Bull. Korean Math. Soc. **52** (2015), No. 2, pp. 377–394
http://dx.doi.org/10.4134/BKMS.2015.52.2.377

BERTRAND CURVES AND RAZZABONI SURFACES IN MINKOWSKI 3-SPACE

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Abstract. In this paper, we generalize some results about Bertrand curves and Razzaboni surfaces in Euclidean 3-space to the case that the ambient space is Minkowski 3-space. Our discussion is divided into three different cases, i.e., the parent Bertrand curve being timelike, spacelike with timelike principal normal, and spacelike with spacelike principal normal. For each case, first we show that Razzaboni surfaces and their mates are related by a reciprocal transformation; then we give Bäcklund transformations for Bertrand curves and for Razzaboni surfaces; finally we prove that the reciprocal and Bäcklund transformations on Razzaboni surfaces commute.

1. Introduction

A Bertrand curve is one of a pair of curves having the same principal normals. It was named after mathematician J. Bertrand, who made the first study in 1850. A surface is termed a Razzaboni surface if it is spanned by a one-parameter family of geodesic Bertrand curves. Much literature is devoted to the study of Bertrand curves and Razzaboni surfaces in Euclidean 3-space. For example, Razzaboni [9] got a Bäcklund transformation for Bertrand curves; Bruke [2] showed that if the binormals of two curves are parallel, then they are both Bertrand curves; Ekmekci and İlarslan [3] got some important characterizations for Bertrand curves; Izumiya and Takeuchi [4] obtained generic properties of Bertrand curves; Schief [11] discussed the integrable nature of Bertrand curves and Razzaboni surfaces in the context of modern soliton theory; Balgetir, Bektas, Ergöt [1] and Külahci, Ergüt [6] focused on the nonnull and AW(k)-type Bertrand curves in Lorentzian 3-space, respectively; Yilmaz and Bektas [12] studied the general properties of Bertrand curves and their characterizations in Riemann-Otsuki space.

In this paper, we discuss Bertrand curves and Razzaboni surfaces in Minkowski 3-space. We focus on three different cases: 1. the parent Bertrand curve

Received April 9, 2013.
2010 *Mathematics Subject Classification.* Primary 53B30; Secondary 53A35.
Key words and phrases. Bertrand curve, Razzaboni surface, Minkowski 3-space, reciprocal transformation, Bäcklund transformation.

is timelike; 2. the parent Bertrand curve is spacelike with timelike principal normal; 3. the parent Bertrand curve is spacelike with spacelike principal normal.

The paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, for the case that the parent Bertrand curve is timelike, we first demonstrate Razzaboni surfaces and their mates are related by a reciprocal transformation which induces an invariance of the Gauss-Mainardi-Codazzi equations, then we give Bäcklund transformations for Bertrand curves and Razzaboni surfaces, and finally we show that the reciprocal and Bäcklund transformation on Razzaboni surfaces commute. For the case that the parent Bertrand curve is spacelike with timelike principal normal or with spacelike principal normal, we have similar results, which are given in Sections 4 and 5, respectively.

2. Preliminaries

In this paper, we are concerned with curves and surfaces in Minkowski 3-space $\mathbb{E}^{2,1}$ with standard metric $\langle , \rangle = dx_1^2 + dx_2^2 - dx_3^2$. Then for two vectors $\mathbf{u} = (u_1, u_2, u_3), \mathbf{v} = (v_1, v_2, v_3) \in \mathbb{E}^{2,1}$, their inner product equals

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2 - u_3 v_3, \tag{2.1}$$

and their vector product is defined as

$$\mathbf{u} \times \mathbf{v} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}, - \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right). \tag{2.2}$$

Definition 2.1 (see [7, 8])**.** A vector $\mathbf{v} \in \mathbb{E}^{2,1}$ is called spacelike if $\langle \mathbf{v}, \mathbf{v} \rangle > 0$ or $\mathbf{v} = 0$, timelike if $\langle \mathbf{v}, \mathbf{v} \rangle < 0$, and lightlike(or null) if $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ and $\mathbf{v} \neq 0$.

By direct calculations, we have the following two propositions.

Proposition 2.2. *Let* $\mathbf{v}, \mathbf{w}$ *be two vectors in* $\mathbb{E}^{2,1}$ *satisfying* $\langle \mathbf{v}, \mathbf{w} \rangle = 0$ *and* $\mathbf{v}, \mathbf{w} \neq 0$. 1. *If* $\mathbf{v}$ *is spacelike, then* $\mathbf{w}$ *may be spacelike, timelike and lightlike*; 2. *If* $\mathbf{v}$ *is timelike, then* $\mathbf{w}$ *must be spacelike*; 3. *If* $\mathbf{v}$ *is lightlike, then* $\mathbf{w}$ *may be spacelike and lightlike.*

Proposition 2.3. *Let* $\mathbf{u}, \mathbf{v} \in \mathbb{E}^{2,1}$ *be two nonzero linearly independent vectors. If both* $\mathbf{u}$ *and* $\mathbf{v}$ *are spacelike* (*resp. timelike, lightlike*) *vectors, then their vector product* $\mathbf{u} \times \mathbf{v}$ *is timelike* (*resp. spacelike, spacelike*).

Let $\mathbf{r} : I \subset \mathbb{R} \to \mathbb{E}^{2,1}$ be a curve C in $\mathbb{E}^{2,1}$ with s as its parameter. Denote by $\mathbf{t} = \mathbf{r}_s$ its tangent vector field and let $\langle \mathbf{t}, \mathbf{t} \rangle = \varepsilon_1$. If $\varepsilon_1 = \pm 1$, then s is called *arc length parameter* and the constant ε_1 is called the *causal character* of C. In the following we will use the arc length parameter.

Definition 2.4 (see [7, 8])**.** A curve C in $\mathbb{E}^{2,1}$ is called spacelike (resp. timelike, lightlike) at s if its tangent vector $\mathbf{r}_s$ is spacelike (resp. timelike, lightlike). The curve C is called spacelike (resp. timelike, lightlike) if it is for any $s \in I$.

For a nonlightlike curve C, assume $\langle \mathbf{t}_s, \mathbf{t}_s \rangle \neq 0$. Then the function $\kappa = \sqrt{|\langle \mathbf{t}_s, \mathbf{t}_s \rangle|}$ is called its *curvature*. Let $\varepsilon_2 = \langle \mathbf{t_s}/\kappa, \mathbf{t_s}/\kappa \rangle$ and call it the *second causal character* of C. Define the *principal normal* vector field $\mathbf{n}$ along C by $\mathbf{t}_s = \kappa \mathbf{n}$. Finally, the *binormal* vector field $\mathbf{b}$ is defined by $\mathbf{b} = -\varepsilon_1\varepsilon_2 \mathbf{t} \times \mathbf{n}$, and $\varepsilon_3 = \langle \mathbf{b}, \mathbf{b} \rangle$ is called the *third causal character* of C. Now we have the well-known Serret-Frenet relations:

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}_s = \begin{pmatrix} 0 & \kappa & 0 \\ -\varepsilon_1\varepsilon_2\kappa & 0 & \tau \\ 0 & -\varepsilon_2\varepsilon_3\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}. \tag{2.3}$$

Note that $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$. When $\varepsilon_1 = -1$, the curve is timelike. When $\varepsilon_1 = 1$, the curve is spacelike with either spacelike principal normal or timelike principal normal. The orthonormal triad $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ satisfies

$$\mathbf{t} = -\varepsilon_2\varepsilon_3 \mathbf{n} \times \mathbf{b}, \qquad \mathbf{n} = -\varepsilon_1\varepsilon_3 \mathbf{b} \times \mathbf{t}, \qquad \mathbf{b} = -\varepsilon_1\varepsilon_2 \mathbf{t} \times \mathbf{n}. \tag{2.4}$$

Definition 2.5 (see [1])**.** A curve $C : I \to \mathbb{E}^{2,1}$ with $\kappa \neq 0$ is called a Bertrand curve if there exists another curve $C^* : I \to \mathbb{E}^{2,1}$ such that at each $s \in I$, C and C^* share the same principal normal lines. In this case C^* is called a Bertrand mate of C.

Sometimes the Bertrand mate C^* of C is also called offset curve.

Example 2.6. All Planar curves in $\mathbb{E}^{2,1}$ are Bertrand curves.

Example 2.7. For any non-zero constants a, b satisfying $a^2 > b^2$, the curves

$$\mathbf{r} = (a\cosh\frac{s}{\sqrt{a^2-b^2}}, \frac{bs}{\sqrt{a^2-b^2}}, a\sinh\frac{s}{\sqrt{a^2-b^2}})$$

and

$$\mathbf{r} = (a\cos\frac{s}{\sqrt{a^2-b^2}}, a\sin\frac{s}{\sqrt{a^2-b^2}}, \frac{bs}{\sqrt{a^2-b^2}})$$

are both Bertrand curves.

For surfaces in $\mathbb{E}^{2,1}$, we have the following definition.

Definition 2.8 (see [5, 8])**.** A surface in $\mathbb{E}^{2,1}$ is called timelike(resp. spacelike) if its normal vector field is spacelike (resp. timelike).

3. Timelike Bertrand curves and timelike Razzaboni surfaces

In this section, we always assume that the parent Bertrand curve is timelike. We discuss Bertrand mates, Razzaboni surfaces, Bäcklund transformations for Bertrand curves and Razzaboni surfaces. We also give a commutativity theorem.

3.1. Bertrand curves and their mates

Let C be a Bertrand curve with position vector $\mathbf{r}$ and C^* its offset curve. Then the position vector $\mathbf{r}^*$ of C^* is given by

$$\mathbf{r}^* = \mathbf{r} + A\mathbf{n}, \tag{3.1}$$

where A is a non-zero constant. Analogous to the classic theorem in Euclidean 3-space, we have:

Theorem 3.1 (see [1, 7])**.** *Let C be a non-planar timelike curve in $\mathbb{E}^{2,1}$. Then C is a Bertrand curve if and only if there exist two constants $A, B(\neq 0)$ such that its curvature κ and torsion τ satisfy a linear relation*

$$-A\kappa + B\tau = 1. \tag{3.2}$$

The orthonormal triad $(\mathbf{t}^*, \mathbf{n}^*, \mathbf{b}^*)$, curvature κ^* and torsion τ^* of the Bertrand mate C^* are uniquely determined by those of its parent Bertrand curve C. In fact, we have:

Theorem 3.2. *Let C be a timelike Bertrand curve with orthonormal triad $(\mathbf{t}, \mathbf{n}, \mathbf{b})$, curvature κ and torsion τ, and C^* be its Bertrand mate. Then the orthonormal triad $(\mathbf{t}^*, \mathbf{n}^*, \mathbf{b}^*)$ along C^* is given by*

$$\mathbf{t}^* = \frac{B\mathbf{t} + A\mathbf{b}}{D}, \quad \mathbf{n}^* = \mathbf{n}, \quad \mathbf{b}^* = \delta\frac{A\mathbf{t} + B\mathbf{b}}{D}, \tag{3.3}$$

where $\delta = 1$ if $B^2 > A^2$, $\delta = -1$ if $B^2 < A^2$, and $D = \sqrt{|B^2 - A^2|}$. The curvature, torsion and arc length of C^* are given by

$$\kappa^* = \frac{B\kappa - A\tau}{\delta(B^2 - A^2)\tau}, \quad \tau^* = \frac{1}{\delta(B^2 - A^2)\tau}, \quad ds^* = D\tau ds. \tag{3.4}$$

Note that the curvature κ^* and torsion τ^* of C^* satisfy

$$A^*\kappa^* + B^*\tau^* = 1, \tag{3.5}$$

where $A^* = \delta A$, $B^* = \delta B$.

Proof. We only prove Theorem 3.2 for the case $B^2 > A^2$. For the other case $B^2 < A^2$, it can be proved similarly. Differentiating with respect to s on both sides of (3.1), and by virtue of (3.2), yields

$$\mathbf{r}^*{}_{s^*}\frac{ds^*}{ds} = B\tau\mathbf{t} + A\tau\mathbf{b}. \tag{3.6}$$

Then we have

$$(\mathbf{r}^*{}_{s^*})^2(\frac{ds^*}{ds})^2 = -(B^2 - A^2)\tau^2. \tag{3.7}$$

So $ds^*/ds = \sqrt{B^2 - A^2}\tau$, and $(\mathbf{r}^*{}_{s^*})^2 = \mathbf{t}^{*2} = -1$. The first equation of (3.3) and the third equation of (3.4) are proved. By virtue of $\mathbf{b}^* = \mathbf{t}^* \times \mathbf{n}^*$ and $\mathbf{n}^* = \mathbf{n}$ we have the third equation of (3.3). From $\kappa^* = \langle \mathbf{t}^*{}_{s^*}, \mathbf{n}^* \rangle$ and $\tau^* = -\langle \mathbf{b}^*{}_{s^*}, \mathbf{n}^* \rangle$ we have the first and second equations of (3.4). □

3.2. Razzaboni surfaces

The following definition is the Minkowski 3-space version of the classic Razzaboni surface in Euclidean 3-space (see [10]).

Definition 3.3. A surface Σ in $\mathbb{E}^{2,1}$ is termed a Razzaboni surface if it is spanned by a one-parameter family of geodesic Bertrand curves with the same parameters A and B.

It is well known that a curve C is a geodesic on a surface Σ in $\mathbb{E}^{2,1}$ if and only if the principal normal $\mathbf{n}$ of C is parallel to the normal $\mathbf{N}$ of Σ. This implies that if a surface Σ is spanned by a one-parameter family of geodesic Bertrand curves $C(t)$ with the same parameters A and B, then the Bertrand mates $C^*(t)$ form a parallel surface Σ^* on which they are likewise geodesics. So we have:

Theorem 3.4. *Any Razzaboni surface Σ in $\mathbb{E}^{2,1}$ with position vector $\mathbf{r}$ admits a parallel (dual) Razzaboni surface Σ^* with position vector $\mathbf{r}^*$ defined by*

$$\mathbf{r}^* = \mathbf{r} + A\mathbf{n}. \tag{3.8}$$

Let the curve C move along its binormal vector $\mathbf{b}$, i.e., $\mathbf{r}_t = g\mathbf{b}$, and Σ be the surface swept out by the moving curve. The first fundamental form of Σ is

$$d\mathbf{r}^2 = -ds^2 + g^2 dt^2. \tag{3.9}$$

Since the principal normal $\mathbf{n}$ of the moving curve C is parallel to the normal $\mathbf{N}$ of the surface Σ, C is a geodesic on Σ. By $\langle \mathbf{r}_s, \mathbf{r}_t \rangle = 0$, the one-parameter family of geodesics and their orthogonal trajectories form coordinate lines on the surface Σ. The variation of the orthonormal triad $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ in s-direction is given by the Serret-Frenet relations (2.3). The t-dependence must be of the general form

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}_t = \begin{pmatrix} 0 & \alpha & \beta \\ \alpha & 0 & \gamma \\ \beta & -\gamma & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}. \tag{3.10}$$

The compatibility of $\mathbf{r}_{st} = \mathbf{r}_{ts}$ yields $\alpha = -\tau g$, $\beta = g_s$. Therefore

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}_t = \begin{pmatrix} 0 & -\tau g & g_s \\ -\tau g & 0 & \gamma \\ g_s & -\gamma & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}. \tag{3.11}$$

$(\mathbf{t}, \mathbf{n}, \mathbf{b})_{st} = (\mathbf{t}, \mathbf{n}, \mathbf{b})_{ts}$ yields that κ, τ, g, γ satisfy

$$g_{ss} = \kappa\gamma + \tau^2 g, \quad \kappa_t = -\tau_s g - 2\tau g_s, \quad \tau_t = \gamma_s - \kappa g_s. \tag{3.12}$$

The systems (3.12) may be regarded as the Gauss-Mainardi-Codazzi equations of the surface Σ. For a given solution of (3.12), the linear systems (2.3), (3.11) is compatible and therefore determine a surface Σ up to its position in Minkowski 3-space $\mathbb{E}^{2,1}$. If, in addition, the constraint (3.2) is imposed, then Σ is guaranteed to be a Razzaboni surface.

The transition from Razzaboni surfaces to their duals induces an invariance of the governing equations (3.12) and (3.2). In fact, we have:

Theorem 3.5. *The nonlinear system* (3.12) *and* (3.2) *are invariant under the reciprocal transformation*

$$
\begin{aligned}
& ds^* = \sqrt{|B^2 - A^2|}\tau ds - \delta \frac{A(B\tau g + g + A\gamma)}{\sqrt{|B^2 - A^2|}} dt, \qquad dt^* = dt, \\
& \kappa^* = \frac{B\kappa - A\tau}{\delta(B^2 - A^2)\tau}, \qquad \tau^* = \frac{1}{\delta(B^2 - A^2)\tau}, \qquad A^* = \delta A, \\
& B^* = \delta B, \qquad g^* = \frac{Bg + AB\gamma + A^2\tau g}{\sqrt{|B^2 - A^2|}}, \\
& \gamma^* = \frac{\delta}{\sqrt{|B^2 - A^2|}}[Ag\tau + B\gamma + \frac{A(B\tau g + g + A\gamma)}{(B^2 - A^2)\tau}],
\end{aligned}
\tag{3.13}
$$

where δ is given in Theorem 3.2.

Proof. We only prove the theorem for the case $B^2 > A^2$. For the other case $B^2 < A^2$, it can be proved similarly. It is readily verified that the differentials ds^* and dt^* defined by the first and second expressions of (3.13) are exact modulo (3.2) and (3.12). This guarantees the existence of the coordinates s^* and t^* and hence the corresponding derivatives read

$$
\partial_{s^*} = \frac{1}{\sqrt{B^2 - A^2}\tau}\partial_s, \quad \partial_{t^*} = \partial_t - \frac{A(B\tau g + g + A\gamma)}{(B^2 - A^2)\tau}\partial_s. \tag{3.14}
$$

Differentiation of the dual position vector (3.8) then shows that

$$
{\mathbf{r}^*}_{s^*} = \mathbf{t}^*, \quad {\mathbf{r}^*}_{t^*} = g^*\mathbf{b}^*, \tag{3.15}
$$

where $\mathbf{t}^*$ and $\mathbf{b}^*$ as given by the first and third expressions of (3.3) constitute the unit tangent and binormal to the Bertrand curves on Σ^*. Accordingly, s^* represents arc length of the Bertrand curves on Σ^* and t^* parametrizes their orthogonal trajectories. The remaining quantity $\gamma^* = \langle {\mathbf{n}^*}_{t^*}, \mathbf{b}^* \rangle$ is readily verified to be the last expression of (3.13). □

In the case $B \neq 0$ in Theorem 3.5, the reciprocal character of the above invariance encoded in $^{**} = id$ is illustrated by the compact relations

$$
\begin{pmatrix} g^* \\ h^* \end{pmatrix} = S \begin{pmatrix} g \\ h \end{pmatrix}, \qquad S^*S = I, \tag{3.16}
$$

where the constant matrix S is given by

$$
S = \delta \frac{\sqrt{|B^2 - A^2|}}{B} \begin{pmatrix} 1 & \frac{AB^2}{B^2 - A^2} \\ \delta \frac{A}{B^2} & 1 \end{pmatrix} \tag{3.17}
$$

and

$$
h = \gamma + \frac{A(Bg\tau + g)}{B^2}. \tag{3.18}
$$

3.3. Bäcklund transformations

Taking $A = a\sinh\sigma$, $B = a\cosh\sigma$, (3.2) becomes

$$-\kappa\sinh\sigma + \tau\cosh\sigma = \frac{1}{a}. \tag{3.19}$$

Then we have:

Theorem 3.6. *Let $C : \mathbf{r} = \mathbf{r}(s)$ be a timelike Bertrand curve and $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ be its orthonormal triad. Then for any non-zero constant λ,*

$$\mathbf{r}' = \mathbf{r} + a\sinh\lambda(\cosh\sigma\sinh\phi\mathbf{t} + \cosh\phi\mathbf{n} + \sinh\sigma\sinh\phi\mathbf{b}) \tag{3.20}$$

is the position vector of another Bertrand curve $C'(\lambda)$, where the function ϕ satisfies the first-order ordinary differential equation

$$\begin{aligned}\phi_s =& \frac{\sinh\lambda}{a(\cosh\sigma + \cosh\lambda)}\cosh\phi - \kappa\cosh\sigma + \tau\sinh\sigma \\ &- \frac{\sinh\sigma}{a(\cosh\sigma + \cosh\lambda)}.\end{aligned} \tag{3.21}$$

The orthonormal triad $(\mathbf{t}', \mathbf{n}', \mathbf{b}')$ of $C'(\lambda)$ is given by

$$\begin{aligned}\mathbf{t}' =& \frac{\cosh\sigma + \cosh\lambda + \sinh\lambda\cosh\phi(\cosh\lambda\sinh\sigma + \sinh\lambda\cosh\sigma\cosh\phi)}{1 + \cosh\lambda\cosh\sigma + \sinh\lambda\sinh\sigma\cosh\phi}\mathbf{t} \\ &+ \frac{\sinh\lambda\sinh\phi(\sinh\lambda\cosh\phi - \sinh\sigma)}{1 + \cosh\lambda\cosh\sigma + \sinh\lambda\sinh\sigma\cosh\phi}\mathbf{n} + \sinh\lambda\cosh\phi\mathbf{b},\end{aligned} \tag{3.22}$$

$$\begin{aligned}\mathbf{n}' =& -\frac{\sinh\lambda\sinh\phi(\sinh\lambda\cosh\sigma\cosh\phi + \cosh\lambda\sinh\sigma)}{1 + \cosh\lambda\cosh\sigma + \sinh\lambda\sinh\sigma\cosh\phi}\mathbf{t} \\ &- \frac{\sinh\lambda\cosh\phi(\sinh\lambda\cosh\phi - \sinh\sigma) - \cosh\lambda(\cosh\sigma + \cosh\lambda)}{1 + \cosh\lambda\cosh\sigma + \sinh\lambda\sinh\sigma\cosh\phi}\mathbf{n} \\ &- \sinh\lambda\sinh\phi\mathbf{b},\end{aligned} \tag{3.23}$$

$$\begin{aligned}\mathbf{b}' =& \frac{\sinh\lambda(\sinh\sigma\sinh\lambda + \cosh\sigma\cosh\lambda\cosh\phi + \cosh\phi)}{1 + \cosh\lambda\cosh\sigma + \sinh\lambda\sinh\sigma\cosh\phi}\mathbf{t} \\ &+ \frac{\sinh\lambda\sinh\phi(\cosh\sigma + \cosh\lambda)}{1 + \cosh\lambda\cosh\sigma + \sinh\lambda\sinh\sigma\cosh\phi}\mathbf{n} + \cosh\lambda\mathbf{b}.\end{aligned} \tag{3.24}$$

So (3.21) *defines a Bäcklund transformation $C \to C'(\lambda)$ on Bertrand curves, which obeys the constant length property, that is, the distance between corresponding points of C and $C'(\lambda)$ only depends on the parameter λ.*

Proof. Differentiation with respect to s on both sides of (3.20) yields

$$\mathbf{r}'_s = f\mathbf{t}', \tag{3.25}$$

where $\mathbf{t}'$ is given by (3.22), and

$$f = \frac{\sinh\lambda\sinh\sigma}{\cosh\sigma + \cosh\lambda}\cosh\phi + \frac{1 + \cosh\lambda\cosh\sigma}{\cosh\sigma + \cosh\lambda}. \tag{3.26}$$

Differentiation with respect to s on both sides of (3.25) gives

$$\mathbf{t}'_s = f\kappa'\mathbf{n}', \tag{3.27}$$

where $\mathbf{n}'$ is given by (3.23), and

$$\kappa' = -\frac{1}{a\sinh\sigma} + \coth\sigma\frac{\tau}{f^2}. \tag{3.28}$$

Note that for $\sigma = 0$, i.e., $\tau = 1/a$, the above expression for κ' is still valid. In fact, in this case, we have

$$\kappa' = -\kappa + \frac{2\sinh\lambda\cosh\phi}{a(1+\cosh\lambda)}. \tag{3.29}$$

From $\mathbf{b}' = \mathbf{t}' \times \mathbf{n}'$, we have (3.24). Then

$$\mathbf{b}'_s = -f\tau'\mathbf{n}', \qquad \tau' = \frac{\tau}{f^2}. \tag{3.30}$$

From (3.26) and the second equation of (3.30) we have

$$-\kappa'\sinh\sigma' + \tau'\cosh\sigma' = \frac{1}{a}. \tag{3.31}$$

So the curves $C'(\lambda)$ indeed constitute a family of Bertrand curves with $a' = a$ and $\sigma' = \sigma$. □

By (3.24),

$$\langle \mathbf{b}, \mathbf{b}' \rangle = \cosh\lambda. \tag{3.32}$$

So we have:

Corollary 3.7. *The angle between the binormals* $\mathbf{b}$ *and* $\mathbf{b}'$ *of* C *and* $C'(\lambda)$ *is constant.*

Now we give a Bäcklund transformation on Razzaboni surfaces.

Theorem 3.8. *Let* $\Sigma : \mathbf{r} = \mathbf{r}(s,t)$ *be a Razzaboni surface parametrized in terms of geodesic coordinates* s, t*. Then for any non-zero constant* λ*, the system on* ϕ *composed of* (3.21) *and*

$$\begin{aligned}\phi_t = {} & \frac{a\sinh\lambda(\cosh\lambda+\cosh\sigma)(\cosh\sigma\gamma+\sinh\sigma g\tau)+\sinh\lambda\sinh\sigma g}{a\cosh\lambda(\cosh\sigma+\cosh\lambda)}\cosh\phi \\ & + g_s\tanh\lambda\sinh\phi + \sinh\sigma\gamma + \cosh\sigma g\tau \\ & + \frac{1+\cosh\lambda\cosh\sigma}{a\cosh\lambda(\cosh\sigma+\cosh\lambda)}g\end{aligned} \tag{3.33}$$

is integrable; *moreover, substituting any solution* ϕ *of the integrable system into* (3.20) *gives another Razzaboni surface* $\Sigma'(\lambda)$.

Proof. Firstly, from (3.12) and (3.19), by a tedious calculation, we have the compatibility condition $\phi_{st} = \phi_{ts}$.

Secondly, it is readily verified that

$$\mathbf{r}'_t = g'\mathbf{b}', \tag{3.34}$$

where

$$g' = \frac{g}{\cosh\lambda} + a\tanh\lambda\cosh\sigma(\sinh\phi g_s + \cosh\sigma\cosh\phi h),$$

and h is given by (3.18).

Let $\mathbf{N}'$ be the normal vector field to the surface Σ'. Then

$$\mathbf{N}' = \mathbf{r}'_s \times \mathbf{r}'_t = (f\mathbf{t}') \times (g'\mathbf{b}') = -fg'\mathbf{n}'.$$

So Σ' is a Razzaboni surface. □

Example 3.9. Let C be the curve

$$\mathbf{r} = (a\sinh\sigma\cos\frac{s}{a}, a\sinh\sigma\sin\frac{s}{a}, s\cosh\sigma), \tag{3.35}$$

where $a(>0), \sigma$ are arbitrary constants, s is its arc parameter. Its unit tangent vector field is given by

$$\mathbf{t} = (-\sinh\sigma\sin\frac{s}{a}, \sinh\sigma\cos\frac{s}{a}, \cosh\sigma). \tag{3.36}$$

Note that $\langle\mathbf{t},\mathbf{t}\rangle = -1$. So C is timelike. Its curvature κ and torsion τ are given by

$$\kappa = \frac{\sinh\sigma}{a}, \quad \tau = \frac{\cosh\sigma}{a}, \tag{3.37}$$

and they satisfy (3.19).

Let C move along its binormal vector $\mathbf{b}$, i.e., $\mathbf{r}_t = g\mathbf{b}$, and let Σ be the corresponding Razzaboni surface. Solving (3.12) gives

$$g = \text{constant}, \quad \gamma = -\frac{\cosh^2\sigma}{a\sinh\sigma}g. \tag{3.38}$$

Substituting (3.37) and (3.38) into (2.3) and (3.11) yields

$$\begin{aligned}\mathbf{t} =& (-\sinh\sigma\sin\theta, \sinh\sigma\cos\theta, \cosh\sigma),\\ \mathbf{n} =& (-\cos\theta, -\sin\theta, 0),\\ \mathbf{b} =& (\cosh\sigma\sin\theta, -\cosh\sigma\cos\theta, -\sinh\sigma),\end{aligned} \tag{3.39}$$

where

$$\theta = \frac{s}{a} - \frac{\cosh\sigma g}{a\sinh\sigma}t.$$

Then by

$$\mathbf{r}_s = \mathbf{t}, \quad \mathbf{r}_t = g\mathbf{b} \tag{3.40}$$

we get the position vector of Σ

$$\mathbf{r} = (a\cosh\sigma\cos\theta, a\cosh\sigma\sin\theta, a\cosh\sigma\theta - \frac{g}{\sinh\sigma}t). \tag{3.41}$$

If the parameter λ satisfies $\sinh^2\sigma > \sinh^2\lambda$, then substituting (3.37) and (3.38) into (3.21) and (3.33) yields

$$\phi = 2\,\mathrm{arctanh}\left(\sqrt{\frac{\sinh\sigma-\sinh\lambda}{\sinh\sigma+\sinh\lambda}}\tanh\frac{\sqrt{\sinh^2\sigma-\sinh^2\lambda}}{2}\varphi\right), \tag{3.42}$$

where

$$\varphi = \frac{\sinh\sigma s-(1+\cosh\sigma\cosh\lambda)gt}{a\sinh\sigma\cosh\lambda(\cosh\sigma+\cosh\lambda)}.$$

Substituting (3.35), (3.39) and (3.42) into (3.20) one get the position vector of another Razzaboni surface Σ'.

If we take $A = a\cosh\sigma$, $B = a\sinh\sigma$, then (3.2) becomes

$$-\kappa\cosh\sigma+\tau\sinh\sigma = \frac{1}{a}. \tag{3.43}$$

Analogous to Theorems 3.6 and 3.8, we have:

Theorem 3.10. *Let $C:\mathbf{r}=\mathbf{r}(s)$ be a timelike Bertrand curve and $(\mathbf{t},\mathbf{n},\mathbf{b})$ be its orthonormal triad. Then for any non-zero constant λ,*

$$\mathbf{r}' = \mathbf{r}+a\cosh\lambda(\sinh\sigma\sin\phi\mathbf{t}+\cos\phi\mathbf{n}+\cosh\sigma\sin\phi\mathbf{b}) \tag{3.44}$$

is the position vector of another Bertrand curve $C'(\lambda)$, where the function ϕ satisfies the first-order ordinary differential equation

$$\begin{aligned}\phi_s =& \frac{-\cosh\lambda}{a(\sinh\sigma+\sinh\lambda)}\cos\phi+\kappa\sinh\sigma-\tau\cosh\sigma\\ &+\frac{\cosh\sigma}{a(\sinh\sigma+\sinh\lambda)}.\end{aligned} \tag{3.45}$$

The orthonormal triad $(\mathbf{t}',\mathbf{n}',\mathbf{b}')$ of $C'(\lambda)$ is given by

$$\begin{aligned}\mathbf{t}' =& \frac{\sinh\sigma+\sinh\lambda-\cosh\lambda\cos\phi(\sinh\lambda\cosh\sigma+\cosh\lambda\sinh\sigma\cos\phi)}{1-\sinh\lambda\sinh\sigma-\cosh\lambda\cosh\sigma\cos\phi}\mathbf{t}\\ &+\frac{\cosh\lambda\sin\phi(\cosh\lambda\cos\phi-\cosh\sigma)}{1-\sinh\lambda\sinh\sigma-\cosh\lambda\cosh\sigma\cos\phi}\mathbf{n}+\cosh\lambda\cos\phi\mathbf{b},\end{aligned} \tag{3.46}$$

$$\begin{aligned}\mathbf{n}' =& -\frac{\cosh\lambda\sin\phi(\cosh\lambda\sinh\sigma\cos\phi+\sinh\lambda\cosh\sigma)}{1-\sinh\lambda\sinh\sigma-\cosh\lambda\cosh\sigma\cos\phi}\mathbf{t}\\ &-\frac{\cosh\lambda\cos\phi(\cosh\lambda\cos\phi-\cosh\sigma)-\sinh\lambda(\sinh\sigma+\sinh\lambda)}{1-\sinh\lambda\sinh\sigma-\cosh\lambda\cosh\sigma\cos\phi}\mathbf{n}\\ &+\cosh\lambda\sin\phi\mathbf{b},\end{aligned} \tag{3.47}$$

$$\begin{aligned}\mathbf{b}' =& \frac{\cosh\lambda(\sinh\sigma\sinh\lambda\cos\phi+\cosh\sigma\cosh\lambda-\cos\phi)}{1-\sinh\lambda\sinh\sigma-\cosh\lambda\cosh\sigma\cos\phi}\mathbf{t}\\ &-\frac{\cosh\lambda\sin\phi(\sinh\sigma+\sinh\lambda)}{1-\sinh\lambda\sinh\sigma-\cosh\lambda\cosh\sigma\cos\phi}\mathbf{n}+\sinh\lambda\mathbf{b}.\end{aligned} \tag{3.48}$$

So (3.45) *defines a Bäcklund transformation* $C \to C'(\lambda)$ *on Bertrand curves, which obeys the constant length property, that is, the distance between corresponding points of* C *and* $C'(\lambda)$ *only depends on the parameter* λ.

Theorem 3.11. *Let* $\Sigma : \mathbf{r} = \mathbf{r}(s,t)$ *be a Razzaboni surface parametrized in terms of geodesic coordinate* s,t. *Then for any nonzero constant* $\lambda(\neq -\sigma)$, *the system on* ϕ *composed of* (3.45) *and*

$$\begin{aligned}\phi_t = &-\frac{a\cosh\lambda(\sinh\lambda+\sinh\sigma)(\sinh\sigma\gamma+\cosh\sigma g\tau)+\cosh\lambda\cosh\sigma g}{a\sinh\lambda(\sinh\sigma+\sinh\lambda)}\cos\phi\\ &+g_s\coth\lambda\sin\phi-\cosh\sigma\gamma-\sinh\sigma g\tau\\ &+\frac{1-\sinh\lambda\sinh\sigma}{a\sinh\lambda(\sinh\sigma+\sinh\lambda)}g\end{aligned}\tag{3.49}$$

is integrable; *moreover, substituting any solution* ϕ *of the integrable system into* (3.44) *gives another Razzaboni surface* $\Sigma'(\lambda)$.

Remark 3.12. As λ approaches $-\sigma$, $|\mathbf{r}'-\mathbf{r}|$ approaches $a\cosh\sigma = A$, which is precisely the distance between the Bertrand curve C and its dual C^*. So $C \mapsto C^*$ may be regarded as a particular Bäcklund transformation.

Remark 3.13. For $A = a\sinh\sigma$, $B = a\cosh\sigma$, the parent Bertrand curve C is timelike. From (3.22), $\langle \mathbf{t}', \mathbf{t}'\rangle = -1$. So all the Bertrand curves $C'(\lambda)$ in Theorem 3.6 are timelike. However, for $A = a\cosh\sigma$, $B = a\sinh\sigma$, the parent Bertrand curve C is timelike, from (3.46)-(3.48) we have $\langle \mathbf{t}', \mathbf{t}'\rangle = 1$, $\langle \mathbf{n}', \mathbf{n}'\rangle = 1$. So all the Bertrand curves $C'(\lambda)$ in Theorem 3.10 are spacelike with spacelike principal normal.

Remark 3.14. Since the Bertrand curve C is timelike, the Razzaboni surfaces Σ in Theorems 3.8 and 3.11 are timelike. From (3.23) and (3.47), $\langle \mathbf{N}', \mathbf{N}'\rangle = \langle \mathbf{n}', \mathbf{n}'\rangle = 1$, i.e., the Razzaboni surfaces $\Sigma'(\lambda)$ in Theorems 3.8 and 3.11 are also timelike. So we have two different Bäcklund transformations on timelike Razzaboni surfaces.

3.4. A commutativity theorem

From a Razzaboni surface Σ, by Theorem 3.4, we have the dual transformation $\mathcal{R} : \Sigma \mapsto \Sigma^*$, and by Theorem 3.8, we have the Bäcklund transformation $\mathcal{B} : \Sigma^* \mapsto \Sigma^{*\prime}$. Similarly, we have $\mathcal{B} : \Sigma \mapsto \Sigma'$ and $\mathcal{R} : \Sigma' \mapsto \Sigma'^*$. In the following, we prove that the two transformations $\mathcal{R}$ and $\mathcal{B}$ commute for the case $A = a\sinh\sigma, B = a\cosh\sigma$.

Let $s, s^*, s', s'^*, s^{*\prime}$ be the arc lengths of the geodesic Bertrand curves on surfaces $\Sigma, \Sigma^*, \Sigma', \Sigma'^*, \Sigma^{*\prime}$ respectively. From the third equation of (3.4) we have

$$s^*_s = a\tau = 1 - a\sinh\sigma\phi_s|_{\lambda=0}. \tag{3.50}$$

So we may take

$$s^* = s - a\sinh\sigma\phi|_{\lambda=0}. \tag{3.51}$$

On the other hand, from (3.25) we have

$$s'_s = f = a\tau + a\sinh\sigma\phi_s. \tag{3.52}$$

So we may take

$$s' = s^* + a\sinh\sigma\phi. \tag{3.53}$$

Let ϕ^* be a solution of the integrable system composed of (3.21) and (3.33) with $(\kappa, \tau, a, \sigma)$ replaced by $(\kappa^*, \tau^*, a^*, \sigma^*)$. By (3.4),

$$\sigma^* = -\sigma,\ a^* = a.$$

Then

$$s^{*\prime} = s^{**} + a^*\sinh\sigma^*\phi^* = s - a\sinh\sigma\phi^*,$$

and the surface $\Sigma^{*\prime}$ is given by

$$\mathbf{r}^{*\prime} = \mathbf{r}^* + a\sinh\lambda(\cosh\sigma\sinh\phi^*\mathbf{t}^* + \cosh\phi^*\mathbf{n}^* - \sinh\sigma\sinh\phi^*\mathbf{b}^*). \tag{3.54}$$

The surface Σ'^* is given by

$$\mathbf{r}'^* = \mathbf{r}' + a'\sinh\sigma'\mathbf{n}' = \mathbf{r}' + a\sinh\sigma\mathbf{n}'. \tag{3.55}$$

Note that

$$\phi^* = 2\,\mathrm{arctanh}\left(\frac{1+\cosh(\lambda-\sigma)}{\cosh\lambda+\cosh\sigma}\tanh\frac{\phi}{2}\right) \tag{3.56}$$

is a solution of the integrable system composed of (3.21) and (3.33) with $(\kappa, \tau, a, \sigma)$ replaced by $(\kappa^*, \tau^*, a^*, \sigma^*)$. Substituting (3.56) into (3.54) yields that the two surfaces $\Sigma^{*\prime}$ and Σ'^* coincide.

In summary, we have proved the following result.

Theorem 3.15. *The transformations $\mathcal{B}$ and $\mathcal{R}$ in Theorems 3.4 and 3.8 commute, i.e.,*

$$\mathcal{B}\circ\mathcal{R} = \mathcal{R}\circ\mathcal{B}, \tag{3.57}$$

where the associated functions ϕ and ϕ^* are related by (3.56).

Remark 3.16. For the case $A = a\cosh\sigma, B = a\sinh\sigma$, in order that the two transformation $\mathcal{B}$ and $\mathcal{R}$ commute, the parameter λ should be purely imaginary. We do not consider this case.

4. Spacelike Bertrand curves with timelike principal normal

In this section, we consider the case that the parent Bertrand curve is spacelike with timelike principal normal. We have similar results as Section 3. In the following, we list the main results but omit all the proofs.

Theorem 4.1. *Let C be a spacelike curve with timelike principal normal with its curvature κ and torsion τ satisfying* (3.2)*, and C^* be its Bertrand mate.*

Then the orthonormal triad $(\mathbf{t}^*, \mathbf{n}^*, \mathbf{b}^*)$*, curvature* κ^**, torsion* τ^* *and arc length* s^* *of* C^* *are given by*

$$\begin{aligned} &\mathbf{t}^* = \frac{B\mathbf{t} + A\mathbf{b}}{D}, \quad \mathbf{n}^* = \mathbf{n}, \quad \mathbf{b}^* = \frac{-A\mathbf{t} + B\mathbf{b}}{D}, \\ &\kappa^* = \frac{B\kappa + A\tau}{(A^2+B^2)\tau}, \quad \tau^* = \frac{1}{(A^2+B^2)\tau}, \quad ds^* = D\tau ds, \end{aligned} \tag{4.1}$$

where $D = \sqrt{A^2+B^2}$.

Note that the curvature κ^* and torsion τ^* of C^* satisfy

$$-A^*\kappa^* + B^*\tau^* = 1, \tag{4.2}$$

where $A^* = -A$, $B^* = B$.

Let the curve C move along its binormal vector $\mathbf{b}$, i.e., $\mathbf{r}_t = g\mathbf{b}$. Then the surface Σ swept out by the moving curve has its first fundamental form

$$d\mathbf{r}^2 = ds^2 + g^2 dt^2. \tag{4.3}$$

The compatibility condition $\mathbf{r}_{st} = \mathbf{r}_{ts}$ yields

$$\alpha = \tau g, \quad \beta = g_s,$$

and $(\mathbf{t}, \mathbf{n}, \mathbf{b})_{st} = (\mathbf{t}, \mathbf{n}, \mathbf{b})_{ts}$ yield that κ, τ, g, γ satisfy the following system:

$$g_{ss} = \kappa\gamma - \tau^2 g, \qquad \kappa_t = \tau_s g + 2\tau g_s, \qquad \tau_t = \gamma_s - \kappa g_s, \tag{4.4}$$

where α, β, γ satisfy (3.10). Then we have:

Theorem 4.2. *The nonlinear system* (4.4) *and* (3.2) *are invariant under the reciprocal transformation*

$$\begin{aligned} &ds^* = \sqrt{A^2+B^2}\tau ds + \frac{A(B\tau g + g + A\gamma)}{\sqrt{A^2+B^2}}dt, \quad dt^* = dt, \\ &\kappa^* = \frac{B\kappa + A\tau}{(A^2+B^2)\tau}, \quad \tau^* = \frac{1}{(A^2+B^2)\tau}, \quad A^* = -A, \\ &B^* = B, \qquad g^* = \frac{Bg + AB\gamma - A^2\tau g}{\sqrt{A^2+B^2}}, \\ &\gamma^* = \frac{1}{\sqrt{A^2+B^2}}[B\gamma - Ag\tau - \frac{A(B\tau g + g + A\gamma)}{(B^2+A^2)\tau}]. \end{aligned} \tag{4.5}$$

Taking $A = a\sin\sigma, B = a\cos\sigma$, then (3.2) reduces to

$$-\kappa\sin\sigma + \tau\cos\sigma = \frac{1}{a}. \tag{4.6}$$

Then we have the following Bäcklund transformation.

Theorem 4.3. *Let* $\Sigma : \mathbf{r} = \mathbf{r}(s,t)$ *be a Razzaboni surface parametrized in terms of geodesic coordinate* s, t*. Then for any non-zero constant* $\lambda(\neq \pi/2 \pm \sigma)$*,*

$$\mathbf{r}' = \mathbf{r} + a\cos\lambda(\cos\sigma\sinh\phi\mathbf{t} + \cosh\phi\mathbf{n} + \sin\sigma\sinh\phi\mathbf{b}) \tag{4.7}$$

gives another Razzaboni surface $\Sigma'(\lambda)$, where the function ϕ is a solution of the integrable system

$$\begin{aligned}\phi_s =& -\frac{\cos\lambda}{a(\cos\sigma+\sin\lambda)}\cosh\phi-\kappa\cos\sigma-\tau\sin\sigma+\frac{\sin\sigma}{a(\cos\sigma+\sin\lambda)},\\ \phi_t =&\frac{a\cos\lambda(\sin\lambda+\cos\sigma)(\sin\sigma g\tau-\cosh\sigma\gamma)+\cos\lambda\sin\sigma g}{a\sin\lambda(\cos\sigma+\sin\lambda)}\cosh\phi\end{aligned} \tag{4.8}$$

$$-g_s\cot\lambda\sinh\phi-\sin\sigma\gamma-\cos\sigma g\tau-\frac{1+\sin\lambda\cos\sigma}{a\sin\lambda(\cos\sigma+\sin\lambda)}g. \tag{4.9}$$

The Bäcklund transformation $\Sigma\mapsto\Sigma'$ obeys the constant length property, i.e., the distance between corresponding points on Σ and Σ' only depends on the Bäcklund parameter λ.

Remark 4.4. As λ approaches $-\pi/2\pm\sigma$, $|\mathbf{r}'-\mathbf{r}|^2$ approaches $-a^2\sin^2\sigma=-A^2$, which is precisely the distance between the parent Bertrand curve C and its mate C^*. So $C\mapsto C^*$ may be regarded as a particular Bäcklund transformation.

For the same $A=a\sin\sigma, B=a\cos\sigma$, if we take alternative parameters, then we can obtain another Bäcklund transformation.

Theorem 4.5. *Let $\Sigma:\mathbf{r}=\mathbf{r}(s,t)$ be a Razzaboni surface parametrized in terms of geodesic coordinate s,t. Then for any constant λ,*

$$\mathbf{r}'=\mathbf{r}+a\sinh\lambda(\cos\sigma\cosh\phi\mathbf{t}+\sinh\phi\mathbf{n}+\sin\sigma\cosh\phi\mathbf{b}) \tag{4.10}$$

gives another Razzaboni surface $\Sigma'(\lambda)$, where the function ϕ is a solution of the integrable system

$$\begin{aligned}\phi_s =& -\frac{\sinh\lambda}{a(\cos\sigma+\cosh\lambda)}\sinh\phi-\kappa\cos\sigma-\tau\sin\sigma\\ &+\frac{\sin\sigma}{a(\cos\sigma+\cosh\lambda)},\end{aligned} \tag{4.11}$$

$$\begin{aligned}\phi_t =&\frac{a\sinh\lambda(\cosh\lambda+\cos\sigma)(\sin\sigma g\tau-\cos\sigma\gamma)+\sinh\lambda\sin\sigma g}{a\cosh\lambda(\cos\sigma+\cosh\lambda)}\sinh\phi\\ &-g_s\tanh\lambda\cosh\phi-\sin\sigma\gamma-\cos\sigma g\tau\\ &-\frac{1+\cosh\lambda\cos\sigma}{a\cosh\lambda(\cos\sigma+\cosh\lambda)}g.\end{aligned} \tag{4.12}$$

The Bäcklund transformation $\Sigma\mapsto\Sigma'$ obeys the constant length property, i.e., the distance between corresponding points on Σ and Σ' only depends on the Bäcklund parameter λ.

Remark 4.6. Since the parent Bertrand curve C is spacelike with timelike principal normal, we have $\langle\mathbf{t}',\mathbf{t}'\rangle=1$, $\langle\mathbf{n}',\mathbf{n}'\rangle=-1$. So all the Bertrand curves $C'(\lambda)$ in Theorems 4.3 and 4.5 are spacelike with timelike principal normal. Therefore we have two different Bäcklund transformations on spacelike Bertrand curves with timelike principal normal.

Remark 4.7. Since the parent Bertrand curve C is spacelike with timelike principal normal, the Razzaboni surfaces Σ in Theorems 5.4 and 5.5 are spacelike. Analogous to Theorems 3.8 and 3.11, $\langle \mathbf{N}', \mathbf{N}' \rangle = \langle \mathbf{n}', \mathbf{n}' \rangle = -1$, i.e., the Razzaboni surfaces $\Sigma'(\lambda)$ in Theorems 5.4 and 5.5 are also spacelike. So we have two different Bäcklund transformations on spacelike Razzaboni surfaces.

Similar to Theorem 3.15, the reciprocal transformation in Theorem 4.2 and Bäcklund transformation in Theorem 4.3 commute, provided the associated functions ϕ and ϕ^* satisfy

$$\phi^* = 2 \operatorname{arctanh} \left(\frac{1 + \sin(\lambda + \sigma)}{\sin \lambda + \cos \sigma} \tanh \frac{\phi}{2} \right). \tag{4.13}$$

5. Spacelike Bertrand curves with spacelike principal normal

In this section, we consider the case that the parent Bertrand curve is spacelike with spacelike principal normal. Just as Section 4, we list the main results but omit all the proofs.

Theorem 5.1. *Let C be a non-planar spacelike curve with spacelike principal normal in $\mathbb{E}^{2,1}$. Then C is a Bertrand curve if and only if there exist constants A, $B(\neq 0)$ such that its curvature κ and torsion τ satisfy a linear relation*

$$A\kappa + B\tau = 1. \tag{5.1}$$

Theorem 5.2. *Let C be a spacelike curve with spacelike principal normal, C^* be its Bertrand mate. Then the orthonormal triad $(\mathbf{t}^*, \mathbf{n}^*, \mathbf{b}^*)$ of C^* is related to that of C by*

$$\mathbf{t}^* = \frac{B\mathbf{t} + A\mathbf{b}}{D}, \quad \mathbf{n}^* = \mathbf{n}, \quad \mathbf{b}^* = \delta \frac{A\mathbf{t} + B\mathbf{b}}{D}, \tag{5.2}$$

where $\delta = 1$ if $B^2 > A^2$, $\delta = -1$ if $B^2 < A^2$, and $D = \sqrt{|B^2 - A^2|}$. The curvature, torsion and arc length of C^* are given by

$$\kappa^* = \frac{B\kappa + A\tau}{\delta(B^2 - A^2)\tau}, \quad \tau^* = \frac{1}{\delta(B^2 - A^2)\tau}, \quad ds^* = D\tau ds. \tag{5.3}$$

Note that the curvature κ^* and torsion τ^* of C^* satisfy

$$A^*\kappa^* + B^*\tau^* = 1, \tag{5.4}$$

where $A^* = -\delta A$, $B^* = \delta B$.

Let the curve C move along its binormal vector $\mathbf{b}$, i.e., $\mathbf{r}_t = g\mathbf{b}$. Then the surface Σ swept out by the moving curve has its first fundamental form

$$d\mathbf{r}^2 = ds^2 - g^2 dt^2. \tag{5.5}$$

The compatibility of $\mathbf{r}_{st} = \mathbf{r}_{ts}$ yields

$$\alpha = \tau g, \quad \beta = g_s,$$

and $(\mathbf{t}, \mathbf{n}, \mathbf{b})_{st} = (\mathbf{t}, \mathbf{n}, \mathbf{b})_{ts}$ yield that κ, τ, g, γ satisfy the system

$$g_{ss} = \kappa\gamma - \tau^2 g, \qquad \kappa_t = \tau_s g + 2\tau g_s, \qquad \tau_t = \gamma_s + \kappa g_s, \tag{5.6}$$

where α, β, γ satisfy (3.10). Then we have:

Theorem 5.3. *The nonlinear system* (5.1) *and* (5.6) *are invariant under the reciprocal transformation*

$$
\begin{aligned}
& ds^* = \sqrt{|B^2 - A^2|}\tau ds - \delta \frac{A(B\tau g + g + A\gamma)}{\sqrt{|B^2 - A^2|}} dt, \qquad dt^* = dt, \\
& \kappa^* = \frac{B\kappa + A\tau}{\delta(B^2 - A^2)\tau}, \qquad \tau^* = \frac{1}{\delta(B^2 - A^2)\tau}, \qquad A^* = -\delta A, \\
& B^* = \delta B, \qquad g^* = \frac{Bg + AB\gamma + A^2\tau g}{\sqrt{|B^2 - A^2|}}, \\
& \gamma^* = -\frac{\delta}{\sqrt{|B^2 - A^2|}}[Ag\tau + B\gamma + \frac{A(B\tau g + g + A\gamma)}{(B^2 - A^2)\tau}],
\end{aligned} \tag{5.7}
$$

where δ is given in Theorem 5.2.

If taking $A = a\sinh\sigma, B = a\cosh\sigma$, then (5.1) reduces to

$$\kappa \sinh\sigma + \tau\cosh\sigma = \frac{1}{a}. \tag{5.8}$$

We have:

Theorem 5.4. *Let $\Sigma : \mathbf{r} = \mathbf{r}(s,t)$ be a Razzaboni surface parametrized in terms of geodesic coordinate s, t. Then for any non-zero constant λ,*

$$\mathbf{r}' = \mathbf{r} + a\sinh\lambda(\cosh\sigma\sin\phi\mathbf{t} + \cos\phi\mathbf{n} + \sinh\sigma\sin\phi\mathbf{b}) \tag{5.9}$$

gives another Razzaboni surfaces $\Sigma'(\lambda)$, where the function ϕ is a solution of the integrable system

$$
\begin{aligned}
\phi_s =& \frac{\sinh\lambda}{a(\cosh\sigma + \cosh\lambda)}\cos\phi + \kappa\cosh\sigma + \tau\sinh\sigma \\
& - \frac{\sinh\sigma}{a(\cosh\sigma + \cosh\lambda)},
\end{aligned} \tag{5.10}
$$

$$
\begin{aligned}
\phi_t =& \frac{a\sinh\lambda(\cosh\lambda + \cosh\sigma)(\cosh\sigma\gamma + \sinh\sigma g\tau) + \sinh\lambda\sin\sigma g}{a\cosh\lambda(\cosh\sigma + \cosh\lambda)}\cos\phi \\
& + g_s\tanh\lambda\sinh\phi + \sinh\sigma\gamma + \cosh\sigma g\tau \\
& + \frac{1 + \cosh\lambda\cosh\sigma}{a\cosh\lambda(\cosh\sigma + \cosh\lambda)} g.
\end{aligned} \tag{5.11}
$$

The Bäcklund transformation $\Sigma \mapsto \Sigma'$ obeys the constant length property, i.e., the distance between corresponding points on Σ and Σ' only depends on the Bäcklund parameter λ.

If taking $A = a\cosh\sigma$, then (5.1) reduces to

$$\kappa\cosh\sigma + \tau\sinh\sigma = \frac{1}{a}. \tag{5.12}$$

We have:

Theorem 5.5. *Let $\Sigma : \mathbf{r} = \mathbf{r}(s,t)$ be a Razzaboni surface parametrized in terms of geodesic coordinate s,t. Then for any constant λ,*

$$\mathbf{r}' = \mathbf{r} + a\cosh\lambda(\sinh\sigma\sinh\phi\mathbf{t} + \cosh\phi\mathbf{n} + \cosh\sigma\sinh\phi\mathbf{b}) \tag{5.13}$$

gives another Razzaboni surface $\Sigma'(\lambda)$, where the function ϕ is a solution of the integrable system

$$\begin{aligned}\phi_s =& -\frac{\cosh\lambda}{a(\sinh\sigma+\sinh\lambda)}\cosh\phi - \kappa\sinh\sigma - \tau\cosh\sigma\\ &+\frac{\cosh\sigma}{a(\sinh\sigma+\sinh\lambda)},\end{aligned} \tag{5.14}$$

$$\begin{aligned}\phi_t =& -\frac{a\cosh\lambda(\sinh\lambda+\sinh\sigma)(\sinh\sigma\gamma+\cosh\sigma g\tau)+\cosh\lambda\cosh\sigma g}{a\sinh\lambda(\sinh\sigma+\sinh\lambda)}\cosh\phi\\ &+g_s\coth\lambda\sinh\phi-\cosh\sigma\gamma-\sinh\sigma g\tau\\ &+\frac{1-\sinh\lambda\sinh\sigma}{a\sinh\lambda(\sinh\sigma+\sinh\lambda)}g.\end{aligned} \tag{5.15}$$

The Bäcklund transformation $\Sigma \mapsto \Sigma'$ obeys the constant length property, i.e., the distance between corresponding points on Σ and Σ' only depends on the Bäcklund parameter λ.

Remark 5.6. For $A = a\sinh\sigma$, $B = a\cosh\sigma$, the parent Bertrand curve C is spacelike with spacelike principal normal. Analogous to Theorems 3.6 and 3.8, one can obtain $\langle \mathbf{t}', \mathbf{t}'\rangle = 1$, $\langle \mathbf{b}', \mathbf{b}'\rangle = -1$. So all the Bertrand curves $C'(\lambda)$ in Theorem 5.4 are spacelike with spacelike principal normal. However, for $A = a\cosh\sigma$, $B = a\sinh\sigma$, the parent Bertrand curve C is spacelike with spacelike principal normal, we have $\langle \mathbf{t}', \mathbf{t}'\rangle = -1$. So all the Bertrand curves $C'(\lambda)$ in Theorem 5.5 are timelike.

Remark 5.7. Since the parent Bertrand curve C is spacelike with spacelike principal normal, the Razzaboni surfaces Σ in Theorems 5.4 and 5.5 are timelike. Analogous to Theorems 3.8 and 3.11, $\langle \mathbf{N}', \mathbf{N}'\rangle = \langle \mathbf{n}', \mathbf{n}'\rangle = 1$, i.e., the Razzaboni surfaces $\Sigma'(\lambda)$ in Theorems 5.4 and 5.5 are also timelike. So we have two different Bäcklund transformations on timelike Razzaboni surfaces.

For the case $A = a\sinh\sigma, B = a\cosh\sigma$, similar to Theorem 3.15, the reciprocal transformation in Theorem 5.3 and Bäcklund transformation in Theorem 5.4 commute, provided the associated functions ϕ and ϕ^* satisfy

$$\phi^* = 2\arctan\left(\frac{1+\cosh(\lambda-\sigma)}{\cosh\lambda+\cosh\sigma}\tan\frac{\phi}{2}\right). \tag{5.16}$$

Acknowledgments. Xu was supported by NSFC Grant 11401104, Fuyang Teachers College Science Foundation FSB201301010 and Anhui Provincial Natural Science Foundation of University KJ2014A196. Zhu was partially supported by NSFC Grants 11471145, 11101352 and Qing Lan Project.

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