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# CAYLEY-SYMMETRIC SEMIGROUPS

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Abstract. The concept of Cayley-symmetric semigroups is introduced, and several equivalent conditions of a Cayley-symmetric semigroup are given so that an open problem proposed by Zhu [19] is resolved generally. Furthermore, it is proved that a strong semilattice of self-decomposable semigroups $S_\alpha$ is Cayley-symmetric if and only if each $S_\alpha$ is Cayley-symmetric. This enables us to present more Cayley-symmetric semigroups, which would be non-regular. This result extends the main result of Wang [14], which stated that a regular semigroup is Cayley-symmetric if and only if it is a Clifford semigroup. In addition, we discuss Cayley-symmetry of Rees matrix semigroups over a semigroup or over a 0-semigroup.

## 1. Introduction

Many research papers were devoted to Cayley graphs of semigroups (see, for example, [1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 17]). Based on these works, the author first introduced the concept of generalized Cayley graphs of semigroups in [18], where some fundamental properties of generalized Cayley graphs of semigroups were studied. This work was extended in [19], where various combinatorial issues relating to generalized Cayley graphs were addressed. Especially, in Remark 3.8 of [19], the author proposed the following question: It may be interesting to characterize semigroups $S$ such that $Cay(S, S_l) = Cay(S, S_r)$, where $S_l$ and $S_r$ denote the left and right universal relations on $S^1$ respectively. This problem was partially solved by Wang in [14], where it was proved that for any regular semigroup $S$, $Cay(S, S_l) = Cay(S, S_r)$ if and only if $S$ is a Clifford semigroup.

Following [14, 18, 19], we continue to study the generalized Cayley graphs of semigroups in the present paper. Based on the problem mentioned as above, we introduce naturally the concept of Cayley-symmetric semigroups, in view of which the above problem may be restated as follows: When is a semigroup

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Cayley-symmetric? Consequently, the main result of Wang [14] would be restated as follows: A regular semigroup is Cayley-symmetric if and only if it is a Clifford semigroup. As the most general answers to the above problem, several equivalent conditions of a Cayley-symmetric semigroup are given in this paper. We also generalize the notion of Clifford semigroups and establish a necessary and sufficient condition for a semilattices of semigroups to be Cayley-symmetric. Consequently, we can display some Cayley-symmetric semigroups which are not regular. In this sense, our results also extend the main result of [14]. In addition, the Cayley-symmetry of Rees matrix semigroups over a semigroup or over a 0-semigroup is also discussed so that the corresponding equivalent conditions are given.

## 2. Preliminaries

Recall that if $S$ is an ideal of a semigroup $T$, then we call $T$ an *ideal extension* of $S$. Let $T^1$ be the semigroup $T$ with an identity adjoined if necessary.

**Definition 2.1** ([18, 19]). Let $T$ be an ideal extension of a semigroup $S$ and $\rho \subseteq T^1 \times T^1$. The *Cayley graph* $Cay(S,\rho)$ *of* $S$ *relative to* $\rho$ is defined as the graph with vertex set $S$ and edge set $E(\mathrm{Cay}(S,\rho))$ consisting of those ordered pairs $(a,b)$, where $xay = b$ for some $(x,y) \in \rho$. We also call the Cayley graphs defined in this way the *generalized Cayley graphs*, in order to distinguish them from the usual ones.

Assume that $S$ is a semigroup and $a \in S$. Then $J(a) = S^1aS^1$ ($L(a) = S^1a$, $R(a) = aS^1$) is the principal (left, right) ideal generated by $a$ (cf. [2, 20]). Let $S_l = S^1 \times \{1\}$(the left universal relation on $S^1$), $S_r = \{1\} \times S^1$(the right universal relation on $S^1$) and $\omega_S = S^1 \times S^1$(the universal relation on $S^1$). Then the generalized Cayley graphs $\mathrm{C}ay(S,S_l)$, $\mathrm{C}ay(S,S_r)$ and $\mathrm{C}ay(S,S_\omega)$ are called the *left universal, right universal* and *universal Cayley graphs* of $S$, respectively.

As mentioned in [19, Remark 3.8], it would be interesting to characterize semigroups $S$ such that $\mathrm{Cay}(S,S_l) = \mathrm{Cay}(S,S_r)$. We shall answer this question in general in this paper. It would be convenient to give the following definition.

**Definition 2.2.** A semigroup $S$ is called *Cayley-symmetric* if $\mathrm{Cay}(S,S_l) = \mathrm{Cay}(S,S_r)$.

Assume that $T$ is an ideal extension of a semigroup $S$. Then the generalized Cayley graphs $\mathrm{C}ay(S,T_l)$, $\mathrm{C}ay(S,T_r)$ and $\mathrm{C}ay(S,T_\omega)$ are called the *left $T$-universal, right $T$-universal* and *$T$-universal Cayley graphs* of $S$, respectively.

We also consider a more general symmetry problem in this paper. As a natural generalization of Definition 2.2, we have the following definition.

**Definition 2.3.** Let $T$ be an ideal extension of a semigroup $S$. If $\mathrm{C}ay(S,T_l) = \mathrm{C}ay(S,T_r)$, then we say that $S$ is *Cayley-symmetric in $T$*.

Throughout the paper, graphs are directed graphs without multiple edges, but possibly with loops, or digraphs in terms of [1, 11]. As in [18, 19], we always equate two graphs isomorphic to each other if no confusion occurs. For a graph $\Gamma$, denote by $V(\Gamma)$ and $E(\Gamma)$ its vertex set and edge set, respectively. A graph $\Gamma_0$ is called a *subgraph* of a graph $\Gamma$ if $V(\Gamma_0) \subseteq V(\Gamma)$ and $E(\Gamma_0) \subseteq E(\Gamma)$. A graph $\Gamma_0$ is called an *induced subgraph* of a graph $\Gamma$ if $\Gamma_0$ is a subgraph of $\Gamma$ and the following condition is satisfied: for any $a, b \in V(\Gamma_0)$, $(a, b) \in E(\Gamma_0)$ if and only if $(a, b) \in E(\Gamma)$. Any book on graph theory, e.g., [4, 15], will provide terminology which may be used in this paper without definition.

For notions and notations of semigroup theory, we refer the reader to [2].

## 3. Cayley-symmetric semigroups

In this section, we answer the following question in general: When is a semigroup Cayley-symmetric? More generally, we answer the following question: When is a semigroup Cayley-symmetric in its a given ideal extension?

Assume that $T$ is an ideal extension of a semigroup $S$ and that $A$ is a subset of $S$. Then the ideal, left ideal and right ideal generated by $A$ in $T$ are denoted by $J_T(A)$, $L_T(A)$ and $R_T(A)$, respectively. If $A = \{a\}$, then instead of writing $J_T(A)$, $L_T(A)$ and $R_T(A)$, we write $J_T(a)$, $L_T(a)$ and $R_T(a)$, which are called the ideal, left ideal and right ideal *generated by $a$ in $T$*, respectively. In the case that $S = T$, we write $J(A)$, $L(A)$, $R(A)$, $J(a)$, $L(a)$ and $R(a)$ instead of writing $J_T(A)$, $L_T(A)$, $R_T(A)$, $J_T(a)$, $L_T(a)$ and $R_T(a)$, respectively. It is clear that $J_T(a) = T^1aT^1$, $L_T(a) = T^1a$ and $R_T(a) = aT^1$. Thus $J(a) = S^1aS^1$, $L(a) = S^1a$ and $R(a) = aS^1$ (cf. [2]). It is easy to show the following lemma.

**Lemma 3.1.** *Let $T$ be an ideal extension of a semigroup $S$ and $A$ a subset of $S$. Then $J_T(A) = \bigcup_{a\in A} J_T(a)$, $L_T(A) = \bigcup_{a\in A} L_T(a)$, and $R_T(A) = \bigcup_{a\in A} R_T(a)$.*

Furthermore, we can prove the next lemma.

**Lemma 3.2.** *If $T$ is an ideal extension of a semigroup $S$, then the following statements are equivalent*:

(1) *$L_T(a) = R_T(a)$ for every $a \in S$;*
(2) *$L_T(A) = R_T(A)$ for every $A \subseteq S$;*
(3) *$L_T(a)$ is a right ideal of $T$ and $R_T(a)$ is a left ideal of $T$ for every $a \in S$;*
(4) *$L_T(A)$ is a right ideal of $T$ and $R_T(A)$ is a left ideal of $T$ for every $A \subseteq S$.*

*Proof.* (1)$\Rightarrow$ (2): In view of Lemma 3.1, we have $L_T(A) = \bigcup_{a\in A} L_T(a)$ and $R_T(A) = \bigcup_{a\in A} R_T(a)$, which together with the condition (1) implies that

$$L_T(A) = \bigcup_{a\in A} L_T(a) = \bigcup_{a\in A} R_T(a) = R_T(A).$$

(2)$\Rightarrow$ (1): This is trivial.

(1)$\Rightarrow$ (3): This is obvious since $L_T(a)$ is a left ideal and $R_T(a)$ is a right ideal of $T$.

(3)⇒ (1): Note that $L_T(a) \subseteq J_T(a)$ by definitions. Since $L_T(a)$ is a right ideal of $T$, $J_T(a) = T^1aT^1 = L_T(a)T^1 \subseteq L_T(a) \subseteq J_T(a)$. Hence $L_T(a) = J_T(a)$. A similar argument shows that $R_T(a) = J_T(a)$. So $L_T(a) = J_T(a) = R_T(a)$.

(4)⇒ (3): This is trivial.

(2)⇒ (4): This is obvious since $L_T(A)$ is a left ideal and $R_T(A)$ is a right ideal of $T$. □

As the main result of this section, the following theorem characterizes the semigroups that are Cayley-symmetric in their ideal extensions.

**Theorem 3.3.** *If $T$ is an ideal extension of a semigroup $S$, then the following statements are equivalent*:

(1) *$S$ is Cayley-symmetric in $T$*;
(2) *$L_T(a) = R_T(a)$ for every $a \in S$*;
(3) *$L_T(A) = R_T(A)$ for every $A \subseteq S$*;
(4) *$L_T(a)$ is a right ideal of $T$ and $R_T(a)$ is a left ideal of $T$ for every $a \in S$*;
(5) *$L_T(A)$ is a right ideal of $T$ and $R_T(A)$ is a left ideal of $T$ for every $A \subseteq S$.*

*Proof.* In light of Lemma 3.2, we only need to prove the equivalence of (1) and (2). According to the definition of generalized Cayley graphs, we know that

$$\begin{aligned} & (a,b) \in E(\mathrm{Cay}(S,T_l)) \\ \Leftrightarrow\quad & b = xa \quad \text{for some} \quad x \in T^1 \\ \Leftrightarrow\quad & b \in T^1a = L_T(a) \end{aligned}$$

and that

$$\begin{aligned} & (a,b) \in E(\mathrm{Cay}(S,T_r)) \\ \Leftrightarrow\quad & b = ay \quad \text{for some} \quad y \in T^1 \\ \Leftrightarrow\quad & b \in aT^1 = R_T(a). \end{aligned}$$

If $\mathrm{Cay}(S,T_l) = \mathrm{Cay}(S,T_r)$, then for any $a, b \in S$, $(a,b) \in E(\mathrm{Cay}(S,T_l))$ if and only if $(a,b) \in E(\mathrm{Cay}(S,T_r))$. Thus for all $a, b \in S$ we have that $b \in L_T(a)$ if and only if $b \in R_T(a)$. Since $S$ is an ideal of $T$, $L_T(a) = T^1a \subseteq S$ and $R_T(a) = aT^1 \subseteq S$ for every $a \in S$. So $L_T(a) = R_T(a)$ for every $a \in S$.

Conversely, suppose that $L_T(a) = R_T(a)$ for every $a \in S$. Then for all $a, b \in S$ we have that $b \in L_T(a)$ if and only if $b \in R_T(a)$. Hence $(a,b) \in E(\mathrm{Cay}(S,T_l))$ if and only if $(a,b) \in E(\mathrm{Cay}(S,T_r))$. That is to say that $\mathrm{Cay}(S,T_l) = \mathrm{Cay}(S,T_r)$. □

Using Theorem 3.3 with $T = S$, we immediately have the following corollary, which answers the open question raised by the author in [19].

**Corollary 3.4.** *For any semigroup $S$, the following statements are equivalent*:

(1) *$S$ is Cayley-symmetric*;
(2) *$L(a) = R(a)$ for every $a \in S$*;

(3) *$L(A) = R(A)$ for every $A \subseteq S$;*
(4) *$L(a)$ is a right ideal and $R(a)$ is a left ideal for every $a \in S$;*
(5) *$L(A)$ is a right ideal and $R(A)$ is a left ideal for every $A \subseteq S$;*
(6) *Every left ideal is a right ideal and every right ideal is a left ideal.*

In the remainder of this section, we present some Cayley-symmetric semigroups.

Suppose that $S$ is a semigroup with zero 0 and that $S = \bigcup_{\alpha \in I} S_\alpha$, where each $S_\alpha$ is a subsemigroup of $S$, and where

$$S_i S_j = S_i \cap S_j = \{0\}.$$

Then we call $S$ is a 0-*direct union* of $S_\alpha$'s ([2]). If we further suppose that every $S_\alpha$ is Cayley-symmetric, then $S_\alpha^1 a = a S_\alpha^1$ for all $a \in S_\alpha$ by Corollary 3.4. It follows that $S^1 a = a S^1$ for all $a \in S$. Hence $S$ is Cayley-symmetric by Corollary 3.4 again. Consequently, we obtain the following corollary.

**Corollary 3.5.** *Assume that $S$ is a 0-direct union of semigroups $S_\alpha$'s. If every $S_\alpha$ is Cayley-symmetric, then so is $S$.*

If $G$ is a group, then we have $aG = G = Ga$ for every $a \in G$, which means that $G$ is Cayley-symmetric by Corollary 3.4. Let $G^0 = G \cup \{0\}$, where 0 is an adjoined zero. Then we call $G^0$ a 0-*group.* Also we have $aG^0 = G^0 a$ for every $a \in G^0$, which means that $G^0$ is also Cayley-symmetric by Corollary 3.4 again. As a direct consequence of Corollary 3.5, we have the following corollary.

**Corollary 3.6.** *If $S$ is a 0-direct union of some 0-groups, then $S$ is Cayley-symmetric.*

## 4. Cayley-symmetry of Rees matrix semigroups

This section is devoted to the Cayley-symmetry of Rees matrix semigroups so that a necessary and sufficient condition is given for a Rees matrix semigroup to be Cayley-symmetric. The main results of this section are Theorems 4.4 and 4.7.

Let $S$ be a semigroup, let $I$, $\Lambda$ be nonempty sets and let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix with entries in $S^1$. (Note that here is $S^1$ rather than $S$.) Let $T = I \times S \times \Lambda$, and define a multiplication on $T$ by

$$(i, a, \lambda)(j, b, \mu) = (i, a p_{\lambda j} b, \mu). \tag{4.1}$$

Then $T$ is a semigroup, which is called the $I \times \Lambda$ *Rees matrix semigroup over the semigroup $S$ with the sandwich matrix $P$* and denoted by $\mathcal{M}[S; I, \Lambda; P]$. Recall that a semigroup is called *completely simple* if it is simple and if it contains a primitive idempotent. By [2, Theorem 3.3.1], a semigroup is completely simple if and only if it is isomorphic to a Rees matrix semigroup $\mathcal{M}[G; I, \Lambda; P]$ over a group $G$.

To study the Cayley-symmetry of a Rees matrix semigroup over a semigroup, we need some new terminologies. As a generalization of the identity of a semigroup, a mid-identity of a semigroup is defined as follows.

**Definition 4.1.** An element $u$ of a semigroup $S$ is called a *mid-identity*, if for all $x, y \in S$, $xuy = xy$.

The next terms generalize the concept of invertible elements of a group.

**Definition 4.2.** Let $S$ be a semigroup and $p \in S$. If there exists $q \in S$ such that $pq$ ($qp$) is a mid-identity, then $p$ is called *left* (*right*) *factor* of a mid-identity. If $p$ is not only a left factor of a mid-identity but also a right factor of a mid-identity, then $p$ is called *mid-invertible.*

It is clear that the identity of a semigroup is a mid-identity and it is also mid-invertible. Note that if $S$ is a semigroup with an identity 1 and with a mid-identity $u$, then $1u1 = 11$ which means that $u = 1$. A non-trivial example is as follows.

**Example 4.3.** Let $S = \langle p, q\rangle$ be the matrix semigroup (under the usual multiplication of matrices) generated by two matrices $p = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $q = \begin{pmatrix} 2^{-1} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Then $u = pq = qp = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is a mid-identity rather than an identity of $S$. Furthermore, both $p$ and $q$ are mid-invertible.

As one of two main results of this section, the next theorem gives an equivalent condition for a Rees matrix semigroup over a semigroup to be Cayley-symmetric.

**Theorem 4.4.** *Let $T = \mathcal{M}[S; I, \Lambda; P]$ be an $I \times \Lambda$ Rees matrix semigroup over a semigroup $S$, where the sandwich matrix $P = (p_{\lambda i})$ is a $\Lambda \times I$ matrix with entries in $S^1$. Assume that $p_{11} = p$ is a mid-invertible element of $S^1$. Then $T$ is Cayley-symmetric if and only if $|I| = |\Lambda| = 1$ and $S$ is Cayley-symmetric.*

*Proof. Necessity.* Suppose that $T$ is Cayley-symmetric. By (4.1), one can deduce that for any $i \in I$, $\lambda \in \Lambda$, $a \in S$,

$$(i, a, \lambda)T^1 = \{(i, ap_{\lambda j}b, \mu) \mid j \in I,\ \mu \in \Lambda,\ b \in S)\}, \tag{4.2}$$

$$T^1(i, a, \lambda) = \{(j, bp_{\mu i}a, \lambda) \mid j \in I,\ \mu \in \Lambda,\ b \in S)\}. \tag{4.3}$$

Since $T$ is Cayley-symmetric, we have

$$(i, a, \lambda)T^1 = T^1(i, a, \lambda) \tag{4.4}$$

by Corollary 3.4. From (4.2), (4.3) and (4.4), it follows that $|I| = |\Lambda| = 1$. Thus we may set $I = \{1\}$ and $\Lambda = \{1\}$. Then we have that

$$\{(1, a, 1), (1, apb, 1) \mid b \in S\} = \{(1, a, 1), (1, cpa, 1) \mid c \in S\} \tag{4.5}$$

holds for all $a \in S$. It follows that for every $a \in S$,

$$\{a, apb \mid b \in S\} = \{a, cpa \mid c \in S\}. \tag{4.6}$$

By the assumption of the theorem, we can suppose that there exist $u, v, q, r \in S$ such that $u, v$ are mid-identity, and that $pq = u$ and $rp = v$. Thus for any $a, b \in S$, according to (4.6), we have that

$$\begin{aligned} ab &= aub \\ &= apqb \\ &= \begin{cases} a \\ cpa. \end{cases} \end{aligned}$$

This shows that $aS^1 \subseteq S^1 a$. A symmetric argument shows that $S^1 a \subseteq aS^1$. Hence $aS^1 = S^1 a$. So we have that $S$ is Cayley-symmetric by Corollary 3.4.

*Sufficiency.* Assume that $I = \{1\}$ and $\Lambda = \{1\}$ and that $S$ is Cayley-symmetric. To show that $T$ is Cayley-symmetric, it only remains to prove that (4.6) holds. For this, let $a, b \in S$. By Corollary 3.4 again, we obtain that

$$\{a, ab \mid b \in S\} = \{a, ca \mid c \in S\}. \tag{4.7}$$

Then $apb = a$ or $apb = ca$ (for some $c \in S$). If the case is the former, then it is easy to see that

$$\{a, apb \mid b \in S\} \subseteq \{a, cpa \mid c \in S\}. \tag{4.8}$$

If it is the latter case, i.e., if there exists $c \in S$ such that $apb = ca$, then we have that

$$apb = ca = cva = arpa,$$

which shows that (4.8) remains true. By a symmetric argument, we can prove the inverse conclusion of (4.8). Therefore, (4.6) holds. This completes the proof. □

Note that each element of a group is mid-invertible. So we have the following corollary immediately.

**Corollary 4.5.** *A completely simple semigroup is Cayley-symmetric if and only it is isomorphic to a group.*

Observe that if $p_{\lambda i} = 1$ for all $i \in I$ and $\lambda \in \Lambda$, then $T = \mathcal{M}[S; I, \Lambda; P]$ is isomorphic to the direct product of $S$ and the rectangular band $B = I \times \Lambda$. Consequently, we obtain the following corollary.

**Corollary 4.6.** *A direct product of a semigroup $S$ and a rectangular band $B$ is Cayley-symmetric if and only if $|B| = 1$ and $S$ is Cayley-symmetric.*

Let us turn our attention to the Cayley-symmetry of a Rees matrix semigroup over a 0-semigroup $S^0$ ($S^0 = S \cup \{0\}$ where 0 is an adjoined zero).

Assume that $S$ is a semigroup. Let $I$, $\Lambda$ be nonempty sets and let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix with entries in $S^{1,0}$ ($S^{1,0} = S^1 \cup \{0\}$). Suppose that $P$ is *regular*, in the sense that no row or column of $P$ consists entirely of zeros. Formally,

$$\begin{aligned} &(\forall i \in I)(\exists \lambda \in \Lambda) p_{\lambda i} \neq 0, \\ &(\forall \lambda \in \Lambda)(\exists i \in I) p_{\lambda i} \neq 0. \end{aligned} \tag{4.9}$$

Let $T^0 = (I \times S \times \Lambda) \cup \{0\}$, and define a composition on $T^0$ by

$$(i, a, \lambda)(j, b, \mu) = \begin{cases} (i, ap_{\lambda j}b, \mu), & \text{if}\, p_{\lambda j} \neq 0 \\ 0, & \text{if}\, p_{\lambda j} = 0. \end{cases} \tag{4.10}$$

Then $T^0$ is a semigroup, which is called the $I \times \Lambda$ *Rees matrix semigroup over the* 0*-semigroup* $S^0$ *with the regular sandwich matrix* $P$ and denoted by $\mathcal{M}^0[S; I, \Lambda; P]$. Recall that a semigroup is called *completely* 0*-simple* if it is 0-simple and if it contains a primitive idempotent. By Rees Theorem (see [2, Theorem 3.2.3]), a semigroup is completely 0-simple if and only if it is isomorphic to a Rees matrix semigroup $\mathcal{M}^0[G; I, \Lambda; P]$ over a 0-group $G^0$.

As the other main result of this section, the next theorem gives an equivalent condition of a Cayley-symmetric Rees matrix semigroup over a 0-semigroup.

**Theorem 4.7.** *Let* $T^0 = \mathcal{M}^0[S; I, \Lambda; P]$ *be an* $I \times \Lambda$ *Rees matrix semigroup over a* 0*-semigroup* $S^0$*, where the sandwich matrix* $P = (p_{\lambda i})$ *is a regular* $\Lambda \times I$ *matrix with entries in* $S^{1,0}$*. Assume that* $p_{11} = p$ *is a mid-invertible element of* $S^{1,0}$*. Then* $T^0$ *is Cayley-symmetric if and only if* $|I| = |\Lambda| = 1$ *and* $S$ *is Cayley-symmetric.*

*Proof.* It is analogous to the proof of Theorem 4.4. □

As a direct consequence of Theorem 4.7, we have the following corollary.

**Corollary 4.8.** *A completely* 0*-simple semigroup is Cayley-symmetric if and only it is isomorphic to a* 0*-group.*

## 5. Cayley-symmetry of strong semilattices of semigroups

In this section, we shall introduce the concept of strong semilattices of semigroups, which is a natural generalization of the notion of Clifford semigroups. The main result of this section is Theorem 5.3, which gives a necessary and sufficient condition for a strong semilattice of semigroups to be Cayley-symmetric. Consequently, we can display more Cayley-symmetric semigroups which would not be regular. This means that our construction will provide a more universal class of Cayley-symmetric semigroups than the Clifford semigroups. In this sense, our result generalizes that of [14], which stated that a regular semigroup is Cayley-symmetric if and only if it is a Clifford semigroup.

Suppose that we have a semilattice $Y$ and a set of semigroups $S_\alpha$ indexed by $Y$, and suppose that, for all $\alpha \geq \beta$ in $Y$ there exists a morphism $\phi_{\alpha,\beta} : \ S_\alpha \to S_\beta$ such that: (1) for each $\alpha \in Y, \phi_{\alpha,\alpha} = 1_{S_\alpha}$, the identity mapping on $S_\alpha$; (2) $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$ for all $\alpha, \beta, \gamma \in Y$ such that $\alpha \geq \beta \geq \gamma$. Let $S = \dot{\bigcup}_{\alpha \in Y} S_\alpha$, the disjoint union of $S_\alpha$'s. Define a multiplication on $S$ by the rule that, for each $a \in S_\alpha$ and $b \in S_\beta$,

$$ab = (a)\phi_{\alpha,\alpha\beta}(b)\phi_{\beta,\alpha\beta}. \tag{5.1}$$

Then $S$ is a semigroup, called the *strong semilattice of semigroups* $S_\alpha$ (cf. [18, Definition 3.11]). We write

$$S = \mathcal{S}[Y; S_\alpha; \phi_{\alpha,\beta}].$$

To study Cayley-symmetry of a strong semilattice of semigroups, we need a new term, so called self-decomposable semigroups, which is also a generalization of the notion of monoids or that of regular semigroups.

**Definition 5.1.** Let $S$ be a semigroup. If for every $a \in S$, $a \in Sa \cap aS$, then we call $S$ *self-decomposable*.

Note that if $S$ is a self-decomposable semigroup, then $L(a) = S^1a = Sa$ and $R(a) = aS^1 = aS$. Thus according to Corollary 3.4, we immediately obtain the following lemma, which provides a general answer to that problem in [19] for the self-decomposable semigroup class and will be used repeatedly later.

**Lemma 5.2.** *Let $S$ be a self-decomposable semigroup. Then $S$ is Cayley-symmetric if and only if $Sa = aS$ for all $a \in S$.*

As our main result of this section, the next theorem gives a necessary and sufficient condition for a strong semilattice of self-decomposable semigroups to be Cayley-symmetric.

**Theorem 5.3.** *Suppose that $S = \mathcal{S}[Y; S_\alpha; \phi_{\alpha,\beta}]$, where each $S_\alpha$ is self-decomposable. Then $S$ is Cayley-symmetric if and only if for every $\alpha \in Y$, $S_\alpha$ is Cayley-symmetric.*

*Proof.* Observe that $S = \mathcal{S}[Y; S_\alpha; \phi_{\alpha,\beta}]$ is self-decomposable since for every $\alpha$, $S_\alpha$ is self-decomposable.

*Necessity.* Assume that $S$ is Cayley-symmetric. Let $a, b \in S_\alpha$ with $\alpha \in Y$. According to Lemma 5.2, there exist $\beta \in Y$ and $c \in S_\beta$ such that $ab = ca$. It is clear that $\beta \geq \alpha$ since $ab \in S_\alpha$. Since $S_\alpha$ is self-decomposable, there exists $u \in S_\alpha$ such that $a = ua$. It follows that $ab = ca = c(ua) = (cu)a$, where $cu \in S_\alpha$. We have proved that $aS_\alpha \subseteq S_\alpha a$. Similarly, we have that $S_\alpha a \subseteq aS_\alpha$, which means that $S_\alpha a = aS_\alpha$. Thus by Lemma 5.2 again, $S_\alpha$ is Cayley-symmetric.

*Sufficiency.* Suppose that for all $\alpha$, $S_\alpha$ is Cayley-symmetric. Let $a \in S_\alpha$ and $b \in S_\beta$ with $\alpha, \beta \in Y$. Set $\alpha\beta = \gamma \in Y$, then $\alpha, \beta \geq \gamma$. Since $ab \in S_\gamma$ and $S_\gamma$ is self-decomposable, there exist $x, y \in S_\gamma$ such that $ab = x(ab) = (ab)y$. It follows that $ab = x(ab)y = (xa)(by)$, where $xa, by \in S_\gamma$. Since $S_\gamma$ is Cayley-symmetric, from Lemma 5.2 we deduce that there exists $z \in S_\gamma$ such that $(xa)(by) = z(xa) = (zx)a$, which implies that $ab = (zx)a \in Sa$. We have proved that $aS \subseteq Sa$. The inverse conclusion may be proved in a similar way. Hence $aS = Sa$. Therefore, by Lemma 5.2 again, $S$ is Cayley-symmetric. This completes the proof. □

Now suppose that $S$ is a Clifford semigroup. According to [2, Theorem 4.1], $S = \mathcal{S}[Y; G_\alpha; \phi_{\alpha,\beta}]$, a strong semilattice of groups $G_\alpha$'s. Of course, every group

$G_\alpha$ is Cayley-symmetric. Thus using Theorem 5.3, we immediately obtain the following corollary, which is a half of the main theorem of Wang [14].

**Corollary 5.4** ([14])**.** *Every Clifford semigroup is Cayley-symmetric.*

Let us conclude our discussion by the following example, which shows that in light of Theorem 5.3, we can construct non-regular, non-commutative but Cayley-symmetric semigroups.

**Example 5.5.** Let $Y = (\{\alpha, \beta, \gamma\}, \geq)$ be a semilattice where the partial order $\geq$ is defined by

$$\geq = \{(\alpha, \alpha), (\beta, \beta), (\gamma, \gamma), (\alpha, \gamma), (\beta, \gamma)\}.$$

Let $S_\alpha$, $S_\beta$ be two dihedral groups. Let $S_\gamma = \{1\} \cup \{2n \mid n \in \mathbb{Z}\}$. It is seen that $S_\gamma$ becomes a monoid with respect to the usual multiplication of integers. Define homomorphisms as follows:

$$\phi_{\alpha,\gamma} : S_\alpha \to S_\gamma,\ a \mapsto 1 \ \text{ for all}\, a \in S_\alpha,$$

$$\phi_{\beta,\gamma} : S_\beta \to S_\gamma,\ b \mapsto 1 \ \text{ for all}\, b \in S_\beta,$$

$$\phi_{\delta,\delta} = 1_{S_\delta} \ \text{ for all}\, \delta \in Y.$$

Let $S$ be the strong semilattice of semigroups $\{S_\delta\}_{\delta \in Y}$, that is,

$$S = \mathcal{S}[Y; S_\delta; \phi_{\delta,\lambda}].$$

It is clear that for each $\delta \in Y$, $S_\delta$ is not only self-decomposable but also Cayley-symmetric. By Theorem 5.3, $S$ is Cayley-symmetric. But note that the semigroup $S_\gamma$ is not regular. Furthermore, $S$ is also not regular. At last, $S$ is not commutative since nor is the dihedral group $S_\alpha$.

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