

THE DISTINGUISHING NUMBERS OF MERGED JOHNSON GRAPHS

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ABSTRACT. In present article, we determine the distinguishing number of the merged Johnson graphs which are generalization of both the Kneser graphs and the Johnson graphs.

1. Introduction

The *distinguishing number* of a graph G is the minimum number of colors for which there exists an assignment of colors to the vertices of G such that the identity is the only color-preserving automorphism of G . Generally, for a permutation group Γ acting on X , the *distinguishing number* of Γ is the minimum number of cells of a partition π of X satisfying that the identity is the only element of Γ fixing each cell of π . Albertson and Collins first introduced the distinguishing number of a graph [3] and there have been many interesting results on the distinguishing numbers of graphs and permutation groups in last few years [1, 2, 3, 4, 5, 7, 8, 10, 11].

Here we consider a class of graphs based on the Johnson graphs. For positive integers k, n such that $1 \leq k \leq \frac{n}{2}$, the *Johnson graph* $J(n, k)$ has vertices given by the k -subsets of $[n] = \{1, 2, \dots, n\}$ and there exists an edge between two vertices if and only if their intersection has size $k - 1$. Given a nonempty subset $I \subseteq \{1, 2, \dots, k\}$, the *merged Johnson graph* $J(n, k)_I$ is the union of the distance i graphs $J(n, k)_i$ of $J(n, k)$ for all $i \in I$, namely, two k -subsets are adjacent in $J(n, k)_I$ if and only if their intersection has $k - i$ elements for some $i \in I$. The merged Johnson graphs $J(n, k)_I$ include many interesting graphs such as the Johnson graph $J(n, k) = J(n, k)_{\{1\}}$ and the Kneser graph $K(n, k) = J(n, k)_{\{k\}}$.

In [2], M. O. Albertson and D. L. Boutin determined the distinguishing number of the Kneser graphs. The aim of the present article is to determine the distinguishing number of the merged Johnson graphs.

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The outline of this paper is as follows. In Section 2, we review some preliminaries regarding the distinguishing numbers and the merged Johnson graphs. In Section 3, we find Theorem 3.2 which addresses the distinguishing numbers of the merged Johnson graphs. We also prove lemmas which are used for a proof of the main theorem. At last, we provide a proof of Theorem 3.2 in Section 4.

2. Preliminaries

For a given graph G , a coloring $f : V(G) \rightarrow \{1, 2, \dots, r\}$ is said to be *r-distinguishing* if the identity is the only graph automorphism ϕ satisfying $f(\phi(v)) = f(v)$ for all $v \in V(G)$. This means that the distinguishing coloring is a symmetry-breaking coloring of G . The distinguishing number, denoted by $\text{Dist}(G)$, is the minimum r that G has an r -distinguishing coloring. One can easily see that $\text{Dist}(G) = \text{Dist}(G^c)$ where G^c is the complement of G . If G is an *asymmetric graph*, namely if the identity is the only automorphism of G , then $\text{Dist}(G) = 1$. In fact, the converse is also true.

For a graph G and for a subset $S \subseteq V(G)$, a coloring $f : S \rightarrow \{1, 2, \dots, s\}$ is said to be *s-distinguishing* if for any graph automorphism ϕ of G fixing S set-wisely and satisfying $f(\phi(v)) = f(v)$ for all $v \in S$ fixes all elements of S . In this case, ϕ does not need to fix other vertices outside of the given set S . If there exists an s -distinguishing coloring for S , the set S is called an *s-distinguishable set*.

For a graph G , a subset $S \subseteq V(G)$ is called a *determining set* if the identity is the only automorphism fixing every element of S . Note that if $S \subseteq V(G)$ is a determining set, then any subset $T \subseteq V(G)$ containing S is also a determining set. The determining sets provide a useful tool for finding the distinguishing number of G as stated in the following theorem.

Proposition 2.1 ([2]). *For a given graph G , G has an r -distinguishable determining set if and only if G has an $(r + 1)$ -distinguishing coloring.*

Consequently, we find the following corollary which will be used in the proof of our results.

Corollary 2.2. *For a given graph G , if there is an asymmetric subgraph of G induced by a determining set S , then $\text{Dist}(G) = 1$ or 2 .*

Proof. For a graph automorphism ϕ of G fixing S set-wisely, the restriction of ϕ on the induced subgraph $\langle S \rangle$ is a graph automorphism of $\langle S \rangle$. Since $\langle S \rangle$ is asymmetric, the coloring $f(v) = 1$ for all $v \in S$ is a 1-distinguishing. Since S is a determining set, G has a 2-distinguishing coloring by Proposition 2.1. \square

Using Proposition 2.1, M. O. Albertson and D. L. Boutin determined the distinguishing number of the Kneser graphs as follows.

Proposition 2.3 ([2]). *For any integers $n > k \geq 2$ with $n > 2k$, $\text{Dist}(K(n, k)) = 2$ except $(n, k) = (5, 2)$; and $\text{Dist}(K(5, 2)) = 3$.*

Note that the Kneser graph $K(5, 2)$ is isomorphic to the Petersen graph and its distinguishing number is 3. $K(n, 1)$ is isomorphic to the complete graph and so $\text{Dist}(K(n, 1)) = n$ for any integer n .

For a graph G , the distinguishing number $\text{Dist}(G)$ is equal to the distinguishing number of $\text{Aut}(G)$ which acts on the vertex set $V(G)$. Hence we have the following lemma. The proof is straightforward and we omit it.

Lemma 2.4. *Let G_1 and G_2 be two graphs having the same vertex set V .*

- (1) *If $\text{Aut}(G_1)$ is a subgroup of $\text{Aut}(G_2)$ as acting groups on V , then $\text{Dist}(G_1) \leq \text{Dist}(G_2)$.*
- (2) *If $\text{Aut}(G_1) = \text{Aut}(G_2)$ as acting groups on V , then $\text{Dist}(G_1) = \text{Dist}(G_2)$.*

3. The distinguishing numbers of the merged Johnson graphs

Let Ω be the set of all k -subsets of $[n]$. The action of S_n on $[n]$ naturally induces an action of S_n on Ω . The Johnson graph $J(n, k)$ is an orbital graph which corresponds to the orbital $\{(M, N) \in \Omega^2 \mid |M \cap N| = k - 1\}$. The distance i graph $J(n, k)_i$ of $J(n, k)$ is an orbital graph corresponding to the orbital

$$\Gamma_i = \{(M, N) \in \Omega^2 \mid |M \cap N| = k - i\}$$

(see [6] for orbital graphs).

For a merged Johnson graph $J = J(n, k)_I$, the complementation of each k -sets $M \rightarrow M^c$ induces an isomorphism from $J(n, k)_I$ to $J(n, n - k)_I$. So we may assume that $k \leq n/2$. For a merged Johnson graph $J = J(n, k)_I$ with $1 \leq k \leq \frac{n}{2}$, if $I = \emptyset$ or $\{1, 2, \dots, k\}$, then J is the null or complete graph and so $\text{Aut}(J) = S_d$, where $d = \binom{n}{k}$. Thus, we further assume that $k \geq 2$ and $\emptyset \subsetneq I \subsetneq \{1, 2, \dots, k\}$. For notational simplicity, let $I' = I \setminus \{k\}$, and for any integer t , let $t - I = \{t - i \mid i \in I\}$ and $t - I' = \{t - i \mid i \in I'\}$. We also denote $I'' = k - I'$, and let $e = \frac{1}{2} \binom{n}{n/2}$.

In [9], G. Jones found the automorphism groups of the merged Johnson graphs as follows.

Theorem 3.1 ([9]). *Let $J = J(n, k)_I$, where $2 \leq k \leq \frac{n}{2}$ and $\emptyset \subsetneq I \subsetneq \{1, 2, \dots, k\}$ and let $A = \text{Aut}(J)$.*

- (1) *If $2 \leq k < \frac{n-1}{2}$, and $J \neq J(12, 4)_I$ with $I = \{1, 3\}$ or $\{2, 4\}$, then $A = S_n$ with orbitals $\Gamma_0, \Gamma_1, \dots, \Gamma_k \subset \Omega^2$.*
- (2) *If $J = J(12, 4)_I$ with $I = \{1, 3\}$ or $\{2, 4\}$, then $A = O_{10}^{-1}(2)$ with orbitals $\Gamma_0, \Gamma_1 \cup \Gamma_3, \Gamma_2 \cup \Gamma_4$.*
- (3) *If $k = \frac{n-1}{2}$ and $I \neq k + 1 - I$, then $A = S_n$ with orbitals $\Gamma_0, \Gamma_1, \dots, \Gamma_k \subset \Omega^2$.*
- (4) *If $k = \frac{n-1}{2}$ and $I = k + 1 - I$, then $A = S_{n+1}$ with orbitals Γ_0 and $\Gamma_i \cup \Gamma_{k+1-i}$ for all $i = 1, 2, \dots, \lfloor \frac{k+1}{2} \rfloor$.*
- (5) *If $k = \frac{n}{2}$ and $I \neq \{k\}$ nor $\{1, 2, \dots, k - 1\}$, and $I' \neq I''$, then $A = S_2 \times S_n$ with orbitals $\Gamma_0, \Gamma_1, \dots, \Gamma_k \subset \Omega^2$.*

- (6) If $k = \frac{n}{2}$ and $I \neq \{k\}$ nor $\{1, 2, \dots, k-1\}$, and $I' = I''$, then $A = S_2^e : S_n$ with orbitals Γ_0 and $\Gamma_i \cup \Gamma_{k-i}$ for all $i = 1, 2, \dots, \lfloor \frac{k}{2} \rfloor$ and Γ_k .
- (7) If $k = \frac{n}{2}$ and $I = \{k\}$ or $\{1, 2, \dots, k-1\}$, then $A = S_2^e : S_e = S_2 \wr S_e$ with orbitals $\Gamma_0, \Gamma_1 \cup \dots \cup \Gamma_{k-1}$ and Γ_k .

To understand the automorphism group S_{n+1} of $J(n, \frac{n-1}{2})_I$ with $I = k + 1 - I$, let $[\tilde{n}] = [n] \cup \{\infty\}$ and let Ψ be the set of equipartitions of $[\tilde{n}]$, by which we mean the unordered partitions $\{P_1, P_2\}$ of $[\tilde{n}]$ satisfying $|P_1| = |P_2| = \frac{n+1}{2}$. There is a bijection $\phi : \Omega \rightarrow \Psi$, sending each M to $\{M \cup \{\infty\}, [n] - M\}$. Note that its inverse sends an equipartition $\{P_1, P_2\}$ to $P_i \setminus \{\infty\}$, where i is chosen so that $\infty \in P_i$. The natural action of S_{n+1} on $[\tilde{n}]$ induces an action of S_{n+1} on Ω . By the condition $I = k + 1 - I$, one can see that this action induces an automorphism group of $J(n, \frac{n-1}{2})_I$ (For a detail information, see the paper [9]). The next theorem is the main theorem of this paper.

Theorem 3.2. *Let $J = J(n, k)_I$, where $2 \leq k \leq \frac{n}{2}$ and $\emptyset \subsetneq I \subsetneq \{1, 2, \dots, k\}$.*

- (1) *If $(n, k) \neq (5, 2)$ and $J \neq J(n, \frac{n}{2})_I$ with $I = \{\frac{n}{2}\}, \{1, 2, \dots, \frac{n}{2} - 1\}$ or $I' = I''$, then $\text{Dist}(J) = 2$.*
- (2) *If $J = J(5, 2)_I$ with $I = \{1\}$ or $\{2\}$; or $J = J(n, \frac{n}{2})_I$ satisfying $I' = I''$ and I is neither $\{\frac{n}{2}\}$ nor $\{1, 2, \dots, \frac{n}{2} - 1\}$, then $\text{Dist}(J) = 3$.*
- (3) *If $J = J(n, \frac{n}{2})_I$ with $I = \{\frac{n}{2}\}$ or $\{1, 2, \dots, \frac{n}{2} - 1\}$, then $\text{Dist}(J) = \lfloor \frac{1 + \sqrt{1 + 4\binom{n}{n/2}}}{2} \rfloor$.*

Corollary 3.3. *Let $G = J(n, k)$ be the Johnson graph with $k \geq 2$.*

- (1) *If $(n, k) \neq (4, 2)$ nor $(5, 2)$, then $\text{Dist}(G) = 2$ and*
- (2) *if $G = J(4, 2)$ or $J(5, 2)$, then $\text{Dist}(J) = 3$.*

We will prove Theorem 3.2 in the next section. For the rest of the section, we will prove lemmas which will be used in the proof of Theorem 3.2. For a set $[n] = \{1, 2, 3, \dots, n\}$ and for any permutation π of $[n]$, π can be represented by (i_1, i_2, \dots, i_n) , where for any $j = 1, \dots, n$, $\pi(j) = i_j$. Throughout the rest of the paper, we use the above representation of permutations. Now for any permutations $\pi = (i_1, i_2, \dots, i_n), \sigma = (j_1, j_2, \dots, j_n)$ and for any integer $t \in [n]$, $\pi\sigma(t) = \pi(\sigma(t)) = \pi(j_t) = i_{j_t}$, and hence $\pi\sigma = (i_{j_1}, i_{j_2}, \dots, i_{j_n})$. For any $j \in [n]$, let τ_j be the transposition of $[n]$ exchanging j and $j + 1$. Note that for any permutation $\pi = (i_1, i_2, \dots, i_n)$, $\pi\tau_j = (i_1, \dots, i_{j-1}, i_{j+1}, i_j, i_{j+2}, \dots, i_n)$.

For any permutation $\pi = (i_1, i_2, \dots, i_n)$ and for any $k, \ell \in [n]$ with $1 \leq k \leq \frac{n}{2}$, let $V_{k,\ell}^\pi$ be the k -subset $\{i_\ell, i_{\ell+1}, \dots, i_{\ell+k-1}\}$, where the subscripts are considered as their residue classes modulo n . For our convenience, as a vertex of $J(n, k)_I$, we denote $V_{k,\ell}^\pi$ by V_ℓ^π and if π is the identity, we denote V_ℓ^π simply by V_ℓ .

For any permutation $\pi = (i_1, i_2, \dots, i_n)$, let ϕ_π be the permutation of the vertex set of $J(n, k)_I$ defined by

$$\phi_\pi(\{a_1, a_2, \dots, a_k\}) = \{\pi(a_1), \pi(a_2), \dots, \pi(a_k)\} = \{i_{a_1}, i_{a_2}, \dots, i_{a_k}\}$$

for any vertex $\{a_1, a_2, \dots, a_k\}$ of $J(n, k)_I$. Now one can check that ϕ_π be an automorphism of $J(n, k)_I$.

Lemma 3.4. *For any permutation $\pi = (i_1, i_2, \dots, i_n)$ of $[n]$ and for any $\ell \in [n]$, $\phi_\pi(V_\ell) = V_\ell^\pi$. Furthermore for any permutation σ of $[n]$, $\phi_\pi(V_\ell^\sigma) = V_\ell^{\pi\sigma}$.*

Proof. Since $V_\ell = \{\ell, \ell + 1, \dots, \ell + k - 1\}$, we have

$$\phi_\pi(V_\ell) = \{\pi(\ell), \pi(\ell + 1), \dots, \pi(\ell + k - 1)\} = \{i_\ell, i_{\ell+1}, \dots, i_{\ell+k-1}\} = V_\ell^\pi.$$

Furthermore for any permutation σ of $[n]$,

$$\begin{aligned} \phi_\pi(V_\ell^\sigma) &= \phi_\pi(\{\sigma(\ell), \sigma(\ell + 1), \dots, \sigma(\ell + k - 1)\}) \\ &= \{\pi\sigma(\ell), \pi\sigma(\ell + 1), \dots, \pi\sigma(\ell + k - 1)\} = V_\ell^{\pi\sigma}. \quad \square \end{aligned}$$

Lemma 3.5. *Let $J = J(12, 4)_I$ with $I = \{1, 3\}$. For any permutation π of X , if an automorphism ϕ of J fixes all vertices in $S = \{V_j^\pi, V_j^{\pi\tau_1}, V_j^{\pi\tau_2} \mid j = 1, 2, \dots, 12\}$, then ϕ also fixes $V_j^{\pi\tau_i}$ for all $i, j = 1, 2, \dots, 12$.*

Proof. By Lemma 3.4, we may assume that π is the identity. Note that $S = \{V_j \mid j = 1, 2, \dots, 12\} \cup \{V_2^{\tau_1}, V_{10}^{\tau_1}\} \cup \{V_3^{\tau_2}, V_{11}^{\tau_2}\}$, where $V_2^{\tau_1} = \{1, 3, 4, 5\}$, $V_{10}^{\tau_1} = \{10, 11, 12, 2\}$ and $V_3^{\tau_2} = \{2, 4, 5, 6\}$, $V_{11}^{\tau_2} = \{11, 12, 1, 3\}$.

Let ϕ be an automorphism of J fixing all vertices in S . Since $\tau_3 = (1, 2, 4, 3, 5, 6, \dots, 12)$, we have $\{V_j^{\tau_3} \mid j = 1, 2, \dots, 12\} \setminus S = \{V_4^{\tau_3}, V_{12}^{\tau_3}\}$, where $V_4^{\tau_3} = \{3, 5, 6, 7\}$ and $V_{12}^{\tau_3} = \{12, 1, 2, 4\}$. Let $A = N(V_4^{\tau_3}) \cap S$ and $B = N(V_{12}^{\tau_3}) \cap S$. Now we have

$$A = \{V_1, V_3, V_4, V_5, V_7, V_{12}, V_{11}^{\tau_2}\} \quad \text{and} \quad B = \{V_1, V_3, V_4, V_9, V_{11}, V_{12}\}.$$

Let X be a vertex in J such that $N(X) \cap S = A$. For the first case, assume that $|X \cap V_1| = 3$. If $X \cap V_1 = \{1, 2, 3\}$, then $4, 5, 6 \notin X$ and $7 \in X$ because X is adjacent to both V_3 and V_4 but not to V_2 . In this case, X is adjacent to V_6 , a contradiction. If $X \cap V_1 = \{1, 2, 4\}$, then $5, 6, 7 \notin X$ and $8 \in X$, and hence X is adjacent to V_6 , a contradiction. Similarly, one can show that $X \cap V_1$ is neither $\{1, 3, 4\}$ nor $\{2, 3, 4\}$. Therefore we have $|X \cap V_1| = 1$.

Assume that $X \cap V_1 = \{1\}$. Then $2, 3, 4, 5 \notin X$ and $6 \in X$ because $V_2 \notin N(X)$ and $V_3 \in N(X)$. Since X is adjacent to V_4, V_5, V_7, V_{12} but not to V_2, V_6, V_8 , the only possible X is $\{1, 6, 9, 11\}$. But $\{1, 6, 9, 11\}$ is adjacent to $V_2^{\tau_1} = \{1, 3, 4, 5\}$ which is an element in $S \setminus A$, a contradiction. Similarly, one can show that if $X \cap V_1 = \{2\}$ or $X \cap V_1 = \{4\}$, then a contradiction occurs. Furthermore when $X \cap V_1 = \{3\}$, the only possible X is $\{3, 5, 6, 7\}$, which satisfies $N(X) \cap S = A$. Hence $X = \{3, 5, 6, 7\} = V_4^{\tau_3}$. This implies that ϕ also fixes $V_4^{\tau_3}$.

Let Y be a vertex in J such that $N(Y) \cap S = B$. By considering the fact that Y is adjacent to $V_1, V_3, V_4, V_9, V_{11}, V_{12}$ but not to $V_2, V_5, V_6, V_7, V_8, V_{10}$, one can show that $Y = \{1, 6, 8, 10\}, \{2, 5, 8, 9\}$ or $\{12, 1, 2, 4\}$. Since Y is not adjacent to $V_2^{\tau_1} = \{1, 3, 4, 5\}$, Y is $V_{12}^{\tau_3} = \{12, 1, 2, 4\}$. This implies that ϕ fixes $V_{12}^{\tau_3}$. Up to now, we showed that ϕ fixes $V_j^{\tau_3}$ for all $j = 1, 2, \dots, 12$.

Since ϕ fixes all elements in $\{V_j, V_j^{\tau_2}, V_j^{\tau_3} \mid j = 1, 2, \dots, 12\}$, one can show that ϕ fixes $V_j^{\tau_4}$ for all $j = 1, 2, \dots, 12$ by a similar way. Continuing the similar process, one can show that ϕ fixes $V_j^{\tau_i}$ for all $i, j = 1, 2, \dots, 12$. \square

Lemma 3.6. *Let $J = J(12, 4)_I$ with $I = \{1, 3\}$. For any permutation π of $[n]$, if an automorphism ϕ of J fixes all vertices in $S_1 = \{V_j^\pi, V_j^{\pi\tau_i} \mid i, j = 1, 2, \dots, 12\}$, then ϕ also fixes $V_j^{\pi\tau_i\tau_k}$ for all $i, j, k = 1, 2, \dots, 12$.*

Proof. By Lemma 3.4, assume that π is the identity. Let ϕ be an automorphism of J fixing all vertices in S_1 . Note that $\{V_j^{\tau_1\tau_2} \mid j = 1, 2, \dots, 12\} - S_1 = \{V_3^{\tau_1\tau_2}, V_{11}^{\tau_1\tau_2}\}$ because $\tau_1\tau_2 = (2, 3, 1, 4, 5, \dots, 12)$, where $V_3^{\tau_1\tau_2} = \{1, 4, 5, 6\}$ and $V_{11}^{\tau_1\tau_2} = \{11, 12, 2, 3\}$. Let $C = N(V_3^{\tau_1\tau_2}) \cap S_1$ and $D = N(V_{11}^{\tau_1\tau_2}) \cap S_1$. Now we have

$$\begin{aligned} \{V_3, V_4, V_6, V_{10}, V_{11}, V_{12}, V_2^{\tau_1}, V_3^{\tau_2}\} &\subset C, \quad \{V_1, V_2, V_5, V_7, V_8, V_9\} \cap C = \emptyset \quad \text{and} \\ \{V_3, V_8, V_{11}, V_{12}, V_2^{\tau_1}, V_3^{\tau_2}\} &\subset D, \quad \{V_1, V_2, V_4, V_5, V_6, V_7, V_9, V_{10}\} \cap D = \emptyset. \end{aligned}$$

Let X be a vertex in J such that $N(X) \cap S_1 = C$. By considering the fact that X is adjacent to $V_3, V_4, V_6, V_{10}, V_{11}, V_{12}$ but not to $V_1, V_2, V_5, V_7, V_8, V_9$, we have that $X = \{6, 8, 9, 12\}, \{1, 4, 5, 6\}$ or $\{2, 4, 9, 10\}$. Since X is adjacent to both $V_2^{\tau_1} = \{1, 3, 4, 5\}$ and $V_3^{\tau_2} = \{2, 4, 5, 6\}$, X is $V_3^{\tau_1\tau_2} = \{1, 4, 5, 6\}$. This implies that ϕ also fixes $V_3^{\tau_1\tau_2}$.

Let Y be a vertex in J such that $N(Y) \cap S_1 = D$. By considering the fact that Y is adjacent to V_3, V_8, V_{11}, V_{12} but not to $V_1, V_2, V_4, V_5, V_6, V_7, V_9, V_{10}$, one can show that $Y = \{6, 7, 10, 12\}, \{2, 4, 7, 8\}$ or $\{11, 12, 2, 3\}$. Since Y is adjacent to both $V_2^{\tau_1} = \{1, 3, 4, 5\}$ and $V_3^{\tau_2} = \{2, 4, 5, 6\}$, Y is $V_{11}^{\tau_1\tau_2} = \{11, 12, 2, 3\}$. This implies that ϕ fixes $V_{11}^{\tau_1\tau_2}$. Therefore ϕ fixes $V_j^{\tau_1\tau_2}$ for all $j = 1, 2, \dots, 12$.

Since ϕ fixes all vertices in $\{V_j^\pi, V_j^{\pi\tau_1}, V_j^{\pi\tau_2} \mid j = 1, 2, \dots, 12\}$ with $\pi = \tau_1$, ϕ also fixes $V_j^{\tau_1\tau_i}$ for all $i, j = 1, 2, \dots, 12$ by Lemma 3.5. By a similar way, one can show that ϕ fixes $V_j^{\tau_i\tau_k}$ for all $i, j, k = 1, 2, \dots, 12$. \square

Lemma 3.7. *For the merged Johnson graph $J = J(12, 4)_I$ with $I = \{1, 3\}$, $S = \{V_j, V_j^{\tau_1}, V_j^{\tau_2} \mid j = 1, 2, \dots, 12\}$ is a determining set.*

Proof. Let ϕ be an automorphism of J fixing all vertices in $S = \{V_j, V_j^{\tau_1}, V_j^{\tau_2} \mid j = 1, 2, \dots, 12\}$. By Lemma 3.5, ϕ fixes $V_j^{\tau_i}$ for all $i, j = 1, 2, \dots, 12$. Furthermore, ϕ fixes $V_j^{\tau_i\tau_k}$ for all $i, j, k = 1, 2, \dots, 12$ by Lemma 3.6. By applying Lemma 3.6 again with $\pi = \tau_i$, one can show that ϕ fixes $V_j^{\tau_i\tau_k\tau_\ell}$ for all $i, j, k, \ell = 1, 2, \dots, 12$. Continuing the similar process, one can show that ϕ fixes $V_j^{\tau_{i_1}\tau_{i_2}\dots\tau_{i_t}}$ for any positive integer t and for all i_1, i_2, \dots, i_t, j in $[12]$. Since $\{\tau_i \mid i = 1, 2, \dots, 12\}$ generates symmetric group on $[12]$, ϕ fixes all vertices of J , i.e., S is a determining set. \square

Let J be a merged Johnson graph $J(2m, m)_I$ with $I \subseteq \{1, 2, \dots, m\}$. For any $v \in V(G)$, let \bar{v} be the vertex $[2m] - v$ for convenience.

Lemma 3.8. *Let $J = J(n, k)_I$, where $1 \leq k \leq \frac{n}{2}$ and $\emptyset \subsetneq I \subsetneq \{1, 2, \dots, k\}$.*

- (1) For $(n, k) = (2m + 1, m)$ with $m \geq 3$ and $I = \{1, m\}$, $S_1 = \{V_1, V_2, \dots, V_{m+2}\}$ is a determining set.
- (2) For $(n, k) = (2m, m)$ with $m \geq 3$ and $I = \{1\}$, $S_2 = \{V_1, V_2, \dots, V_{2m}\} \cup \{\{1, 2, \dots, m - 2, m, m + 2\}\}$ is a determining set.

Proof. (1) Note that $\text{Aut}(J)$ is isomorphic to S_{n+1} by Theorem 3.1. For any automorphism ψ of J as a permutation of vertices of J , let ψ' be a corresponding permutation of $[\tilde{n}] = [n] \cup \{\infty\}$. Let ϕ be an automorphism of J fixing all elements in S_1 . Since ϕ fixes V_1 , ϕ' fixes $\{1, 2, \dots, m, \infty\}$ set-wisely or ϕ' sends $\{1, 2, \dots, m, \infty\}$ to $\{m + 1, m + 2, \dots, 2m + 1\}$ set-wisely.

Case 1: ϕ' fixes $\{1, 2, \dots, m, \infty\}$ set-wisely.

Since ϕ fixes V_2 and ϕ' fixes $\{1, 2, \dots, m, \infty\}$ set-wisely, ϕ' also fixes $\{2, 3, \dots, m + 1, \infty\}$ set-wisely. This implies that ϕ' fixes both 1 and $m + 1$. Using the fact that ϕ fixes all elements in S_1 , one can see that ϕ' fixes all elements in $[\tilde{n}]$, namely, ϕ is the identity element.

Case 2: ϕ' sends $\{1, 2, \dots, m, \infty\}$ to $\{m + 1, m + 2, \dots, 2m + 1\}$ set-wisely.

Since ϕ fixes V_2 and ϕ' sends $\{1, 2, \dots, m, \infty\}$ to $\{m + 1, m + 2, \dots, 2m + 1\}$ set-wisely, ϕ' also sends $\{2, 3, \dots, m + 1, \infty\}$ to $\{m + 2, m + 3, \dots, 2m + 1, 1\}$ set-wisely. Furthermore this implies that $\phi'(\infty)$ belongs to $\{m + 2, m + 3, \dots, 2m + 1\}$. By the similar way, one can show that for any $i = 1, 2, \dots, m + 2$, ϕ' sends $\{i, i + 1, \dots, m + i - 1, \infty\}$ to $[n] - \{i, i + 1, \dots, m + i - 1\}$ set-wisely.

Let $\phi'(\infty) = a$. Since a belongs to $\{m + 2, \dots, 2m + 1\}$, ϕ' can not send $\{a - m + 1, a - m + 2, \dots, a, \infty\}$ to $[n] - \{a - m + 1, a - m + 2, \dots, a\}$ set-wisely, which is a contradiction.

Therefore $S_1 = \{V_1, V_2, \dots, V_{m+2}\}$ is a determining set.

(2) Let α be the automorphism which sends v to \bar{v} for all $v \in V(G)$. Then the order of α is 2 and $\text{Aut}(J) \cong \langle \alpha \rangle \times S_n$.

Let ψ be an automorphism of J fixing all elements in S_2 . Since ψ fixes V_1 , ψ is an automorphism induced by a permutation of $[n]$ fixing $\{1, 2, \dots, m\}$ set-wisely or ψ is a product of α and an automorphism induced by a permutation of $[n]$ sending $\{1, 2, \dots, m\}$ to $\{m + 1, m + 2, \dots, 2m\}$ set-wisely.

If ψ is an automorphism induced by a permutation of $[n]$, say ψ_1 , fixing $\{1, 2, \dots, m\}$ set-wisely, then ψ_1 also fixes $\{i, i + 1, \dots, i + m - 1\}$ set-wisely for all $i = 1, 2, \dots, 2m$ because ψ fixes V_i for all $i = 1, 2, \dots, 2m$. This implies that ψ is the identity. Hence we can assume that ψ is a product of α and an automorphism induced by a permutation of $[n]$, say ψ_2 , sending $\{1, 2, \dots, m\}$ to $\{m + 1, m + 2, \dots, 2m\}$ set-wisely. Since ψ fixes V_2 , ψ_2 also sends $\{2, 3, \dots, m + 1\}$ to $\{m + 2, m + 3, \dots, 2m, 1\}$ set-wisely. This implies that ψ_2 exchanges 1 and $m + 1$. By a similar way, one can show that ψ_2 exchanges i and $m + i$ for any $i = 1, 2, \dots, m$. But in this case, ψ does not fix $\{1, 2, \dots, m - 2, m, m + 2\}$. Therefore S_2 is a determining set. \square

Lemma 3.9. *If $J = J(2m, m)_{\{1, m-1\}}$ with $m \geq 4$, then $\text{Dist}(J) > 2$.*

Proof. For any $v \in V(J)$, let β_v be the automorphism of $V(J)$ exchanging v and \bar{v} and fixing all other vertices. Let $f : V(J) \rightarrow \{1, 2\}$ be a coloring. If there exists $u \in V(J)$ such that $f(u) = f(\bar{u})$, then β_u is a color-preserving automorphism, and hence f is not 2-distinguishing. Assume that for all $v \in V(J)$, $f(v)$ and $f(\bar{v})$ are distinct. Let ϕ be an automorphism of J induced by a non-identity permutation of $[2m]$. Let

$$\psi = \left(\prod_{\{u, \bar{u}\} \text{ with } f(u) \neq f(\phi(u))} \beta_u \right) \phi.$$

Then ψ is a color-preserving automorphism. Therefore there is no 2-distinguishing coloring, and hence $\text{Dist}(J) > 2$. \square

For $n = 2m$ with $m \geq 4$, let Φ be the set of equipartitions of $[n]$. Note that the size of Φ is $\frac{1}{2} \binom{n}{m}$. The natural action of S_n on $[n]$ induces an action of S_n on Φ . Let π be a permutation in S_n satisfying that for all non-identity permutation $\sigma \in S_n$, the number of equipartitions in Φ fixed by π is greater than or equal to the number of equipartitions in Φ fixed by σ . Suppose that there exist a permutation $\gamma \in S_n$ and $i_1, i_2, \dots, i_t \in [n]$ with $t \geq 3$ such that $\gamma(i_j) = i_{j+1}$ for all $j = 1, 2, \dots, t-1$ and $\gamma(i_t) = i_1$. Let $\tilde{\gamma} \in S_n$ be a permutation that $\tilde{\gamma}(i_2) = i_1, \tilde{\gamma}(i_t) = i_3$ and $\tilde{\gamma}(\ell) = \gamma(\ell)$ for all $\ell \in [n] \setminus \{i_2, i_t\}$. Note that all equipartitions in Φ fixed by γ are also fixed by $\tilde{\gamma}$. Furthermore there is an equipartition fixed by $\tilde{\gamma}$ but not by γ . This implies that π is a product of disjoint transpositions. For two permutations $\gamma_1, \gamma_2 \in S_n$, suppose that there exist $i_1, i_2, \dots, i_6 \in [n]$ such that

$$\begin{aligned} \gamma_1(i_1) &= i_2, \gamma_1(i_2) = i_1, \gamma_1(i_3) = i_4, \gamma_1(i_4) = i_3, \gamma_1(i_5) = i_5, \gamma_1(i_6) = i_6, \\ \gamma_2(i_1) &= i_2, \gamma_2(i_2) = i_1, \gamma_2(i_3) = i_3, \gamma_2(i_4) = i_4, \gamma_2(i_5) = i_5, \gamma_2(i_6) = i_6, \end{aligned}$$

and for all $j \in [n] \setminus \{i_1, i_2, \dots, i_6\}$, $\gamma_1(j) = \gamma_2(j)$. Then one can check that all equipartitions fixed by γ_1 are also fixed by γ_2 . This implies that π is a transposition or a product of m disjoint transpositions. Note that the number of equipartitions fixed by a transposition is $\binom{2m-2}{m-2}$ and the number of equipartitions fixed by a product of m disjoint transpositions is 2^{m-1} if m is odd; $2^{m-1} + \binom{m}{m/2}$ if m is even. We aim to show inductively that $\binom{2m-2}{m-2} > 2^{m-1} + \binom{m}{m/2}$ for all even m with $m \geq 4$. For $m = 4$,

$$\binom{2m-2}{m-2} = 15 > 14 = 2^{m-1} + \binom{m}{m/2}.$$

Suppose that for some even $m \geq 4$, $\binom{2m-2}{m-2} > 2^{m-1} + \binom{m}{m/2}$. Then

$$\begin{aligned} \binom{2m}{m-1} &= \frac{2m(2m-1)}{(m+1)(m-1)} \binom{2m-2}{m-2} > 2 \left(2^{m-1} + \binom{m}{m/2} \right) > 2^m \text{ and} \\ \binom{2m+2}{m} &= \frac{(2m+2)(2m+1)2m(2m-1)}{(m+2)(m+1)m(m-1)} \binom{2m-2}{m-2} \end{aligned}$$

$$> 2^2 \frac{(2m+1)(2m-1)}{(m+2)(m-1)} \left(2^{m-1} + \binom{m}{m/2} \right) > 2^{m+1} + \binom{m+2}{m/2+1}.$$

Hence π is a transposition. Furthermore for any non-identity permutation $\sigma \in S_n$, the number of equipartitions fixed by σ is at most $\binom{2m-2}{m-2}$.

Lemma 3.10. *For $n = 2m$ with $m \geq 4$, let Φ be the set of equipartitions of $[n]$. Now there is a 3-coloring $c : \Phi \rightarrow \{B, R, Y\}$ of Φ such that only identity permutation in S_n preserves all colors under the induced action of S_n on Φ .*

Proof. Give a random coloring on Φ with three colors $\{B, R, Y\}$. For any non-identity permutation σ of $[2m]$, let A_σ be the event that σ preserves colors of all equipartitions of $[2m]$. Note that the number of equipartitions fixed by σ is at most $\binom{2m-2}{m-2}$. Namely, the number of equipartitions which are not fixed by σ is at least

$$|\Phi| - \binom{2m-2}{m-2} = \frac{1}{2} \binom{2m}{m} - \binom{2m-2}{m-2} = \frac{m}{m-1} \binom{2m-2}{m-2}.$$

For any orbit O of σ whose size is t with $t \geq 2$ under the action of S_{2m} on Φ , the probability that σ preserves colors of all equipartitions in O is 3^{-t+1} , which is less than $3^{-\frac{t}{2}}$. Hence we have

$$Pr(A_\sigma) \leq 3^{-\frac{m}{2(m-1)} \binom{2m-2}{m-2}}.$$

Therefore

$$\begin{aligned} Pr\left(\bigcup_{\sigma \in S_n \setminus \{id\}} A_\sigma\right) &\leq \sum_{\sigma \in S_n \setminus \{id\}} Pr(A_\sigma) < ((2m)! - 1) 3^{-\frac{m}{2(m-1)} \binom{2m-2}{m-2}} \\ &\leq (2m)! 3^{-\frac{m}{2(m-1)} \binom{2m-2}{m-2}}, \end{aligned}$$

where id is the identity permutation of $[n]$. For $n = 8$, the number

$$n! 3^{-\frac{m}{2(m-1)} \binom{2m-2}{m-2}}$$

is $\frac{4480}{3^8}$, and it is less than 1. Furthermore for any $m \geq 4$,

$$\frac{(2m+2)! 3^{-\frac{m+1}{2m} \binom{2m}{m-1}}}{(2m)! 3^{-\frac{m}{2(m-1)} \binom{2m-2}{m-2}}} = \frac{(2m+2)(2m+1)}{3^{\frac{3m-2}{2(m-1)} \binom{2m-2}{m-2}}} < \frac{(2m+2)(2m+1)}{3^{\frac{3}{2} \binom{2m-2}{m-2}}} < 1$$

because

$$\begin{aligned} 3^{\frac{3}{2} \binom{2m-2}{m-2}} &= 3^{\frac{3(2m-2)(2m-3)\dots(m+1)}{2 \cdot (m-2)!}} = 3^{\frac{12(m-1)(m-2)(2m-3)(2m-5)(2m-6)\dots(m+1)}{12(m-2)(m-3)(m-4)\dots 5 \cdot 4}} \\ &= 3^{\frac{(m-1)(2m-3)(2m-5)(2m-6)\dots(m+1)}{(m-3)(m-4)\dots 5 \cdot 4}} \\ &> 3^{(m-1)(2m-3)(2m-5)} > (2m+2)(2m+1). \end{aligned}$$

This implies that for any $m \geq 4$,

$$Pr\left(\bigcup_{\sigma \in S_n \setminus \{id\}} A_\sigma\right) < 1$$

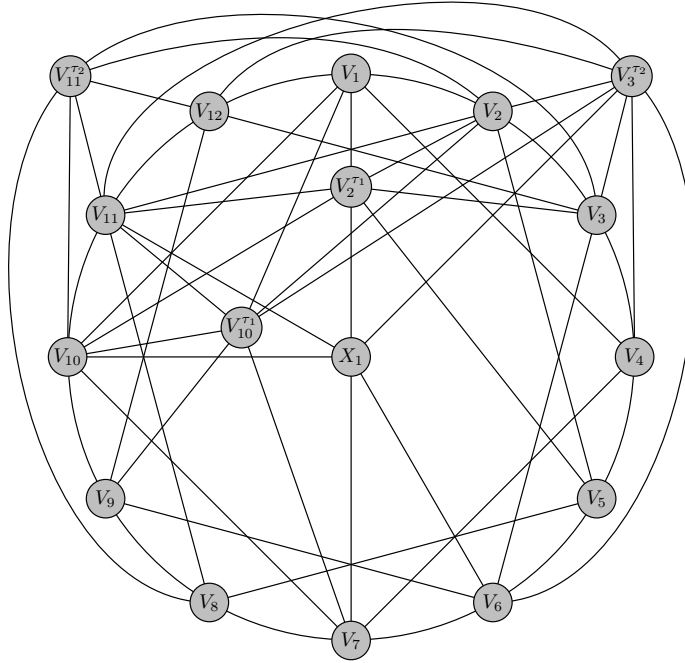


FIGURE 1. An asymmetric subgraph induced by a determining set of $J(12, 4)_{\{1,3\}}$.

and hence there exists a 3-coloring $c : \Phi \rightarrow \{B, R, Y\}$ such that the identity permutation in S_{2m} is the only color-preserving permutation under the induced action of S_{2m} on Φ . □

4. A proof of the main theorem

In this section, we prove Theorem 3.2 which is the main result in this paper. For any positive integer i , let D_i be the set of all vertices whose degrees are i .

Let $J = J(n, k)_I$, where $2 \leq k \leq \frac{n}{2}$ and $\emptyset \subsetneq I \subsetneq \{1, 2, \dots, k\}$.

Case 1: $(n, k) \neq (5, 2)$ and $2 \leq k < \frac{n-1}{2}$, and $J \neq J(12, 4)_I$ with $I = \{1, 3\}$ or $\{2, 4\}$.

In this case, $\text{Aut}(J) = \text{Aut}(K(n, k))$ as an acting group on the vertex set, and hence $\text{Dist}(J) = \text{Dist}(K(n, k)) = 2$ by Proposition 2.3 and Lemma 2.4.

Case 2: $J = J(12, 4)_I$ with $I = \{1, 3\}$ or $\{2, 4\}$.

Assume that $J = J(12, 4)_I$ with $I = \{1, 3\}$. Let $S_1 = \{V_j, V_j^{T_1}, V_j^{T_2} \mid j = 1, 2, \dots, 12\} \cup \{X_1 = \{1, 3, 5, 7\}\}$ and let H_1 be the subgraph of J induced by S_1 as illustrated in Figure 1. Then S_1 is a determining set by Lemma 3.7. Let

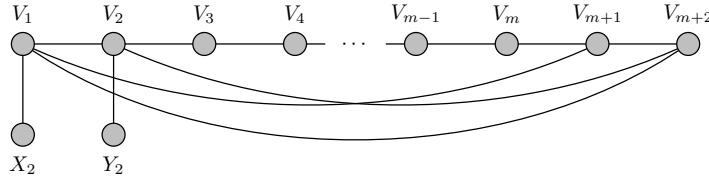


FIGURE 2. An asymmetric subgraph induced by a determining set of $J(2m + 1, m)_{\{1,m\}}$.

ψ_1 be an automorphism of H_1 . Note that the order of H_1 is 17 and

$$D_5 = \{V_4, V_5, V_8, V_9\}, D_6 = \{V_1, V_6, V_7, V_{12}, V_{11}^{T_2}, X_1\},$$

$$D_7 = \{V_3, V_2^{T_1}, V_{10}^{T_1}\}, D_8 = \{V_2, V_{10}, V_3^{T_2}\}, D_9 = \{V_{11}\}.$$

Since $D_9 = \{V_{11}\}$, $\psi_1(V_{11}) = V_{11}$. The fact $N(V_{11}) \cap D_5 = \{V_8\}$ implies that $\psi_1(V_8) = V_8$. Since $N(V_8) \cap D_5 = \{V_5, V_9\}$, V_4 is the only vertex in D_5 which is not contained to $N(V_8) \cup \{V_8\}$. Hence ψ_1 fixes V_4 . By the fact that $N(V_4) \cap N(V_8) \cap D_5 = \{V_5\}$, ψ_1 fixes both V_5 and V_9 . Note that $N(V_8) \cap N(V_{11}) = \{V_{11}^{T_2}\}$. So ψ_1 fixes $V_{11}^{T_2}$. The fact $N(V_4) \cap N(V_8) \cap D_6 = \{V_7\}$ and $N(V_5) \cap N(V_9) \cap D_6 = \{V_6\}$ implies that ψ_1 fixes both V_6 and V_7 . Furthermore ψ_1 fixes $X_1 = \{1, 3, 5, 7\}$ because $N(V_6) \cap N(V_7) = \{X_1\}$. Since $N(V_7) \cap N(V_{11}) \cap D_7 = \{V_{10}^{T_1}\}$ and $N(V_7) \cap N(V_{11}) \cap D_8 = \{V_{10}\}$, ψ_1 also fixes both V_{10} and $V_{10}^{T_1}$. Note that $N(V_4) \cap N(V_6) \cap D_7 = \{V_3\}$ and $N(V_4) \cap N(V_6) \cap D_8 = \{V_3^{T_2}\}$. This implies that $\psi_1(V_3) = V_3$ and $\psi_1(V_3^{T_2}) = V_3^{T_2}$. By the fact $N(V_5) \cap N(V_{11}) = \{V_2^{T_1}\}$ and $N(V_9) \cap N(V_{11}^{T_2}) \cap D_6 = \{V_{12}\}$, ψ_1 fixes both $V_2^{T_1}$ and V_{12} . Up to now, we know that ψ_1 fixes all vertices in $V(H)$ except V_1 and V_2 . Since the degree of V_1 is 6 and that of V_2 is 8, ψ_1 also fixes both V_1 and V_2 . Therefore ψ_1 is the identity, which implies that H is an asymmetric graph. By Corollary 2.2, we have $\text{Dist}(J) = 2$.

For any $J_1 = J(12, 4)_I$ with $I = \{2, 4\}$, $\text{Aut}(J_1) = \text{Aut}(J)$. Therefore $\text{Dist}(J_1) = \text{Dist}(J) = 2$.

Case 3: $J = J(5, 2)_I$ with $I = \{1\}$ or $\{2\}$.

Since $J(5, 2)_{\{2\}}$ is the Kneser graph $K(5, 2)$ and $J(5, 2)_{\{1\}}$ is its complement, we have $\text{Dist}(J) = 3$ by Proposition 2.3.

Case 4: $k = \frac{n-1}{2}$ and $I \neq k + 1 - I$.

If $n = 5$, then $I = \{1\}$ or $\{2\}$, which means that $I = k + 1 - I$. Hence we may assume that $n \geq 7$. In this case, since $\text{Aut}(J) = \text{Aut}(K(n, \frac{n-1}{2}))$ as an acting group on the vertex set, one can find that $\text{Dist}(J) = \text{Dist}(K(n, k)) = 2$ by Proposition 2.3 and Lemma 2.4.

Case 5: $k = \frac{n-1}{2}$ and $I = k + 1 - I$.

Let $J = J(2m + 1, m)_{\{1,m\}}$ with $m \geq 3$. Let

$$S_2 = \{V_1, V_2, \dots, V_{m+2}\} \cup \{X_2, Y_2\},$$

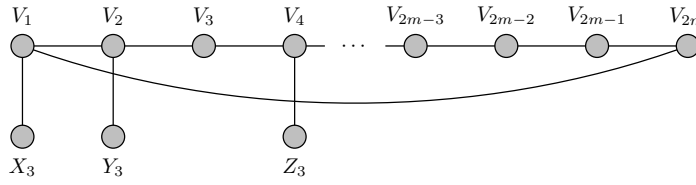


FIGURE 3. An asymmetric subgraph induced by a determining set of $J(2m, m)_{\{1\}}$.

where $X_2 = \{1, 2, \dots, m - 2, m, m + 2\}$ and $Y_2 = \{2, 3, \dots, m - 1, m + 1, m + 3\}$; and let H_2 be the subgraph of J induced by S_2 as depicted in Figure 2. Now S_2 is a determining set of J by Lemma 3.8(1). Let ψ_2 be an automorphism of H_2 . Note that

$$D_1 = \{X_2, Y_2\}, D_2 = \{V_3, V_4, \dots, V_m\}, D_3 = \{V_{m+1}, V_{m+2}\}, D_4 = \{V_1, V_2\}.$$

Since V_1 is the only vertex adjacent to all vertices in D_3 , $\psi_2(V_1) = V_1$. This implies that ψ_2 fixes V_2, X_2 and Y_2 because $D_4 = \{V_1, V_2\}$; and X_2 and Y_2 are only adjacent to V_1 and V_2 , respectively. By the fact that V_{m+2} is the only vertex adjacent to all vertices in D_4 , $\psi_2(V_{m+2}) = V_{m+2}$. This implies that ψ_2 also fixes V_{m+1} because $D_3 = \{V_{m+1}, V_{m+2}\}$. Since $N(V_2) \cap D_2 = \{V_3\}$, $\psi_2(V_3) = V_3$. By a similar way, one can show that ψ_2 fixes all vertices in H_2 , and hence ψ_2 is the identity. This means that H_2 is asymmetric. By Corollary 2.2, we have $\text{Dist}(J) = 2$.

For any $J_2 = J(2m + 1, m)_I$ satisfying $I = k + 1 - I$, $\text{Aut}(J_2) = \text{Aut}(J)$. Therefore $\text{Dist}(J_2) = \text{Dist}(J) = 2$.

Case 6: $k = \frac{n}{2}$ and I is neither $\{k\}$ nor $\{1, 2, \dots, k - 1\}$, and $I' \neq I''$.

Let $J = J(2m, m)_{\{1\}}$ with $m \geq 3$. Let

$$S_3 = \{V_1, V_2, \dots, V_{2m}\} \cup \{X_3, Y_3, Z_3\},$$

where $X_3 = \{1, 2, \dots, m - 2, m, m + 2\}$, $Y_3 = \{2, 3, \dots, m - 1, m + 1, m + 3\}$ and $Z_3 = \{4, 5, \dots, m + 1, m + 3, m + 5\}$; and let H_3 be the subgraph of J induced by S_3 as shown in Figure 3. Now S_3 is a determining set of J by Lemma 3.8(2). Let ψ_3 be an automorphism of H_3 . Note that

$$D_1 = \{X_3, Y_3, Z_3\}, D_3 = \{V_1, V_2, V_4\} \text{ and } D_2 = V(H_3) - (D_1 \cup D_3).$$

By a similar way with cases 2 and 5, one can show that ψ_3 fixes all vertices in H_3 , and hence ψ_3 is the identity. So H_3 is an asymmetric graph. By Corollary 2.2, we have $\text{Dist}(J) = 2$.

For any $J_3 = J(2m, m)_I$ satisfying $I \neq \{k\}$ nor $\{1, 2, \dots, k - 1\}$, and $I' \neq I''$, $\text{Aut}(J_3) = \text{Aut}(J)$. Therefore we have $\text{Dist}(J_3) = \text{Dist}(J) = 2$.

Case 7: $k = \frac{n}{2}$, $I' = I''$ and I is neither $\{k\}$ nor $\{1, 2, \dots, k - 1\}$.

Note that for $n \leq 6$, this case can not occur. Hence assume that $n \geq 8$. Let $J = J(2m, m)_{\{1, m-1\}}$ with $m \geq 4$. By Lemma 3.9, we have $\text{Dist}(J) \geq 3$. Let

$f : V(J) \rightarrow \{1, 2, 3\}$ be a random coloring satisfying that for any $u \in V(J)$, $f(u)$ and $f(\bar{u})$ are distinct. Note that for any $u \in V(J)$,

$$\begin{aligned} Pr(\{f(u), f(\bar{u})\} = \{1, 2\}) &= Pr(\{f(u), f(\bar{u})\} = \{1, 3\}) \\ &= Pr(\{f(u), f(\bar{u})\} = \{2, 3\}) = \frac{1}{3}. \end{aligned}$$

Let Φ be the set of equipartitions of $[n]$. Using a random coloring $f : V(J) \rightarrow \{1, 2, 3\}$, we can define the coloring $\tilde{f} : \Phi \rightarrow \{B, R, Y\}$ as follows: for any $\{u, \bar{u}\} \in \Phi$, let $\tilde{f}(\{u, \bar{u}\}) = B$ if $\{f(u), f(\bar{u})\} = \{1, 2\}$; let $\tilde{f}(\{u, \bar{u}\}) = R$ if $\{f(u), f(\bar{u})\} = \{1, 3\}$; and let $\tilde{f}(\{u, \bar{u}\}) = Y$ if $\{f(u), f(\bar{u})\} = \{2, 3\}$. Then we can consider \tilde{f} as a random 3-coloring of Φ . By Lemma 3.10, there is a 3-coloring $f : V(J) \rightarrow \{1, 2, 3\}$ such that only the identity permutation in S_n preserves all colors of equipartitions in its corresponding 3-coloring $\tilde{f} : \Phi \rightarrow \{B, R, Y\}$. This implies that there exists a 3-distinguishing coloring $f : V(J) \rightarrow \{1, 2, 3\}$. Therefore $\text{Dist}(J) \leq 3$, and hence $\text{Dist}(J) = 3$.

For any $J_4 = J(2m, m)_I$ satisfying $I' = I''$ and I is neither $\{k\}$ nor $\{1, 2, \dots, k-1\}$, $\text{Aut}(J_4) = \text{Aut}(J)$. Therefore $\text{Dist}(J_4) = \text{Dist}(J) = 3$.

Case 8: $k = \frac{n}{2}$ and $I = \{k\}$ or $\{1, 2, \dots, k-1\}$.

Let $J = J(2m, m)_{\{m\}}$. Then J is composed of $\frac{\binom{2m}{m}}{2}$ components which are isomorphic to K_2 . Note that a coloring $f : V(J) \rightarrow \{1, 2, \dots, r\}$ is an r -distinguishing if and only if for any vertex $u \in V(J)$, $f(u)$ and $f(\bar{u})$ are distinct and for any two vertex $v, w \in V(J)$ contained to different components, $\{f(v), f(\bar{v})\} \neq \{f(w), f(\bar{w})\}$. Hence $\text{Dist}(J)$ is the smallest integer r such that $\binom{r}{2} \geq \frac{\binom{2m}{m}}{2}$. Therefore $\text{Dist}(J) = \lceil \frac{1 + \sqrt{1 + 4\binom{2m}{m}}}{2} \rceil$.

For any $J_5 = J(2m, m)_I$ with $I = \{1, 2, \dots, m-1\}$, J_5 is the complement of J . Hence $\text{Dist}(J_5) = \text{Dist}(J) = \lceil \frac{1 + \sqrt{1 + 4\binom{2m}{m}}}{2} \rceil$.

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