ is the set of vertices of G.

From the equations (1)–(5), for $|V(G)| = n$, it holds that

$$\mathrm{sa}(G) = \mathrm{sa}(G; 0), \qquad \mathrm{a}(G) = (-1)^{n/2}\,\mathrm{sa}(G; 0),$$
$$\mathrm{b}(G) = \mathrm{sa}(G; 1), \qquad \mathrm{c}_i(G) = (-1)^{i/2}[t^{n-i}]\,\mathrm{sa}(G; t).$$

Thus, $\mathrm{sa}(G; t)$ is represented as the sum of $\mathrm{c}_i(G)$'s by

$$\mathrm{sa}(G; t) = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j c_{2j}(G) t^{n-2j}. \tag{6}$$

For example, if G is a graph obtained by deleting one edge from the complete graph K_4, then

$$\mathrm{sa}(G;t) = t^4 - 5t^2 + 4.$$

Thus, $\mathrm{sa}(G) = \mathrm{a}(G) = 4$, $\mathrm{b}(G) = 0$, and $\{\mathrm{c}_i(G)\}_{i=0}^4 = 1, 0, 5, 0, 4$.

Remark. The *Poincaré polynomial* $P_{M(G)}(z) = \sum_{i\geq 0} \beta_i(M(G))z^i$ is the generating function for the *Betti numbers* $\beta_i(M(G))$ of the real toric manifold $M(G)$. Since $\beta_i(M(G)) = \mathrm{c}_{2i}(G)$ in [3, Theorem 1.1], it holds that

$$P_{M(G)}(z) = (\sqrt{-z})^{|V|}\,\mathrm{sa}\left(G; \frac{1}{\sqrt{-z}}\right). \tag{7}$$

In the rest of the section, we compute the generating functions for signed a-polynomials of path graphs, cycle graphs, complete graphs, and star graphs.

Theorem 1. *Let P_n be the path graph with n vertices, which is a tree with exactly $n-2$ vertices of degree 2. Then the generating function for signed a-polynomials of P_n is given by*

$$\sum_{n\geq 0} \mathrm{sa}(P_n;t)x^n = \frac{-1+2tx+\sqrt{1+4x^2}}{2tx-2(t^2-1)x^2}. \tag{8}$$

Proof. From Theorem 2.5 in [3], it is known that

$$\mathrm{c}_{2i}(P_n) = \binom{n}{i} - \binom{n}{i-1} = \mathrm{Cat}_{n-i,i},$$

with Catalan triangle numbers $\mathrm{Cat}_{n,k} = \binom{n+k}{k} - \binom{n+k}{k-1}$. Using formula (6), we have

$$\mathrm{sa}(P_n;t) = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j\,\mathrm{Cat}_{n-j,j}\,t^{n-2j}.$$

Thus, we obtain

$$\begin{aligned} \sum_{n\geq 0} \mathrm{sa}(P_n;t)x^n &= \sum_{n\geq 0}\sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j\,\mathrm{Cat}_{n-j,j}\,t^{n-2j}x^n \\ &= \sum_{k\geq 0}\sum_{j\geq 0} \mathrm{Cat}_{k,j}\,(-x/t)^j(tx)^k. \end{aligned} \tag{9}$$

Since the generating function for Catalan triangle numbers is

$$\sum_{n\geq 0}\sum_{i\geq 0} \mathrm{Cat}_{n,i}\,w^i z^n = \frac{\mathrm{Cat}(wz)}{1 - z\,\mathrm{Cat}(wz)},$$

where $\mathrm{Cat}(x) = \frac{1-\sqrt{1-4x}}{2x}$, therefore (9) becomes (8). □

TABLE 1. Numbers for path graphs P_n, where Catalan triangle numbers $\mathrm{Cat}_{n,k} = \binom{n+k}{k} - \binom{n+k}{k-1}$ and Catalan numbers $\mathrm{Cat}_n = \mathrm{Cat}_{n,n} = \frac{1}{n+1}\binom{2n}{n}$.

G	P_0	P_{2n}	P_{2n+1}
$\mathrm{sa}(G)$	1	$(-1)^n \mathrm{Cat}_n$	0
$\mathrm{a}(G)$	1	Cat_n	0
$\mathrm{b}(G)$	1	0	$(-1)^n \mathrm{Cat}_n$
$\mathrm{c}_{2i}(G)$	$\delta_{i,0}$	$\mathrm{Cat}_{2n-i,i}$	$\mathrm{Cat}_{2n+1-i,i}$
g.f. for $\mathrm{sa}(G;t)$	$\sum_{n\geq 0} \mathrm{sa}(P_n;t)x^n = \dfrac{-1+2tx+\sqrt{1+4x^2}}{2tx-2(t^2-1)x^2}$		

Remark. For two given sequences $\sigma = (s_0, s_1, s_2, \ldots)$ and $\tau = (t_1, t_2, t_3, \ldots)$, define the generalized Catalan number B_n by the sum of weighted Motzkin paths from $(0,0)$ to $(n,0)$ with up steps $(1,1)$, horizontal steps $(1,0)$, and down steps $(1,-1)$ where we associate weight 1 to each up step, weight s_k to each horizontal step on the line $y = k$, and weight t_k to each down step between two lines $y = k-1$ and $y = k$. For example, if $\sigma \equiv 0$ and $\tau \equiv 1$, then $B_{2n} = \mathrm{Cat}_n$. In Section 7.4 in [1], the generating function $B(z) = \sum_{n\geq 0} B_n z^n$ of the generalized Catalan number B_n with $\sigma = (a, s, s, \ldots)$ and $\tau = (b, u, u, \ldots)$ is equal to

$$B(z) = \frac{(2u-b) + (bs-2au)z - b\sqrt{1-2sx+(s^2-4u)z^2}}{2(u-b)+2(bs-2au+ab)z+2(a^2u-abs+b^2)z^2}. \tag{10}$$

For $(a, s, b, u, z) = (t, 0, -1, -1, x)$, the formula (10) gives a combinatorial interpretation of the following formula

$$\sum_{n\geq 0} \mathrm{sa}(P_n;t)x^n = \frac{-1+2tx+\sqrt{1+4x^2}}{2tx-2(t^2-1)x^2}$$

and for $(a, s, b, u, z) = (0, 0, -1, t^2-1, x)$, the formula (10) gives a combinatorial interpretation of the following formula

$$\sum_{n\geq 0} \mathrm{sa}(P_{2n};t)x^{2n} = \frac{-(t^2+1)-(t^2-1)\sqrt{1+4x^2}}{-2t^2+2(t^2-1)^2x^2}.$$

Theorem 2. *Let C_n be the cycle graph with n vertices, which is a connected graph with all vertices of degree 2. Then the generating function for signed a-polynomials of C_n is given by*

$$\sum_{n\geq 0} \mathrm{sa}(C_n;t)x^n = \frac{1}{2} + \frac{1}{2\sqrt{1+4x^2}} \cdot \frac{(t^2+1)x+t\sqrt{1+4x^2}}{t-(t^2-1)x}. \tag{11}$$

TABLE 2. Numbers for cycle graphs C_n.

G	C_0	C_{2n}	C_{2n+1}
$\mathrm{sa}(G)$	1	$\dfrac{(-1)^n}{2}\dbinom{2n}{n}$	0
$\mathrm{a}(G)$	1	$\dfrac{1}{2}\dbinom{2n}{n}$	0
$\mathrm{b}(G)$	1	0	$(-1)^n\dbinom{2n}{n}$
$\mathrm{c}_{2i}(G)$	$\delta_{i,0}$	$\begin{cases}\frac{1}{2}\binom{2n}{n}, & \text{if } i=n\\ \binom{2n}{i}, & \text{if } i<n\end{cases}$	$\dbinom{2n+1}{i}$
g.f. for $\mathrm{sa}(G;t)$	$\displaystyle\sum_{n\geq 0}\mathrm{sa}(C_n;t)x^n = \frac{1}{2}+\frac{1}{2\sqrt{1+4x^2}}\cdot\frac{(t^2+1)x+t\sqrt{1+4x^2}}{t-(t^2-1)x}$		

Proof. From Theorem 2.6 in [3], it is known that

$$\mathrm{c}_{2i}(C_n) = \begin{cases} 1 & \text{if } i=n=0,\\ \frac{1}{2}\binom{n}{n/2} & \text{if } 2i=n>0,\\ \binom{n}{i} & \text{if } 2i<n. \end{cases}$$

Using formula (6), we obtain

$$\begin{aligned}
\sum_{n\geq 0}\mathrm{sa}(C_n;t)x^n &= \sum_{n\geq 0}\sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\,\mathrm{c}_{2j}(C_n)\ t^{n-2j}x^n\\
&= \sum_{k\geq 0}\sum_{j\geq 0}(-1)^j\,\mathrm{c}_{2j}(C_{2j+k})\ t^k x^{2j+k}\\
&= \frac{1}{2}-\frac{1}{2}\sum_{j\geq 0}\binom{2j}{j}(-x^2)^j+\sum_{k\geq 0}\sum_{j\geq 0}\binom{2j+k}{j}\ (tx)^k(-x^2)^j. \qquad (12)
\end{aligned}$$

From $\sum_{n\geq 0}\binom{2n+k}{n}z^n = \frac{1}{1-\sqrt{1-4z}}\left(\frac{1-\sqrt{1-4z}}{2z}\right)^k$, we have two generating functions:

$$\sum_{n\geq 0}\binom{2n}{n}z^n = \frac{1}{1-\sqrt{1-4z}},$$

$$\sum_{n\geq 0}\sum_{k\geq 0}\binom{2n+k}{n}w^k z^n = \frac{1}{1-\sqrt{1-4z}}\cdot\frac{1}{1-w\left(\frac{1-\sqrt{1-4z}}{2z}\right)}.$$

Using the above two generating functions, (12) becomes (11). □

Table 3. Numbers for complete graphs K_n, where $\sum_{n\geq 0} A_n \frac{z^n}{n!} = \sec z + \tan z$.

G	K_0	K_{2n}	K_{2n+1}
$\mathrm{sa}(G)$	1	$(-1)^n A_{2n}$	0
$\mathrm{a}(G)$	1	A_{2n}	0
$\mathrm{b}(G)$	1	0	$(-1)^n A_{2n+1}$
$\mathrm{c}_{2i}(G)$	$\delta_{i,0}$	$\binom{2n}{2i} A_{2i}$	$\binom{2n+1}{2i} A_{2i}$
e.g.f. for $\mathrm{sa}(G;t)$	$\sum_{n\geq 0} \mathrm{sa}(K_n;t)\frac{x^n}{n!} = e^{tx}\,\mathrm{sech}(x)$		

Let A_n be the *n-th Euler zigzag number* for which the exponential generating function is

$$\sum_{n\geq 0} A_n \frac{z^n}{n!} = \sec z + \tan z. \tag{13}$$

Theorem 3. *Let K_n be the complete graph with n vertices. Then the exponential generating function for signed a-polynomials of K_n is given by*

$$\sum_{n\geq 0} \mathrm{sa}(K_n;t)\frac{x^n}{n!} = e^{tx}\,\mathrm{sech}\, x. \tag{14}$$

Proof. From Theorem 2.8 in [3], it is known that

$$\mathrm{sa}(K_{2n}) = (-1)^n A_{2n}.$$

Using formula (5), we obtain

$$\begin{aligned}
\sum_{n\geq 0} \mathrm{sa}(K_n;t)\frac{x^n}{n!} &= \sum_{n\geq 0}\sum_{j=0}^{\lfloor n/2\rfloor} \binom{n}{2j} \mathrm{sa}(K_{2j})\ t^{n-2j}\frac{x^n}{n!} \\
&= \sum_{k\geq 0}\sum_{j\geq 0} \binom{k+2j}{2j} (-1)^j A_{2j}\ t^k \frac{x^{k+2j}}{(k+2j)!} \\
&= \left(\sum_{k\geq 0} \frac{(tx)^k}{k!}\right)\left(\sum_{j\geq 0} A_{2j}\frac{(\imath x)^{2j}}{(2j)!}\right). \qquad (15)
\end{aligned}$$

By (13), formula (15) becomes formula (14). □

Remark. The Euler polynomials $E_n(t)$ is defined by the exponential generating function $\sum_{n\geq 0} E_n(t)\frac{x^n}{n!} = \left(\frac{2}{e^x+1}\right)e^{xt}$. See [4, p. 48]. Then it follows

$$\mathrm{sa}(K_n;t) = E_n\left(\frac{t+1}{2}\right)2^n$$

TABLE 4. Numbers for star graphs $K_{1,n-1}$, where $\sum_{n\ge0} A_n \frac{z^n}{n!} = \sec z + \tan z$.

G	$K_{1,2n-1}$	$K_{1,2n}$
$\mathrm{sa}(G)$	$(-1)^n A_{2n-1}$	0
$\mathrm{a}(G)$	A_{2n-1}	0
$\mathrm{b}(G)$	0	$(-1)^n A_{2n}$
$\mathrm{c}_{2i}(G)$	$\begin{cases} \binom{2n-1}{2i-1} A_{2i-1}, & \text{if } i>0 \\ 1, & \text{if } i=0 \end{cases}$	$\begin{cases} \binom{2n}{2i-1} A_{2i-1}, & \text{if } i>0 \\ 1, & \text{if } i=0 \end{cases}$
e.g.f. for $\mathrm{sa}(G;t)$	$\displaystyle\sum_{n\ge0} \mathrm{sa}(K_{1,n};t)\frac{x^n}{n!} = e^{tx}(t-\tanh x)$	

from $\sum_{n\ge0} \mathrm{sa}(K_n;t)\frac{x^n}{n!} = e^{tx}\,\mathrm{sech}\, x = \left(\frac{2}{e^{2x}+1}\right) e^{2x\left(\frac{t+1}{2}\right)} = \sum_{n\ge0} E_n\left(\frac{t+1}{2}\right)\frac{(2x)^n}{n!}$.

Theorem 4. *Let $K_{1,n}$ be the star graph with $n+1$ vertices, which is a tree with at least one vertex of degree n. Then the exponential generating function for signed a-polynomials of $K_{1,n}$ is given by*

$$\sum_{n\ge0} \mathrm{sa}(K_{1,n};t)\frac{x^n}{n!} = e^{tx}(t-\tanh x). \tag{16}$$

Proof. From Theorem 2.9 in [3], it is known that

$$\mathrm{sa}(K_{1,2n+1}) = (-1)^{n+1} A_{2n+1}.$$

Using formula (5), we obtain

$$\begin{aligned}
&\sum_{n\ge0} \mathrm{sa}(K_{1,n};t)\frac{x^n}{n!} \\
&= \sum_{n\ge0}\left(\mathrm{sa}(\emptyset)t^{n+1} + \sum_{j=0}^{\lfloor n/2\rfloor}\binom{n}{2j+1}\mathrm{sa}(K_{1,2j+1})\ t^{n-(2j+1)}\right)\frac{x^n}{n!} \\
&= \sum_{n\ge0} t^{n+1}\frac{x^n}{n!} + \sum_{k\ge0}\sum_{j\ge0}\binom{k+2j+1}{2j+1}(-1)^{j+1}A_{2j+1}\ t^k\frac{x^{k+2j+1}}{(k+2j+1)!} \\
&= t\left(\sum_{n\ge0}\frac{(tx)^n}{n!}\right) + \left(\sum_{k\ge0}\frac{(tx)^k}{k!}\right)\left(\imath\sum_{j\ge0}A_{2j+1}\frac{(\imath x)^{2j+1}}{(2j+1)!}\right), \qquad (17)
\end{aligned}$$

where $\imath := \sqrt{-1}$. By (13), it follows

$$\sum_{j\ge0} A_{2j+1}\frac{(\imath x)^{2j+1}}{(2j+1)!} = \tan(\imath x) = \imath\tanh x$$

and (17) becomes (16). □

Since $\mathrm{sa}(G) = \mathrm{sa}(G;0)$, putting $t = 0$ in the generating functions (8), (11), (14), and (16) yields the generating functions for signed a-numbers of path graphs, cycle graphs, complete graphs, and star graphs as follows:

$$\sum_{n\geq 0} \mathrm{sa}(P_n)x^n = \frac{-1+\sqrt{1+4x^2}}{2x^2} = \sum_{m\geq 0}(-1)^m \,\mathrm{Cat}_m\, x^{2m},$$

$$\sum_{n\geq 0} \mathrm{sa}(C_n)x^n = \frac{1}{2} + \frac{1}{2\sqrt{1+4x^2}} = 1 + \sum_{m\geq 1}\frac{(-1)^m}{2}\binom{2m}{m}x^{2m},$$

$$\sum_{n\geq 0} \mathrm{sa}(K_n)\frac{x^n}{n!} = \operatorname{sech} x = \sum_{m\geq 0}(-1)^m A_{2m}\frac{x^{2m}}{(2m)!},$$

$$\sum_{n\geq 0} \mathrm{sa}(K_{1,n})\frac{x^n}{n!} = -\tanh x = \sum_{m\geq 1}(-1)^m A_{2m-1}\frac{x^{2m-1}}{(2m-1)!}.$$

Similarly, since $\mathrm{b}(G) = \mathrm{sa}(G;1)$, putting $t = 1$ in the generating functions (8), (11), (14), and (16) yields the generating functions for b-numbers of path graphs, cycle graphs, complete graphs, and star graphs as follows:

$$\sum_{n\geq 0} \mathrm{b}(P_n)x^n = 1 + \frac{-1+\sqrt{1+4x^2}}{2x} = 1 + \sum_{m\geq 0}(-1)^m \,\mathrm{Cat}_m\, x^{2m+1},$$

$$\sum_{n\geq 0} \mathrm{b}(C_n)x^n = 1 + \frac{x}{\sqrt{1+4x^2}} = 1 + \sum_{m\geq 0}(-1)^m\binom{2m}{m}x^{2m+1},$$

$$\sum_{n\geq 0} \mathrm{b}(K_n)\frac{x^n}{n!} = 1 + \tanh x = 1 + \sum_{m\geq 0}(-1)^m A_{2m+1}\frac{x^{2m+1}}{(2m+1)!},$$

$$\sum_{n\geq 0} \mathrm{b}(K_{1,n})\frac{x^n}{n!} = \operatorname{sech} x = \sum_{m\geq 0}(-1)^m A_{2m}\frac{x^{2m}}{(2m)!}.$$

According to (7), putting $t \leftarrow \frac{1}{\sqrt{-z}}$ and $x \leftarrow x\sqrt{-z}$ in the generating functions (8), (11), (14), and (16) yields the next result.

Corollary 5. *Let $P_{M(G)}(z)$ denote the Poincaré polynomials of the real toric manifolds $M(G)$ associated to the graph G. Then the generating functions for Poincaré polynomials of the real toric manifolds associated to path graphs P_n, cycle graphs C_n, complete graphs K_n, and star graphs $K_{1,n}$ are as follows*:

$$\sum_{n\geq 0} P_{M(P_n)}(z)x^n = \frac{-1+2x+\sqrt{1-4zx^2}}{2x-2(1+z)x^2},$$

$$\sum_{n\geq 0} P_{M(C_n)}(z)x^n = \frac{1}{2} + \frac{1}{2\sqrt{1-4zx^2}}\cdot\frac{(1-z)x+\sqrt{1-4zx^2}}{1-(1+z)x},$$

$$\sum_{n\geq 0} P_{M(K_n)}(z)\frac{x^n}{n!} = e^x\sec(x\sqrt{z}),$$

$$\sum_{n\geq 0} P_{M(K_{1,n})}(z)\frac{x^n}{n!} = e^x\left(1+\sqrt{z}\tan(x\sqrt{z})\right).$$

4. Signed a-number of complete multipartite graphs

First, we consider the exponential generating function for signed a-numbers of complete bipartite graphs. Denote by $K_{p,q}$ the *complete bipartite graph* with p-set and q-set.

Theorem 6. *The exponential generating function for signed a-numbers of complete bipartite graphs is*

$$\sum_{p\geq 0}\sum_{q\geq 0}\operatorname{sa}(K_{p,q})\frac{x^p}{p!}\frac{y^q}{q!} = \frac{\cosh x+\cosh y-1}{\cosh(x+y)}. \tag{18}$$

Proof. For two nonnegative integers p and q whose sum is even, there is the recurrence

$$\sum_{i,j\geq 0}\binom{p}{i}\binom{q}{j}\operatorname{sa}(K_{i,j}) = \begin{cases} 0 & \text{if } p \text{ and } q \text{ are positive,}\\ 1 & \text{if } p \text{ or } q \text{ is zero.}\end{cases} \tag{19}$$

The exponential generating function for the right-hand side of (19) is

$$\sum_{\substack{p,q\geq 0\\ p+q=\text{even}}}(RHS)\,\frac{x^p}{p!}\frac{y^q}{q!} = 1+(\cosh x-1)+(\cosh y-1). \tag{20}$$

The exponential generating function for the left-hand side of (19) is

$$\begin{aligned}\sum_{\substack{p,q\geq 0\\ p+q=\text{even}}}(LHS)\,\frac{x^p}{p!}\frac{y^q}{q!} &= \sum_{\substack{p,q\geq 0\\ p+q=\text{even}}}\ \sum_{\substack{0\leq i\leq p\\ 0\leq j\leq q\\ i+j=\text{even}}}\left(\operatorname{sa}(K_{i,j})\frac{x^i}{i!}\frac{y^j}{j!}\right)\left(\frac{x^{p-i}}{(p-i)!}\frac{y^{q-j}}{(q-j)!}\right)\\ &= \left(\sum_{\substack{i,j\geq 0\\ i+j=\text{even}}}\operatorname{sa}(K_{i,j})\frac{x^i}{i!}\frac{y^j}{j!}\right)\left(\sum_{\substack{i,j\geq 0\\ i+j=\text{even}}}\frac{x^i}{i!}\frac{y^j}{j!}\right)\\ &= \left(\sum_{p,q\geq 0}\operatorname{sa}(K_{p,q})\frac{x^p}{p!}\frac{y^q}{q!}\right)\cosh(x+y). \quad (21)\end{aligned}$$

Thus, by (20) and (21), we are done. □

The generating function $SA_q(x)$ is defined by $SA_q(x) = \sum_{p\geq 0}\operatorname{sa}(K_{p,q})\frac{x^p}{p!}$, which is the coefficient of $y^q/q!$ in $\frac{\cosh x+\cosh y-1}{\cosh(x+y)}$. Given a fixed nonnegative q, we can induce the detailed formula $SA_q(x)$ by

$$SA_q(x) = \frac{\partial^q}{\partial y^q}\left(\frac{\cosh x+\cosh y-1}{\cosh(x+y)}\right)\Bigg|_{y=0}.$$

For example, the initial generating functions $A_q(x)$ are listed as follows:

$$\begin{aligned}
SA_0(x) &= 1,\\
SA_1(x) &= -\tanh x,\\
SA_2(x) &= -2\operatorname{sech}^2 x + \operatorname{sech} x + 1,\\
SA_3(x) &= (6\operatorname{sech}^2 x - 3\operatorname{sech} x - 1)\tanh x,\\
SA_4(x) &= 24\operatorname{sech}^4 x - 12\operatorname{sech}^3 x - 20\operatorname{sech}^2 x + 7\operatorname{sech} x + 1.
\end{aligned}$$

Next, we generalize the generating function (18) for complete multipartite graphs. Denote by $K_{p_1,\dots,p_m}$ the *complete m-partite graph* with p_1-set, $\dots$, p_m-set.

Theorem 7. *The exponential generating function for signed a-numbers of complete m-partite graphs is*

(22)
$$\sum_{p_1,\dots,p_m\geq 0} \mathrm{sa}(K_{p_1,\dots,p_m})\frac{x_1^{p_1}}{p_1!}\cdots\frac{x_m^{p_m}}{p_m!} = \frac{(1-m)+\cosh x_1+\cdots+\cosh x_m}{\cosh(x_1+\cdots+x_m)}.$$

Proof. For m nonnegative integers $p_1,\dots,p_m$ whose sum is even, there is the recurrence

$$\begin{aligned}
&\sum_{i_1,\dots,i_m\geq 0}\binom{p_1}{i_1}\cdots\binom{p_m}{i_m}\mathrm{sa}(K_{i_1,\dots,i_m})\\
&= \begin{cases} 0 & \text{if at least two } p_i\text{'s are positive,}\\ 1 & \text{if all } p_i\text{'s are zeros, but at most one.}\end{cases}
\end{aligned} \tag{23}$$

Using both sides of (23), we have the generalized formulas of (20) and (21) as follows:

$$\sum_{\substack{p_i\geq 0\\ p_1+\cdots+p_m=\text{even}}} (RHS)\ \frac{x_1^{p_1}}{p_1!}\cdots\frac{x_m^{p_m}}{p_m!} = 1+(\cosh x_1-1)+\cdots+(\cosh x_m-1)$$

and

$$\begin{aligned}
&\sum_{\substack{p_i\geq 0\\ p_1+\cdots+p_m=\text{even}}} (LHS)\ \frac{x_1^{p_1}}{p_1!}\cdots\frac{x_m^{p_m}}{p_m!}\\
&= \left(\sum_{p_1,\dots,p_m\geq 0}\mathrm{sa}(K_{p_1,\dots,p_m})\frac{x_1^{p_1}}{p_1!}\cdots\frac{x_m^{p_m}}{p_m!}\right)\cosh(x_1+\cdots+x_m),
\end{aligned}$$

which completes the proof. □

Remark. Since $\mathrm{a}(K_{p_1,\dots,p_m}) = (-1)^{\frac{p_1+\cdots+p_m}{2}}\mathrm{sa}(K_{p_1,\dots,p_m})$ and $\cosh(\imath z)=\cos z$, the exponential generating functions for a-numbers of complete bipartite graphs and complete m-partite graphs are equal to

$$\sum_{p\geq 0}\sum_{q\geq 0}\mathrm{a}(K_{p,q})\frac{x^p}{p!}\frac{y^q}{q!}=\frac{\cos x+\cos y-1}{\cos(x+y)},$$

$$\sum_{p_1,\ldots,p_m\geq 0}\mathrm{a}(K_{p_1,\ldots,p_m})\frac{x_1^{p_1}}{p_1!}\cdots\frac{x_m^{p_m}}{p_m!}=\frac{(1-m)+\cos x_1+\cdots+\cos x_m}{\cos(x_1+\cdots+x_m)}.$$

5. Signed a-polynomial of complete multipartite graphs

First, we consider the exponential generating function for signed a-polynomials of complete bipartite graphs.

Theorem 8. *Let $K_{p,q}$ be the complete bipartite graph with p-set and q-set. Then the exponential generating function for signed a-polynomials of $K_{p,q}$ is given by*

$$\sum_{p\geq 0}\sum_{q\geq 0}\mathrm{sa}(K_{p,q};t)\frac{x^p}{p!}\frac{y^q}{q!}=e^{t(x+y)}\left(\frac{\cosh x+\cosh y-1}{\cosh(x+y)}\right). \tag{24}$$

Proof. By definition, we have

(25)

$$\sum_{\substack{p\geq 0\\ q\geq 0}}\mathrm{sa}(K_{p,q};t)\frac{x^p}{p!}\frac{y^q}{q!}=\sum_{\substack{p\geq 0\\ q\geq 0}}\left(\sum_{\substack{0\leq p'\leq p\\ 0\leq q'\leq q}}\binom{p}{p'}\binom{q}{q'}\mathrm{sa}(K_{p',q'})t^{p-p'+q-q'}\right)\frac{x^p}{p!}\frac{y^q}{q!}.$$

Substituting $p''=p-p'$ and $q''=q-q'$, the right-hand side of (25) becomes

$$\sum_{\substack{p''\geq 0\\ q''\geq 0}}\left(\sum_{\substack{p'\geq 0\\ q'\geq 0}}\binom{p'+p''}{p'}\binom{q'+q''}{q'}\mathrm{sa}(K_{p',q'})t^{p''+q''}\right)\frac{x^{p'+p''}}{(p'+p'')!}\frac{y^{q'+q''}}{(q'+q'')!}$$

$$=\left(\sum_{p'\geq 0}\sum_{q'\geq 0}\mathrm{sa}(K_{p',q'})\frac{x^{p'}}{p'!}\frac{y^{q'}}{q'!}\right)\left(\sum_{p''\geq 0}\frac{(tx)^{p''}}{p''!}\right)\left(\sum_{q''\geq 0}\frac{(ty)^{q''}}{q''!}\right).$$

The formula (18) completes the proof. □

Remark. Since the coefficient of $\frac{y^q}{q!}$ in formula (24) is equal to

$$\sum_{n\geq 0}\mathrm{sa}(K_{q,n};t)\frac{x^n}{n!},$$

it holds that

$$\sum_{n\geq 0}\mathrm{sa}(K_{q,n};t)\frac{x^n}{n!}=\frac{\partial^q}{\partial y^q}e^{t(x+y)}\left(\frac{\cosh x+\cosh y-1}{\cosh(x+y)}\right)\Bigg|_{y=0}.$$

In case of $q=1$, we have the exponential generating function (16) for signed a-polynomials of star graphs again.

Similarly, we can deduce the next theorem by the same above method.

Theorem 9. *Let $K_{p_1,\dots,p_m}$ be the complete m-partite graph with p_1-set, ..., p_m-set. Then the exponential generating function for signed a-polynomials of $K_{p_1,\dots,p_m}$ is given by*

$$\begin{aligned}(26)\qquad &\sum_{p_1\dots,p_m\geq 0} \mathrm{sa}(K_{p_1,\dots,p_m};t)\frac{x_1^{p_1}}{p_1!}\cdots\frac{x_m^{p_m}}{p_m!}\\ &= e^{t(x_1+\cdots+x_m)}\left(\frac{(1-m)+\cosh x_1+\cdots+\cosh x_m}{\cosh(x_1+\cdots+x_m)}\right).\end{aligned}$$

Since $\mathrm{sa}(G)=\mathrm{sa}(G;0)$, putting $t=0$ in the generating functions (24) and (26) gives the two formulas (18) and (22), respectively. Also, since $\mathrm{b}(G)=\mathrm{sa}(G;1)$, putting $t=1$ in the generating functions (24) and (26) yields the generating functions for b-numbers of complete bipartite graphs and complete multipartite graphs as follows:

$$\sum_{p\geq 0}\sum_{q\geq 0}\mathrm{b}(K_{p,q})\frac{x^p}{p!}\frac{y^q}{q!} = e^{x+y}\left(\frac{\cosh x+\cosh y-1}{\cosh(x+y)}\right),$$

$$\begin{aligned}&\sum_{p_1\dots,p_m\geq 0} \mathrm{b}(K_{p_1,\dots,p_m})\frac{x_1^{p_1}}{p_1!}\cdots\frac{x_m^{p_m}}{p_m!}\\ &= e^{x_1+\cdots+x_m}\left(\frac{(1-m)+\cosh x_1+\cdots+\cosh x_m}{\cosh(x_1+\cdots+x_m)}\right).\end{aligned}$$

The next result follows from two generating functions (24) and (26) by plugging in (7).

Corollary 10. *Let $P_{M(K_{p,q})}(z)$ and $P_{M(K_{p_1,\dots,p_m})}(z)$ denote the Poincaré polynomials of the real toric manifolds associated to the complete bipartite graph $K_{p,q}$ and the complete m-partite graph $K_{p_1,\dots,p_m}$. Then the generating functions for Poincaré polynomials $P_{M(K_{p,q})}(z)$ and $P_{M(K_{p_1,\dots,p_m})}(z)$ are equal to*

$$\sum_{n\geq 0} P_{M(K_{p,q})}(z)\frac{x^p}{p!}\frac{y^q}{q!} = e^{x+y}\left(\frac{\cos(x\sqrt{z})+\cos(y\sqrt{z})-1}{\cos(x\sqrt{z}+y\sqrt{z})}\right),$$

$$\begin{aligned}&\sum_{n\geq 0} P_{M(K_{p_1,\dots,p_m})}(z)\frac{x_1^{p_1}}{p_1!}\cdots\frac{x_m^{p_m}}{p_m!}\\ &= e^{x_1+\cdots+x_m}\left(\frac{(1-m)+\cos(x_1\sqrt{z})+\cdots+\cos(x_m\sqrt{z})}{\cos(x_1\sqrt{z}+\cdots+x_m\sqrt{z})}\right).\end{aligned}$$

Table 5 shows the Poincaré polynomials $P_{M(K_{p,q})}(z)$ for $p\leq 6$ and $q\leq 3$.

TABLE 5. Table for $P_{M(K_{p,q})}(z)$

$p\backslash q$	0	1	2	3
0	1	1	1	1
1	1	$1+z$	$1+2z$	$1+3z+2z^2$
2	1	$1+2z$	$1+4z+3z^2$	$1+6z+13z^2$
3	1	$1+3z+2z^2$	$1+6z+13z^2$	$1+9z+39z^2+31z^3$
4	1	$1+4z+8z^2$	$1+8z+34z^2+27z^3$	$1+12z+86z^2+205z^3$
5	1	$1+5z+20z^2+16z^3$	$1+10z+70z^2+167z^3$	$1+15z+160z^2+763z^3+617z^4$
6	1	$1+6z+40z^2+96z^3$	$1+12z+125z^2+597z^3+483z^4$	$1+18z+267z^2+2123z^3+5151z^4$

6. Remarks

We have found the signed a-polynomial of complete multipartite graph. For a general graph G, it is not easy to characterize $\mathrm{sa}(G;t)$. Even the case of the tree, it is hard to find the close formula of its signed a-numbers. For example, letting $T_{p,q,r}$ be the tree induced by connecting one vertex and each end-vertices of P_p, P_q, and P_r by three edges, we are not able to find a closed form of $\mathrm{sa}(T_{p,q,r};t)$ with p, q, and r. In order to find the general formulas for the signed a-polynomial of any graph, we need a method to calculate the signed a-numbers of a specific graph without a recursive definition.

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