

NONLINEAR DIFFERENTIAL INCLUSIONS OF SEMIMONOTONE AND CONDENSING TYPE IN HILBERT SPACES

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ABSTRACT. In this paper, we study the existence of classical and generalized solutions for nonlinear differential inclusions $x'(t) \in F(t, x(t))$ in Hilbert spaces in which the multifunction F on the right-hand side is hemicontinuous and satisfies the semimonotone condition or is condensing. Our existence results are obtained via the selection and fixed point methods by reducing the problem to an ordinary differential equation. We first prove the existence theorem in finite dimensional spaces and then we generalize the results to the infinite dimensional separable Hilbert spaces. Then we apply the results to prove the existence of the mild solution for semilinear evolution inclusions. At last, we give an example to illustrate the results obtained in the paper.

1. Introduction

Our aim is to study the nonlinear differential inclusion of first order

$$(1.1) \quad \begin{aligned} x'(t) &\in F(t, x(t)), \\ x(0) &= x_0, \end{aligned}$$

in a Hilbert spaces in which F may be condensing or semimonotone set-valued map. An approach to investigating the existence of solution for a differential inclusion is to reduce it to a problem for ordinary differential equations. In this setting, we are interested to know under what conditions the solution of the corresponding ordinary differential equation belongs to the right side of the given differential inclusion; in other words, under what conditions a continuous function such as $f(\cdot, \cdot)$ exists so that $f(t, x) \in F(t, x)$ for every (t, x) and $x'(t) = f(t, x(t))$. Thus, fixed point theorems and selection theorems to some of which we point out in Section 2, are the main tools in this method. The corresponding ordinary differential equation $x'(t) = f(t, x(t))$ with $x(0) = x_0$, in case that $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous and Lipschitz (or Locally Lipschitz) with

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respect to the second variable, according to the Picard-Lindelof theorem [1], has a unique classical solution (local solution). However, it is well-known that continuity condition alone does not suffice for the problem to have even a local solution in infinite dimensional spaces. In a more general case, the function f is considered to satisfy Caratheodory condition and therefore, solutions of Caratheodory type are obtained. To study the semilinear and nonlinear Cauchy differential equations in Hilbert spaces, we refer to [9, 10, 19, 20, 22].

For differential inclusions, the conditions we impose on set-valued map F are usually a combination of two types: First, regularity of the map F such as various kinds of continuity, semicontinuity and monotonicity condition. Second, geometrical conditions such as compactness, connectedness and convexity of the values of F . Existence of solution for differential inclusions have been studied by many authors in the past half century with different application-directed motivations, including the issues of control and optimization, dynamical systems and even biological sciences [11, 12, 15, 16, 21]. In the earlier works, set-valued maps have been usually considered with convex values [6, 7]. Next, in the case that the images are nonconvex, the existence theorems were also studied by some authors. For example, Bressan [7] proved the existence theorems when the values of set-valued map are completely disconnected subsets of a finite dimensional space. Also, in [2, 3] differential inclusion with convex and nonconvex values on infinite dimensional Banach spaces have systematically been studied. In the case that set-valued map F is nonexpansive or Lipschitz in Hausdorff metric topology, several inequalities such as the Gronwall inequality give us a lot of useful information about properties of the set of trajectories of solutions [2, 7, 21, 24]. However, nonexpansive and Lipschitz conditions are very strong and are not of practical importance. For example, in optimal control problems, set-valued functions are often defined as $F(t, x) = \{f(t, x, u) : u \in U\}$, where U is a metric space and the single-valued function f is defined on $[0, T] \times \mathbb{R}^n \times U$ into \mathbb{R}^n . Here F need not be Lipschitz even if the function f is. Thus, replacing Lipschitz condition with a weaker one, would be very valuable.

When the set-valued map $A = -F$ is maximal monotone on an infinite dimensional Hilbert space, the resolvent map $J_\lambda = (I + \lambda A)^{-1}$ and the Yosida approximation $A_\lambda = \frac{1}{\lambda}(I - J_\lambda)$ of A are single-valued. Also, $\lim_{\lambda \rightarrow 0} A_\lambda(x) \in A(x)$ for every $x \in D(J_\lambda)$. In this case, like nonlinear differential equations with maximal monotone condition [23], first, a set of approximate solutions x_λ are obtained for differential equations $x'_\lambda(t) = A_\lambda(x_\lambda(t))$. The images of these solutions under A_λ constitute a weakly compact subset of the space $L^\infty(0, T; H)$ and finally a Cauchy subsequence $\{x_{\lambda_n}\}_{n=1}^\infty$ is extracted such that x'_{λ_n} converges weakly to x' in $L^2(0, T; H)$ and x_{λ_n} converges strongly to x in $L^2(0, T; H)$. Since A is semiclosed, this subsequence is convergent to the unique solution x of (1.1).

In this paper, we establish the existence and uniqueness of the classical and generalized solution for monotone-type differential inclusion (1.1). Our

novelty is mainly in the conditions we will impose on the nonlinearity F and in the method we will use to prove the existence result which is, in fact, based upon continuous and measurable selection theorems and Kakutani's fixed point theorem.

The paper is organized as follows. In Section 2, we provide some definitions and preliminaries that are required in the next sections. We have allocated Sections 3 and 4 to the existence of generalized solution for problem (1.1). In these sections, we aim to present our main theorems and results when the multifunction F is semimonotone hemicontinuous or condensing. Section 5 includes an application of the results of the previous sections to the existence of the mild solution for semilinear differential inclusions. Finally, we give an example.

2. Definitions and preliminaries

Let X be a Banach space. We assume that $P(X)$ is the family of all non-empty subsets of X and $P_c(X)$ (resp. $P_{c,v}(X)$, $P_{k,v}(X)$ and $P_{b,c,v}(X)$) is the family of all non-empty closed (resp. closed convex, compact convex and bounded closed convex) subsets of X . A set-valued map $F : X \rightarrow P(Y)$ where Y is another Banach space, is called upper semicontinuous (*u.s.c*) if for every open subset U of Y , the set $\{x \in X : F(x) \subseteq U\}$ is open in X . By Proposition 1.2.5 of [18], upper semicontinuity is equivalent to the following: for each sequence $\{x_n\}_{n=1}^\infty$ in X such that $x_n \rightarrow x$ as $n \rightarrow \infty$ and for every $\varepsilon > 0$, there exists a positive integer N such that

$$F(x_n) \subseteq F(x) + \varepsilon B, \quad \forall n \geq N,$$

where B is the open unit ball in Y . The map F is called lower semicontinuous (*l.s.c*) if for every open subset U of Y , the set $\{x \in X : F(x) \cap U \neq \emptyset\}$ is open in X . By Proposition 1.2.6 of [18], lower semicontinuity is equivalent to the following: for each sequence $\{x_n\}_{n=1}^\infty$ in X which converges to x and for each $y \in F(x)$, there exists a sequence $\{y_n\}_{n=1}^\infty$ in Y such that $y_n \in F(x_n)$ for all n and $y_n \rightarrow y$ as $n \rightarrow \infty$. Moreover, the set-valued map F is called closed when its graph $\{(x, y) : y \in F(x)\}$ is closed in the product space $X \times Y$ and it is called compact if $\overline{F(X)}$ is a compact subset of Y . If for each $x \in X$ there exists a neighborhood U_x of x such that $F(U_x)$ is relatively compact in Y , then F is called locally compact. In this paper, we apply Proposition 1.2.23 of [18] to prove upper semicontinuity of set-valued maps:

Proposition 2.1. *Let $F : X \rightarrow P_c(Y)$ be closed and locally compact. Then F is u.s.c.*

Suppose that $T > 0$ and $I = [0, T]$ is an interval on the real line equipped with the σ -field of Lebesgue measurable sets. The set-valued map $F : I \rightarrow P(X)$ is called measurable if for every open subset U of X , the set $\{t \in I : F(t) \cap U \neq \emptyset\}$ is measurable in I . To say that F is measurable is the same as

for every $x \in X$, the map $y : I \rightarrow \mathbb{R}$ defined by $y(t) = d(x, F(t)) = \inf\{d(x, y) : y \in F(t)\}$ is measurable [18].

A single-valued function $f : X \rightarrow Y$ is called a *selection* for the multifunction $F : X \rightarrow P(Y)$ provided that $f(x) \in F(x)$ for all $x \in X$. According to the Michael selection theorem [18], if X is a metric and Y is a Banach space, then any *l.s.c* set-valued map $F : X \rightarrow P_{c,v}(Y)$ has a continuous selection. Moreover, if X is separable, then there exists a sequence of continuous selections $\{f_n\}_{n=1}^\infty$ such that for every $x \in X$, $F(x) = \overline{\{f_n(x)\}}$. One may also be interested in measurable selections as we do in this paper. We use the Kuratowski-Ryll Nardzewski selection theorem [18] which says that if X is a separable complete metric space and the set-valued map $F : I \rightarrow P_c(X)$ is measurable, then F has a measurable selection.

A two-variable multifunction $F : I \times X \rightarrow P(Y)$ is said to satisfy the Caratheodory condition, if (i) for every $x \in X$, the function $t \rightarrow F(t, x)$ is measurable; (ii) for *a.e.* $t \in I$, the function $x \mapsto F(t, x)$ is *u.s.c*. The next theorem which is a parametric version of Michael selection theorem, provides the conditions under which a set-valued map has a selection of Caratheodory type on a separable Banach space.

Theorem 2.2 ([18]). *Let Y be a separable Banach space and the set-valued map $F : I \times X \rightarrow P_{c,v}(Y)$ satisfy the following hypotheses:*

- i) *The map $(t, x) \mapsto F(t, x)$ is measurable;*
- ii) *For *a.e.* $t \in I$, the map $x \rightarrow F(t, x)$ is *l.s.c*.*

Then F has a Caratheodory selection. In other words, there is a function $f : I \times X \rightarrow Y$ which is measurable in the first and continuous in the second variable and $f(t, x) \in F(t, x)$ for all $(t, x) \in I \times X$.

If $F : I \rightarrow P(X)$ is measurable, we denote by S_F the family of all measurable selections of F and for $p \geq 1$, we set $S_F^p = L^p(I, X) \cap S_F$. A set-valued map F is called integrable if $S_F^1 \neq \emptyset$. In that case, the Aumann integral of the set-valued map F on I is defined as

$$\int_I F(t)dt = \left\{ \int_I f(t)dt : f \in S_F^1 \right\}.$$

For the properties of Aumann integral of multifunctions, see e.g. [4] or Chapter 8 of [3].

Next, we assume that H is a real separable Hilbert space. Given a set-valued map $F : I \times H \rightarrow P(H)$, for each continuous function $x : I \rightarrow H$, we define

$$S_{F,x}^p = \{v \in L^p(I; H) : v(t) \in F(t, x(t)) \text{ for a.e. } t \in [0, T]\}.$$

In view of the definition of the integral of set-valued maps, for any continuous function $x : I \rightarrow H$, we have

$$\int_I F(t, x(t))dt = \left\{ \int_I v(t)dt : v \in S_{F,x}^1 \right\}.$$

Consider the following initial value problem for differential inclusion

$$(2.1) \quad \begin{cases} x'(t) \in F(t, x(t)), & t \in [0, T], \\ x(0) = x_0. \end{cases}$$

Here, $x_0 \in H$ and $F : [0, T] \times H \rightarrow P(H)$ is a given set-valued map. An absolutely continuous solution for the above differential inclusion is called *classical solution*. Along with the above differential inclusion, we consider the integral equation

$$(2.2) \quad \begin{cases} x(t) = x_0 + \int_0^t v(s)ds, & t \in [0, T], \\ x(0) = x_0, \end{cases}$$

where $v \in S_{F,x}^1$. If the map v is continuous on $[0, T]$, then the solution x of integral equation (2.2) is continuously differentiable on $[0, T]$. This is called *the strong solution* for problem (2.1). In general, a continuous solution x for the integral equation (2.2) is called a *generalized solution* for the problem (2.1). In order to prove the existence of the generalized solution for problem (2.1), we impose the following hypotheses on set-valued map F :

- (H1) F is semimonotone with constant $M > 0$; that is, for every $x_1, x_2 \in H$, $t \in [0, T]$ and $y_1 \in F(t, x_1)$ and $y_2 \in F(t, x_2)$, we have

$$\langle x_1 - x_2, y_1 - y_2 \rangle \leq M \|x_1 - x_2\|^2;$$

- (H2) There is a continuous function $h : [0, T] \times [0, \infty) \rightarrow [0, \infty)$ such that for each $t \in [0, T]$, the function $u \rightarrow h(t, u)$ is monotone increasing and $\|F(t, x)\| \leq h(t, \|x\|)$ for all $t \in [0, T]$ and $x \in H$;
- (H3) The map F is hemicontinuous; i.e., for each sequence $\{x_n\}_{n=1}^\infty$ in H which is convergent to x and each $t \in [0, T]$, if $y \in F(t, x)$, then there exists a sequence $\{y_n\}_{n=1}^\infty$ in H such that for all $n \geq 1$, $y_n \in F(t, x_n)$ and $y_n \xrightarrow{w} y$ in H .

Remark 2.3. Note that if the set-valued map F is *l.s.c* with respect to the second variable, then F is hemicontinuous.

If the set-valued map F is Caratheodory with closed convex values and satisfies hypothesis (H2), then any generalized solution of the differential inclusion (2.1) is classical solution. Indeed, since $x(t) - x(s) \in \int_s^t F(\tau, x(\tau))d\tau$ for all $s, t \in [0, T]$, we have

$$\|x(t) - x(s)\| \leq \int_s^t \|F(\tau, x(\tau))\| d\tau \leq \int_s^t h(\tau, \|x(\tau)\|)d\tau \leq K |t - s|$$

for some constant $K > 0$. Therefore, x is *a.e.* differentiable. For every $\varepsilon > 0$, there is a positive number δ such that if $|t - s| < \delta$, then $F(s, x(s)) \subseteq F(t, x(t)) + \varepsilon B$ in which B is the unit open ball in H . So,

$$x(t) - x(s) \in \int_s^t (F(\tau, x(\tau)) + \varepsilon B) d\tau = |t - s| (F(t, x(t)) + \varepsilon B).$$

Therefore, letting ε approach to zero, we obtain $x'(t) \in F(t, x(t))$. Thus x is a classical solution.

The Hausdorff measure of noncompactness on the Hilbert space H is a function $\alpha : P_b(H) \rightarrow [0, \infty)$ which is defined by

$$\alpha(A) = \inf \left\{ \varepsilon > 0 : A \subseteq \bigcup_{i=1}^n X_i, \quad \text{diam}(X_i) \leq \varepsilon \right\}, \quad \forall A \in P_b(H).$$

It is not difficult to see that for all $A, B \in P_b(H)$, (1) if $A \subseteq B$, then $\alpha(A) \leq \alpha(B)$; (2) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$; (3) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$; (4) for every $\lambda \in \mathbb{R}$, $\alpha(\lambda A) = |\lambda| \alpha(A)$; (5) $\alpha(\text{co}(A)) = \alpha(A)$, $\alpha(\bar{A}) = \alpha(A)$, where $\text{co}(A)$ is the convex hull of A ; and finally (6) $\alpha(A) = 0$ if and only if A is relatively compact. In general, any function on $P_b(H)$ with the properties (1), (2), (3), (5) and (6), is called a regular measure of noncompactness on H . Let Y be a Banach space and β a regular measure of noncompactness on Y . A set-valued map $G : Y \rightarrow P(Y)$ is called β -(countable) condensing if for every (countable) bounded subset A of Y such that $\beta(A) > 0$, we have $\beta(G(A)) < \beta(A)$. In Section 3, we will need a generalized form of the Monch fixed point theorem (Theorem 4.16 of [1]) for condensing set-valued maps:

Theorem 2.4. *Suppose that C is a closed convex subset of the Banach space Y and let the closed set-valued map $G : C \rightarrow P_{k,v}(C)$ be condensing with respect to the regular measure of noncompactness β . Then G has a fixed point.*

To define the measure of noncompactness for the bounded subsets of the space $C(I; H)$ of all continuous functions $x : I \rightarrow H$ equipped with the sup-norm, we set $\alpha_C(\Omega) = \sup_{t \in I} e^{-Lt} \alpha(\Omega(t))$, for each $\Omega \in P_b(C(I; H))$ where L is a suitable constant to be determined latter in the paper and $\Omega(t) = \{x(t) : x \in \Omega\}$. One can easily check that α_C has all properties of the measure of noncompactness except for the regularity property. In other words, if $\Omega \subseteq C(I; H)$ is a relatively compact set, then $\alpha_C(\Omega) = 0$, but the reverse may not be hold. However, if Ω is a family of equicontinuous functions and $\alpha_C(\Omega) = 0$, then Ω is relatively compact subset of $C(I; H)$. To measure the equicontinuity of the subsets of $C(I; H)$, we use the modulus of equicontinuity. This is defined as

$$\text{mod}_C(\Omega) = \limsup_{\delta \rightarrow 0} \sup_{x \in \Omega} \sup_{|t_1 - t_2| \leq \delta} \|x(t_1) - x(t_2)\|.$$

Note that $\text{mod}_C(\Omega) = 0$ if and only if $\Omega \subseteq C(I; H)$ is an equicontinuous family. Now, we are ready to define a regular measure of noncompactness on $C(I; H)$:

$$\chi(\Omega) = \max_{D \in \Lambda(\Omega)} \{\alpha_C(D), \text{mod}_C(D)\}$$

for all $\Omega \in P_b(C(I; H))$. Here $\Lambda(\Omega)$ is the family of all countable subsets of Ω .

3. Semimonotone set-valued maps

In this section, we first prove the existence theorem for differential inclusion (1.1) when F is a semimonotone set-valued map on a finite dimensional space and then we generalize it to the infinite dimensional Hilbert spaces. Note that without loss of generality, we can assume that $x(0) = 0$.

Theorem 3.1. *Let the Caratheodory set-valued map $F : [0, T] \times \mathbb{R}^n \rightarrow P_{c,v}(\mathbb{R}^n)$ satisfy hypothesis (H2) and the differential equation $u' = h(t, u)$ with $u(0) = 0$ has a solution $u(t)$ existing on $[0, T]$. Then, the differential inclusion*

$$(3.1) \quad \begin{cases} x'(t) \in F(t, x(t)), & t \in [0, T], \\ x(0) = 0, \end{cases}$$

has a classical solution.

Proof. We are to show that there exists a function $x \in C(I; \mathbb{R}^n)$ such that $x(t) \in \int_0^t F(s, x(s))ds$ for all $t \in [0, T]$. To this end, we define the set-valued map Λ on $C(I; \mathbb{R}^n)$ as

$$\Lambda(x) = \{y : I \rightarrow \mathbb{R}^n \mid y(t) = \int_0^t v(s)ds, \forall t \in [0, T] \text{ for some } v \in S_{F,x}^1\}.$$

Since each Caratheodory map is measurable, by Kuratowski-Ryll Nardzewski selection theorem and hypothesis (H2), the map Λ has non-null values. We consider the closed convex subset

$$K = \{x \in C(I; \mathbb{R}^n) : x(0) = 0, \|x(t)\| \leq u(t) \quad \forall t \in [0, T]\},$$

of the space $C(I, \mathbb{R}^n)$. First, we show that $\Lambda : K \rightarrow P(K)$. Indeed, for all $t \in [0, T]$, $x \in K$ and $v \in S_{F,x}^1$ we have

$$\left\| \int_0^t v(s) \right\| \leq \int_0^t \|v(s)\| ds \leq \int_0^t h(s, \|x(s)\|) ds \leq \int_0^t h(s, u(s)) ds = u(t).$$

Next, it is easy to see that the values of Λ are convex. Let $\{y_n\}_{n=1}^\infty$ be a sequence in $\Lambda(x)$ which is convergent to y in $C(I, \mathbb{R}^n)$. Then, there exists a sequence $\{v_n\}_{n=1}^\infty$ in $S_{F,x}^1$ such that $y_n(t) = \int_0^t v_n(s)ds$ for all $t \in [0, T]$. Since $t \mapsto h(t, u(t))$ is a continuous function and

$$\|v_n(t)\| \leq \|F(t, x(t))\| \leq h(t, u(t))$$

for each $t \in [0, T]$ and every $n \geq 1$, we observe that $\{v_n\}_{n=1}^\infty$ is in fact a sequence of elements of the Hilbert space $L^2([0, T]; \mathbb{R}^n)$. So, it has a subsequence (we denote it by the same symbol) that is weakly convergent to a function $v \in L^2([0, T]; \mathbb{R}^n)$, say. Then, we conclude from Corollary V.3.14 of [14] that there is a sequence $\{w_n\}_{n=1}^\infty$ of convex combinations of $\{v_1, v_2, \dots\}$ which is, in particular, convergent to v in the norm topology of the space $L^1([0, T]; \mathbb{R}^n)$. Since F has convex values, $w_n \in S_{F,x}^1$. The sequence $\{w_n\}_{n=1}^\infty$ has a subsequence (we denote it again by the same symbol) which converges to v a.e. on

$[0, T]$. Because the values of F are closed, we have $v \in S_{F,x}^1$ and the dominated convergence theorem gives

$$y(t) = \lim_{n \rightarrow \infty} y_n(t) = \lim_{n \rightarrow \infty} \int_0^t v_n(s) ds = \int_0^t v(s) ds.$$

Thus, $\Lambda(x)$ is a closed set. Now, we prove that Λ is a closed map. Let sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ in K converge to the continuous functions x and y in the space $C(I; \mathbb{R}^n)$, respectively and furthermore $y_n \in \Lambda(x_n)$ for all n . By upper semicontinuity of F with respect to the second variable, for given $\varepsilon > 0$, large enough n and any $t \in [0, T]$, we have $F(t, x_n(t)) \subseteq F(t, x(t)) + \varepsilon B$. So,

$$y_n(t) \in \int_0^t F(s, x_n(s)) ds \subseteq \int_0^t F(s, x(s)) ds + t\varepsilon B.$$

Since ε was arbitrary, we obtain $y_n(t) \in \int_0^t F(s, x(s)) ds = \Lambda(x)$. This together with the result that $\Lambda(x)$ is closed, implies that $y(t) \in \Lambda(x)$ for all $t \in [0, T]$. Finally, we show that $\Lambda(K)$ is uniformly bounded and equicontinuous in the Banach space $C(I; \mathbb{R}^n)$. Uniform boundedness of Λ is a consequence of continuity of the function u on $[0, T]$ and the following dominance relation which by hypothesis H2, is true for all $x \in K$ and $y \in \Lambda(x)$:

$$\sup_{t \in I} \|y(t)\| \leq \int_0^T \|F(s, x(s))\| ds \leq \int_0^T h(s, u(s)) ds = u(T).$$

Also, given $y \in \Lambda(K)$, we can prove similarly that

$$\|y(t_1) - y(t_2)\| \leq |u(t_1) - u(t_2)|$$

for all $t_1, t_2 \in [0, T]$. Thus, by the Arzela-Ascoli theorem, $\Lambda(K)$ is a relatively compact subset of the space $C(I; \mathbb{R}^n)$. So, Λ is compact and it is also *u.s.c* by Proposition 2.1. Now, by Kakutani's fixed point theorem [17], Λ has a fixed point in K which is a classical solution of the differential inclusion (3.1). \square

In the above theorem, if we replaced the upper semicontinuity condition on F by lower semicontinuity, then according to Theorem 2.2, there is a single-valued Caratheodory function $f : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $f(t, x) \in F(t, x)$. Then, Theorem 3.2 of [20], indicates that the ordinary differential equation

$$\begin{cases} x'(t) = f(t, x(t)), & t \in [0, T], \\ x(0) = 0, \end{cases}$$

has a generalized solution on $[0, T]$. Thus, the above differential inclusion has a generalized solution on $[0, T]$. So we have the following theorem.

Theorem 3.2. *Let the integrable set-valued map $F : [0, T] \times \mathbb{R}^n \rightarrow P_{c,v}(\mathbb{R}^n)$ be l.s.c respect to the second variable and satisfy condition (H2). If the differential equation $u' = h(t, u)$ with $u(0) = 0$ has a solution $u(t)$ existing on $[0, T]$, then the differential inclusion (3.1) has a generalized solution.*

Now, we generalize the previous theorem to infinite dimensional real separable Hilbert spaces. Denote by $C_{l_0}(I; H)$ the space of all continuous functions $x : I \rightarrow H$ such that $x(0) = 0$ equipped with sup-norm. Also, $L^2_{l_0}(I; H, M)$ stands for the Hilbert space of all measurable functions $x : I \rightarrow H$ such that

$$\int_0^T e^{-2Mt} \|x(t)\|^2 dt < \infty,$$

with the inner product

$$\langle x, y \rangle_M = \int_0^T e^{-2Mt} \langle x(t), y(t) \rangle dt,$$

and the norm

$$\|x\|_M = \left(\int_0^T e^{-2Mt} \|x(t)\|^2 dt \right)^{\frac{1}{2}}.$$

Proposition 3.3. *Let the Caratheodory multifunction $F : [0, T] \times H \rightarrow P(H)$ satisfy hypothesis (H2). Define the set-valued map G on $C_{l_0}(I; H)$ as*

$$Gx = \{y : I \rightarrow H \mid y \in S^1_{F,x}, y(t) \in F(t, x(t)) \text{ for all } t \in I\}.$$

Then

- (a) $G : C_{l_0}(I; H) \rightarrow P(L^2_{l_0}(I; H, M))$ is bounded on bounded subsets of $C_{l_0}(I; H)$.
- (b) If F satisfies hypothesis (H1), then G is semimonotone with constant M .
- (c) If F satisfies hypothesis (H3), then G is hemicontinuous.

Proof. We refer to [20] for the proof of parts (a) and (b). In particular, if $B \subseteq C_{l_0}(I; H)$ is a bounded set, then there exists a constant $K > 0$ such that $\|Gx\|_M \leq \gamma(T)K$, where $\gamma(T) = \frac{1}{\sqrt{2M}}(1 - e^{-2MT})^{\frac{1}{2}}$. To prove (c), suppose that $\{x_n\}_{n=1}^\infty$ is a sequence which converges to x in $C_{l_0}(I; H)$. Since F is hemicontinuous, if $y \in G(x)$, then there is a sequence of functions $\{y_n\}_{n=1}^\infty$ such that $y_n(t) \in F(t, x_n(t))$ for each $n \geq 1$ and $t \in [0, T]$ and also, $y_n(t) \xrightarrow{w} y(t)$ in H as $n \rightarrow \infty$. According to hypothesis (H2), we have $\{y_n\}_{n=1}^\infty \subseteq L^2_{l_0}(I; H, M)$ and $\langle y_n(t) - y(t), z(t) \rangle \rightarrow 0$ as $n \rightarrow \infty$ for every $z \in L^2_{l_0}(I; H, M)$. Moreover, the Cauchy-Schwarz inequality implies that $|\langle y_n(t) - y(t), z(t) \rangle| \leq K \|z(t)\|$ for some constant $K > 0$. Now, by the dominated convergence theorem, we have

$$|\langle y_n - y, z \rangle_M| \leq \int_0^T e^{-2Mt} |\langle y_n(t) - y(t), z(t) \rangle| dt \rightarrow 0,$$

as $n \rightarrow \infty$. Therefore, $y_n \xrightarrow{w} y$ in $L^2_{l_0}(I; H, M)$ and the proof is complete. \square

Next, we generalize Lemma 1.2 of [9] for the set-valued hemicontinuous maps. It is needed to prove the main theorem of this section.

Lemma 3.4. *Suppose that D is dense subspace of Hilbert space H and the set-valued map $F : D \subseteq H \rightarrow P(H)$ satisfy hypothesis (H3). If $u_0 \in D$, $y_0 \in H$ and $M > 0$ is a constant such that*

$$(3.2) \quad \langle y_0 - y, u_0 - u \rangle \leq M \|u_0 - u\|^2 \quad \text{for all } u \in D \text{ and } y \in F(u),$$

then $y_0 \in F(u_0)$.

Proof. Assume that $y_0 \notin F(u_0)$. Fix $y \in F(u_0)$. Since $y - y_0 \neq 0$ and D is dense in H , we can find $v_y \in D$ such that $\langle y - y_0, v_y \rangle > 0$. Setting $u_n = u_0 + \frac{1}{n}v_y$ for each $n \geq 1$, we have $u_n \in D$ and by hypothesis H3, we can take $y_n \in F(u_n)$ such that $y_n \xrightarrow{w} y$ in H . By (3.2), we have for all $n \in \mathbb{N}$ that

$$\frac{1}{n} \langle y_0 - y_n, v_y \rangle \geq -M \frac{\|v_y\|^2}{n^2}.$$

Therefore,

$$\langle y - y_n, v_y \rangle \geq \langle y - y_0, v_y \rangle - M \frac{\|v_y\|^2}{n}.$$

Now, letting $n \rightarrow \infty$, we obtain $0 < \langle y - y_0, v_y \rangle \leq 0$. This is a contradiction. \square

Our main theorem in this section extends Theorem 3.1 and 3.2 to differential inclusions on infinite dimensional real separable Hilbert spaces.

Theorem 3.5. *Let H be a real separable Hilbert space and the Caratheodory set-valued map $F : [0, T] \times H \rightarrow P_{c,v}(H)$ satisfy hypotheses (H1)–(H3). If the differential equation $u' = h(t, u)$ with $u(0) = 0$ has a solution $u(t)$ existing on $[0, T]$, then the differential inclusion problem*

$$(3.3) \quad \begin{cases} x'(t) \in F(t, x(t)), & t \in [0, T], \\ x(0) = 0, \end{cases}$$

has a unique classical solution.

Proof. Take an orthonormal basis $\{e_n\}_{n=1}^\infty$ for H and let H_n be the subspace of H generated by $\{e_1, \dots, e_n\}$. Then, $\{H_n\}_{n=1}^\infty$ is an increasing sequence of finite dimensional subspaces of H with the property $\overline{\cup_{n=1}^\infty H_n} = H$. If P_n is the orthogonal projection of H onto H_n , then each P_n induces a corresponding projection of $L_{t_0}^2(I; H, M)$ onto subspace $L_{t_0}^2(I; H_n, M)$, we denote it again by P_n :

$$(P_n u)(t) = P_n u(t) \in H_n, \quad \forall t \in I, u \in L_{t_0}^2(I; H, M).$$

We have $P_n^* = P_n \rightarrow I$ as $n \rightarrow \infty$ where P_n^* is the adjoint operator of P_n and I is the identity map on H . For each $n \geq 1$, we define the set-valued map $P_n F : [0, T] \times H_n \rightarrow P_{c,v}(H_n)$ by $P_n F(t, x) = \{P_n(y) \mid y \in F(t, x)\}$. Theorem 1.2.8 of [21] yields that $P_n F$ is a Caratheodory map and since

$$\|P_n F(t, x)\| = \sup_{y \in F(t, x)} \|P_n y\| \leq \|F(t, x)\| \leq h(t, \|x\|),$$

it also satisfies hypothesis (H2). By Theorem 3.1, for each $n \geq 1$, the differential inclusion

$$(3.4) \quad \begin{cases} x'(t) \in P_n F(t, x(t)), & t \in [0, T], \\ x(0) = 0, \end{cases}$$

has a classical solution $x_n : [0, T] \rightarrow H_n$. On the other hand, hypothesis (H1) holds for $P_n F$. In fact, let $x_i \in H_n$ and $y_i \in P_n F(t, x_i)$ for $i = 1, 2$. Then there exist $z_i \in F(t, x_i)$, $i = 1, 2$, such that $y_i = P_n z_i$. Since $P_n = P_n^*$, we get

$$\langle y_2 - y_1, x_2 - x_1 \rangle = \langle z_2 - z_1, x_2 - x_1 \rangle \leq M \|x_2 - x_1\|^2.$$

Thus, just as the proof of Proposition 3.5 of [20], the sequence $\{x_n\}_{n=1}^\infty$ is bounded by a constant which depends only on F and T and we have

$$\|x_n\|_\infty^2 \leq e^{(2M+1)T} \int_0^T \|F(s, 0)\|^2 ds.$$

Therefore, according to part (a) of Proposition 3.3, $G(\{x_n\}_1^\infty)$ is a bounded set in $L^2_{l_0}(I; H, M)$. Now, we define the linear operator $\Gamma : L^2_{l_0}(I; H, M) \rightarrow C_{l_0}(I; H)$ by

$$(\Gamma y)(t) = \int_0^t y(s) ds.$$

Since

$$\|(\Gamma y)(t)\| \leq \gamma(T) \left(\int_0^T e^{-2Mt} \|y(t)\|^2 dt \right)^{\frac{1}{2}} = \gamma(T) \|y\|_M,$$

where $\gamma(T) = \frac{1}{\sqrt{2M}} (1 - e^{-2MT})^{\frac{1}{2}}$, Γ is continuous. Consider the set-valued map $\Gamma P_n Gx = \{\Gamma y \mid y \in P_n Gx\}$ from $C_{l_0}(I; H)$ into $C_{l_0}(I; H_n)$. Let x_n be a solution of the differential inclusion (3.4), so that $x_n \in \Gamma P_n Gx_n$ for all $n \geq 1$. We can find $y_n \in L^2_{l_0}(I; H, M)$ such that $x_n = \Gamma P_n y_n$. Since $y_n \in Gx_n$, the sequence $\{y_n\}_{n=1}^\infty$ is bounded in $L^2_{l_0}(I; H, M)$ and it is possible to extract a subsequence (we denote it again by $\{y_n\}_{n=1}^\infty$) which is weakly convergent in $L^2_{l_0}(I; H, M)$. Suppose that $y_n \xrightarrow{w} y$ as $n \rightarrow \infty$. Then, $x_n \xrightarrow{w} \Gamma y = x \in C_{l_0}(I; H)$. It is enough to show that $y \in Gx$. With the same method as in the proof of Theorem 3.6 of [20], one can show that

$$\langle w - y, x - u \rangle_M \geq -M \|x - u\|_M^2, \quad \forall u \in C_{l_0}(I; H), w \in Gu.$$

Since, by Proposition 3.3, G is hemicontinuous, the result follows from the Lemma 3.4. Finally, we come to prove the uniqueness of solution. If x^1 and x^2 are two solutions of problem (3.3), then we have $x^2(t) - x^1(t) = \int_0^t (v_1(s) - v_2(s)) ds$ for some $v_1 \in S^1_{F, x^1}$ and $v_2 \in S^1_{F, x^2}$ and Lemma 3.4 of [20] shows that

$$\|x^2(t) - x^1(t)\|^2 \leq 2M \int_0^t \|x^2(s) - x^1(s)\|^2 ds.$$

Thus, by the Gronwall inequality, $\|x^2(t) - x^1(t)\| = 0$ for all $t \in [0, T]$. \square

Now, if F is integrable hemicontinuous in the previous theorem, then one can easily see that the set-valued map $P_n F : [0, T] \times H \rightarrow P_{c,v}(H_n)$ is hemicontinuous. So, $P_n F$ is *l.s.c* with respect to the second variable, since weak and strong convergence are equivalent on H_n . Then, by Theorem 3.2 the differential inclusion (3.4) has a generalized solution x_n for all n . Therefore, with a similar proof as Theorem 3.5 we obtain the following theorem.

Theorem 3.6. *Let H be a real separable Hilbert space and the integrable set-valued map $F : [0, T] \times H \rightarrow P_{c,v}(H)$ satisfy hypotheses (H1)–(H3). If the differential equation $u' = h(t, u)$ with $u(0) = 0$ has a solution $u(t)$ existing on $[0, T]$, then the differential inclusion (3.3) has a unique generalized solution.*

4. Condensing set-valued maps

In this section, we discuss the existence of a solution for differential inclusion (3.3) in the case that the set-valued map F is condensing with respect to a regular measure of noncompactness on a real separable Hilbert space. We get a global existence theorem that is the analogous results of the Theorems 5.2.1 of [21], under a continuous bounded. Our main result in this section is the following

Theorem 4.1. *Let H be a real separable Hilbert space. Suppose that the Caratheodory (integrable) set-valued map $F : [0, T] \times H \rightarrow P_{c,v}(H)$ satisfy hypotheses (H2) and there is a function $k \in L^1([0, T]; [0, \infty))$ such that for every countable pointwise bounded subset Ω of $C(I; H)$, we have*

$$\alpha(F(t, \Omega(t))) \leq k(t)\alpha(\Omega(t)),$$

where α is the Hausdorff measure of noncompactness. If the differential equation $u' = h(t, u)$ with $u(0) = 0$ has a solution $u(t)$ existing on $[0, T]$, then the differential inclusion (3.3) has a classical (generalized) solution on $[0, T]$.

To prove the theorem, we need a very useful lemma the proof of which can be found in [21]. Consider the set $K \subset C(I, H)$ as

$$K = \{x \in C(I; H) : x(0) = 0, \|x(t)\| \leq u(t) \quad \forall t \in [0, T]\}.$$

Lemma 4.2. *Suppose that all conditions of Theorem 4.1 hold. Let $\{x_n\}_{n=1}^\infty$ be a sequence in K which converges to x and $\{v_n\}_{n=1}^\infty$ be a sequence in $L^1([0, T]; H)$ such that $v_n \xrightarrow{w} v$ in $L^1([0, T]; H)$. If for each $n \geq 1$ and for a.e. $t \in [0, T]$, $v_n(t) \in F(t, x_n(t))$, then $v \in S_{F,x}^1$. Moreover, $\int_0^t v_n(s) ds$ tends to $\int_0^t v(s) ds$ in the space $C(I; H)$*

Proof. Consider the $K \subseteq C(I; H)$ as above and the set-valued map Λ introduced in the proof of Theorem 3.1. Since H is separable, Λ has non-null values and according to the hypotheses (H2), $\|\Lambda(x)(t)\| \leq u(t)$ for all $(t, x) \in [0, T] \times K$. First, we show that Λ is closed. To this end, let the sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ converge to x and y , respectively, in the space

$C(I; H)$ and $y_n \in \Lambda(x_n)$ for all $n \in \mathbb{N}$. So, there exists a sequence of functions $\{v_n\}_{n=1}^\infty$ in $L^1([0, T]; H)$ such that $v_n \in S_{F, x_n}^1$ for each $n \in \mathbb{N}$ and

$$y_n(t) = \int_0^t v_n(s) ds, \quad \forall t \in [0, T].$$

We obtain from hypothesis (H2) that $\|v_n(t)\| \leq \|F(t, x_n(t))\| \leq h(t, \|x_n(t)\|) \leq h(t, u(t))$. Thus, $\{v_n\}_{n=1}^\infty$ is integrably bounded and moreover since h is continuous, $\{v_n(t)\}_{n=1}^\infty$ is relatively weakly compact in H for all $t \in [0, T]$. Therefore, we can assume that $v_n \xrightarrow{w} v$ in $L^1([0, T]; H)$ (see [5]). Now, by Lemma 4.2, $v \in S_{F, x}^1$ and $y_n(t) \rightarrow \int_0^t v(s) ds$ in $C([0, T]; H)$ as $n \rightarrow \infty$. Consequently, we have $y(t) = \int_0^t v(s) ds$. This proves that Λ is a closed set-valued map and a similar argument indicates that $\Lambda(x)$ is also closed for every $x \in K$. Next, we show that $\Lambda(x)$ is compact for all $x \in K$. Given any sequence $\{y_n\}_{n=1}^\infty \subseteq \Lambda(x)$, we can find for every $n \geq 1$, a $v_n \in S_{F, x}^1$ such that $y_n(t) = \int_0^t v_n(s) ds$ for all $t \in [0, T]$. Again passing to a suitable subsequence, if necessary, we may assume that $v_n \xrightarrow{w} v$ in $L^1(I; H)$. Thus, by Lemma 4.2, $\int_0^t v_n(s) ds \xrightarrow{n \rightarrow \infty} \int_0^t v(s) ds$ in $C([0, T]; H)$. Therefore, $\Lambda(x)$ is compact. Now, let us assume that χ is a regular measure of noncompactness we defined at the end of Section 2. Assume that $\Omega \subseteq C(I; H)$ is bounded and $\chi(\Omega) \leq \chi(\Lambda(\Omega))$. If $\{x_n\}_{n=1}^\infty$ is a sequence in Ω , then $\sup_{t \in I} e^{-Lt} \alpha(\{x_n(t)\}_{n=1}^\infty) \leq \sup_{t \in I} e^{-Lt} \alpha(\{y_n(t)\}_{n=1}^\infty)$, where $y_n \in \Lambda(x_n)$.

Therefore, we obtain

$$\begin{aligned} & \sup_{t \in [0, T]} (e^{-Lt} \alpha(\{x_n(t)\}_{n=1}^\infty)) \\ & \leq \sup_{t \in [0, T]} (e^{-Lt} \alpha(\{\int_0^t v_n(s) ds\}_{n=1}^\infty)) \\ & \leq \sup_{t \in [0, T]} (e^{-Lt} \sup_{u \in [0, T]} (e^{-Lu} \alpha(\{x_n(u)\}_{n=1}^\infty) \int_0^t e^{Ls} k(s) ds)) \\ & \leq \sup_{u \in [0, T]} (e^{-Lu} \alpha(\{x_n(u)\}_{n=1}^\infty)) \sup_{t \in [0, T]} (\int_0^t e^{-L(t-s)} k(s) ds). \end{aligned}$$

If we choose constant L in such a way that $\sup_{t \in [0, T]} \int_0^t e^{-L(t-s)} k(s) ds < 1$, then we obtain $\alpha_C(\{x_n\}_{n=1}^\infty) = 0$. Note that since $\{y_n\}_{n=1}^\infty$ is a relatively compact subset of $C([0, T]; H)$, then $\alpha_C(\{y_n\}_{n=1}^\infty) = 0$. Moreover, $\text{mod}_C(\{y_n\}_{n=1}^\infty) = 0$. Thus, $\chi(\Lambda(\Omega)) = 0$ and consequently, $\chi(\Omega) = 0$. This shows that the set-valued map Λ is χ -condensing. Therefore, according to Theorem 2.4, Λ has a fixed point which is a generalized solution of the differential inclusion (3.3). \square

5. Mild solutions

In this section, we come to prove the existence of mild solution for semilinear differential inclusion

$$(5.1) \quad \begin{cases} x'(t) - Ax(t) \in F(t, x(t)), & t \in [0, T], \\ x(0) = \xi, \end{cases}$$

with $\xi \in H$. By a *mild solution* to (5.1), we mean a continuous function $x : [0, T] \rightarrow H$ such that for every $t \in [0, T]$,

$$x(t) = S(t)\xi + \int_0^t S(t-s)v(s)ds,$$

where $v \in S_{F,x}^1$. If for a.e. $t \in [0, T]$, $v(t) \in D(A)$ and the function $t \mapsto Av(t)$ is integrable, then for every $\xi \in D(A)$, the mild solution is absolutely continuous and therefore, almost- everywhere differentiable.

Theorem 5.1. *Suppose that H is a separable Hilbert space and $F : [0, T] \times H \rightarrow P_{c,v}(H)$ is a integrable (Caratheodory) set-valued map satisfying hypotheses (H1)–(H3). If $A : D(A) \subseteq H \rightarrow H$ is the infinitesimal generator of a contraction C_0 semigroup $\{S(t)\}_{t \geq 0}$ and the differential equation $u' = h(t, u)$ with $u(0) = 0$ has a solution $u(t)$ existing on $[0, T]$, then the semilinear differential inclusion (5.1) has a unique mild solution.*

Proof. By the Hille-Yosida theorem, we have for every $n \in \mathbb{N}$ that

$$\|(nI - A)^{-1}\| \leq \frac{1}{n}.$$

Define $R_n = n(nI - A)^{-1}$ and $A_n = AR_n$. Since $A_n x = (nR_n - nI)x$ for each $x \in H$, we have $\|A_n\| \leq 2n$. Therefore, the bounded linear operator A_n is the infinitesimal generator of the uniformly continuous semigroup $e^{A_n t}$. Then, we define the set-valued map $F_n : [0, T] \times H \rightarrow P_{c,v}(H)$ by $F_n(t, x) = A_n x + F(t, x)$. We show that F_n satisfies hypotheses (H1)–(H3). Since by the Lumer-Philips theorem, A is dissipative and $R_n x \in D(A)$ for every $x \in H$, we have

$$\begin{aligned} \langle A_n x, x \rangle &= \langle AR_n x, x \rangle = \langle AR_n x, R_n x \rangle - \frac{1}{n} \langle AR_n x, AR_n x \rangle \\ &\leq -\frac{1}{n} \|AR_n x\|^2 \leq 0. \end{aligned}$$

Therefore, A_n is also dissipative. Take $x_1, x_2 \in H$, $t \in [0, T]$ and $y_1 \in F_n(t, x_1)$, $y_2 \in F_n(t, x_2)$. Then, there are $z_1 \in F(t, x_1)$ and $z_2 \in F(t, x_2)$ such that $y_i = A_n x_i + z_i$ for $i = 1, 2$. So, we obtain

$$\begin{aligned} \langle y_1 - y_2, x_1 - x_2 \rangle &= \langle A_n x_1 + z_1 - A_n x_2 - z_2, x_1 - x_2 \rangle \\ &= \langle A_n(x_1 - x_2), x_1 - x_2 \rangle + \langle z_1 - z_2, x_1 - x_2 \rangle \\ &\leq M \|x_1 - x_2\|^2. \end{aligned}$$

This implies that F_n is semimonotone with constant M . Define $h_n(t, s) = h(t, s) + 2ns$ for every $(t, s) \in [0, T] \times [0, \infty)$. We have

$$\begin{aligned} \|F_n(t, x)\| &= \sup_{y \in F(t, x)} \|A_n x + y\| \leq \|A_n x\| + \|F(t, x)\| \\ &\leq 2n \|x\| + h(t, \|x\|) = h_n(t, \|x\|). \end{aligned}$$

This shows that F_n satisfies hypothesis (H2). Finally, suppose that $\{x_m\}_1^\infty$ is a sequence in H which is convergent to x . If $y \in F_n(t, x)$, then there is a sequence $\{z_m\}_1^\infty$ in H such that $z_m \in F(t, x_m)$, $z_m \xrightarrow{w} z = y - A_n x \in F(t, y)$. Now, assuming $y_m = z_m + A_n x$, we have $y_m \xrightarrow{w} y$ in H and $y_m \in F_n(t, x_m)$. Consequently, for every $n \in \mathbb{N}$, F_n is hemicontinuous. By hypothesis (H2), F takes the bounded sets in $[0, T] \times H$ to the bounded subsets of H . Since A_n is a bounded operator, F_n has this property also. Now, according to Theorem 3.5 (Theorem 3.5), there exists a unique generalized (classical) solution $x_n : [0, T] \rightarrow H$ for the differential inclusion

$$\begin{cases} x'(t) \in F_n(t, x(t)), & t \in [0, T], \\ x(0) = \xi. \end{cases}$$

Hence, x_n is a strong solution of the integral equation

$$x(t) = \xi + \int_0^t A_n x(s) ds + \int_0^t v(s) ds,$$

where $v \in S_{F, x}^1$. The solution x_n is of the form

$$x_n(t) = e^{A_n t} \xi + \int_0^t e^{A_n(t-s)} v(s) ds, \quad \forall t \in [0, T].$$

Here $v_n \in S_{F, x_n}^1$ for all $n \in \mathbb{N}$. Since $D(A_n) = H$, x_n is a classical solution and we have $\frac{d}{dt} x_n = A_n x_n + v_n$, $x_n(0) = \xi$. By the Hille-Yosida theorem, we know that $S(t)x = \lim_{n \rightarrow \infty} e^{A_n t} x$ for all $x \in H$ and moreover, the convergence is uniform on $[0, T]$. We conclude by Energy-Type inequality in [25] that, the sequence of approximate solutions $\{x_n\}_{n=1}^\infty$ is bounded by a constant which depends only on M and T ; in other words, we have

$$\|x_n(t)\|^2 \leq \gamma_{T, M}, \quad \forall t \in [0, T], \quad n \in \mathbb{N},$$

where

$$\gamma_{T, M} = e^{2M+1} \left(\|\xi\|^2 + \int_0^T h^2(t, 0) dt \right).$$

On the other hand, since $v_n \in S_{F, x_n}^1$, we have $\|v_n(t)\| \leq h(t, \gamma_{T, M})$ for a.e. $t \in [0, T]$. Therefore, we can assume that $\{v_n\}_{n=1}^\infty$ is weakly convergent to v in $L^1(0, T; H)$ and so, there exists a sequence $\{w_n\}_{n=1}^\infty$ of convex combinations of $\{v_n\}$ such that $w_n \rightarrow v$ in $L^1(0, T; H)$ as $n \rightarrow \infty$. We extract a subsequence

of it (denoted by the same symbol) which is *a.e.* convergent to v . Since, F has close convex values, $w_n \in S_{F, x_n}^1$. Define

$$y_n(t) = e^{A_n t} \xi + \int_0^t e^{A_n(t-s)} w_n(s) ds$$

and

$$x(t) = S(t) \xi + \int_0^t S(t-s) v(s) ds,$$

for all $t \in [0, T]$. For every $z \in H$ we obtain

$$\begin{aligned} & \langle z, y_n(t) - x(t) \rangle \\ &= \langle z, e^{A_n t} \xi - S(t) \xi \rangle + \left\langle z, \int_0^t e^{A_n(t-s)} w_n(s) - S(t-s) v(s) ds \right\rangle \\ &= \langle z, e^{A_n t} \xi - S(t) \xi \rangle + \int_0^t \langle z, e^{A_n(t-s)} w_n(s) - S(t-s) v(s) \rangle ds. \end{aligned}$$

By the dominated convergence theorem, we conclude that $y_n(t) \xrightarrow{w} x(t)$ as $n \rightarrow \infty$ for all $t \in [0, T]$ and since the sequence is integrably bounded, $y_n \xrightarrow{w} x$ in $L^1([0, T]; H)$. Eventually, we use again Lemma 3.4 and conclude from the semimonotonicity of G that $v \in Gx$. \square

6. Example

In this section, we give an example to illustrate the application of the results obtained in this paper.

Assume that Ω is an open bounded subset of \mathbb{R}^2 with boundary $\partial\Omega$ of class C^2 . Consider the second order differential inclusion

$$\frac{\partial^2 u}{\partial t^2} \in F(t, x, u, \frac{\partial u}{\partial t}),$$

with the boundary condition $u(t, x) = 0$, for each $(t, x) \in [0, T] \times \partial\Omega$ and initial conditions $\frac{\partial u}{\partial t}(0, x) = \psi(x)$, $u(0, x) = \varphi(x)$. Here, $\varphi \in H_0^1(\Omega)$, $\psi \in L^2(\Omega)$ and the Caratheodory multivalued map $F : [0, T] \times \Omega \times \mathbb{R}^2 \rightarrow P_{b,c,v}(\mathbb{R})$ satisfies the following hypotheses:

- (I) There is a constant $M > 0$ such that for every $t \in [0, T]$, $u_1, u_2 \in \mathbb{R}$ and $v_1, v_2 \in L^2(\Omega)$, we have

$$\int_{\Omega} (\psi_1(x) - \psi_2(x))(v_1(x) - v_2(x)) dx \leq M \|v_1 - v_2\|_{L^2(\Omega)}^2,$$

in which the functions $\psi_1, \psi_2 : \Omega \rightarrow \mathbb{R}$ are such that

$$\psi_1(x) \in F(t, x, u_1, v_1(x))$$

and $\psi_2(x) \in F(t, x, u_2, v_2(x))$ for all $x \in \Omega$.

(II) There is an increasing continuous function $k(t)$ such that

$$\|F(t, x, u, v)\|^2 \leq k(t) (|u|^2 + |v|^2)$$

for all $(t, x, u, v) \in [0, T] \times \Omega \times \mathbb{R}^2$.

(III) For any sequence $\{u_n\}_{n=1}^\infty$ in $H_0^1(\Omega)$ which is convergent to u and sequence $\{v_n\}_{n=1}^\infty$ in $L^2(\Omega)$ which is convergent to v , if there exists a function $z : \Omega \rightarrow \mathbb{R}$ such that $z(x) \in F(t, x, u(x), v(x))$ for all $t \in [0, T]$, then one can find a sequence of functions $z_n : \Omega \rightarrow \mathbb{R}$ such that $z_n \xrightarrow{w} z$ in $L^2(\Omega)$ and moreover, $z_n(x) \in F(t, x, u_n(x), v_n(x))$.

Now, we consider the Hilbert space $H = H_0^1(\Omega) \times L^2(\Omega)$ with the inner product

$$\langle (u_1, v_1), (u_2, v_2) \rangle_H = \int_{\Omega} u_1'(x)u_2'(x)dx + \int_{\Omega} u_1(x)u_2(x)dx + \int_{\Omega} v_1(x)v_2(x)dx.$$

We define the multivalued map G on $[0, T] \times H$ by $G(t, z)(x) = \{0\} \times F(t, x, z(x))$. Then, G is a Caratheodory multivalued map with closed and convex values. It is easy to see that G satisfies hypotheses (H1)–(H3). Therefore, defining $z(t)x = (u(t, x), \frac{\partial u}{\partial t}(t, x))$, we can reduce the above second order differential inclusion to the equivalent first order differential inclusion

$$\begin{cases} z'(t) \in G(t, z(t)), \\ z(0) = (\varphi, \psi). \end{cases}$$

Hence, by Theorem 3.5, it has a unique mild solution $u \in C([0, T]; H_0^1(\Omega))$ such that $\frac{\partial u}{\partial t} \in C([0, T]; L^2(\Omega))$ almost everywhere on $[0, T]$.

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