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FINITE p-GROUPS WHOSE NON-CENTRAL CYCLIC SUBGROUPS HAVE CYCLIC QUOTIENT GROUPS IN THEIR CENTRALIZERS

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Abstract. In this paper, we classified finite p-groups G such that

$$C_G(x)/\langle x\rangle$$

is cyclic for all non-central elements $x \in G$. This solved a problem proposed By Y. Berkovoch.

1. Introduction

If G is a finite group, then $1 \leq \langle x\rangle \leq C_G(x) \leq G$ for all elements $x \in G$. In particular, G is abelian if and only if $|G : C_G(x)| = 1$ for all $x \in G$. Moreover, K. Ishikawa [5] classified finite p-groups with $|G : C_G(x)| \leq p^2$. Along another line, X. H. Li and J. Q. Zhang [6] classified finite p-groups with $|C_G(x) : \langle x\rangle| \leq p^k$, where $k = 1, 2$ and $p > 2$. Y. Berkovich [1] proposed the following:

Problem 116(ii). Classify the p-groups G such that $C_G(H)/H$ is cyclic for all noncentral cyclic $H < G$.

In other words, Problem 116(ii) requires to classify finite p-groups G such that $C_G(x)/\langle x\rangle$ is cyclic for all non-central elements $x \in G$.

For convenience, the groups in Problem 116(ii) are called $\mathcal{P}$-groups. Let $\mathcal{S} = \{G \mid G \text{ is a } \mathcal{P}\text{-group}\}$. In this paper, $\mathcal{S}$ is determined, and hence the Problem 116(ii) is solved.

2. Preliminaries

Assume G is a finite p-group. Let $r(G) = \max\{\log_p|E| \mid E \leq G$ and E is elementary abelian$\}$ and $r_n(G) = \max\{\log_p|E| \mid E \trianglelefteq G$ and E is elementary

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abelian}. $r(G)$ is called the rank of G and $r_n(G)$ is called the normal rank of G.

Let G be a finite p-group. We use C_{p^m}, $C_{p^m}^n$ and $H * K$ to denote the cyclic group of order p^m, the direct product of n cyclic groups of order p^m, and a central product of H and K, respectively. $M <\cdot\, G$ means M is a maximal subgroup of G.

We use $\mathrm{M}_p(m, n)$ to denote the group

$$\langle a, b \mid a^{p^m} = b^{p^n} = 1, a^b = a^{1+p^{m-1}} \rangle, \text{ where } m \geq 2,$$

and $\mathrm{M}_p(m, n, 1)$ to denote the group

$$\langle a, b, c \mid a^{p^m} = b^{p^n} = c^p = 1, [a, b] = c, [c, a] = [c, b] = 1 \rangle,$$

where $m \geq n$, and $m + n \geq 3$ if $p = 2$. For other notation and terminology the reader is referred to [4].

A non-abelian group G is said to be *minimal non-abelian* if every proper subgroup of G is abelian. A concept which is more general than that of minimal non-abelian p-groups was introduced by Y. Berkovich and Z. Janko in [2]. For a positive integer t, a finite p-group G is said to be an $\mathcal{A}_t$*-group* if all subgroups of index p^t of G are abelian, and at least one subgroup of index p^{t-1} of G is not abelian. Obviously, $\mathcal{A}_1$-groups are exactly the minimal non-abelian p-groups.

Lemma 2.1 ([7, Lemma 2.2])**.** *Suppose that G is a finite nonabelian p-group. Then the following conditions are equivalent.*

(1) *G is minimal nonabelian*;
(2) *$d(G) = 2$ and $|G'| = p$*;
(3) *$d(G) = 2$ and $\Phi(G) = Z(G)$.*

Lemma 2.2 ([1, Proposition 72.1])**.** *Assume G is a metacyclic p-group. Then G is an $\mathcal{A}_t$-group if and only if $|G'| = p^t$.*

Lemma 2.3. *Assume G is a finite p-group and $c(G) = 2$. Then G' is elementary abelian if and only if $G/Z(G)$ is elementary abelian.*

Proof. It follows from $[a^p, b] = 1 \Longleftrightarrow [a, b]^p = 1$ for all $a, b \in G$. □

Lemma 2.4 ([1, Section 1, Exercise 69(a)])**.** *Assume G is a finite p-group. If two distinct maximal subgroups of G are abelian, then $|G'| \leq p$.*

Lemma 2.5 ([1, Theorem 41.1])**.** *If G is a minimal non-metacyclic 2-group, then $|G| \leq 2^5$.*

Lemma 2.6 ([3, Theorem 4.1])**.** *Assume G is a group of order p^n with $p > 2$ and $n \geq 5$. If $r_n(G) = 2$, then G is one of the following groups.*

(1) *G is metacyclic*;
(2) *$G \cong \mathrm{M}_p(1, 1, 1) * C_{p^{n-2}}$*;
(3) *G is a 3-group of maximal class of order $\geq 3^5$*;
(4) *$G = \langle a, x, y \mid a^{p^{n-2}} = 1, x^p = y^p = 1, [a, x] = y, [x, y] = a^{ip^{n-3}}, [y, a] = 1 \rangle$, $i = 1$ or σ, where σ is a fixed square non-residue modulo p.*

Lemma 2.7 ([1, Theorem 9.10])**.** *If a group G of order $p^m > p^3$ has a subgroup M of order p^{m-1} of maximal class, then G is either of maximal class or $G/G' \cong C_p^3$.*

Lemma 2.8 ([1, Section 9, Exercise 1(c)])**.** *Assume G is a group of maximal class and order p^n. If $p > 2$ and $n > 3$, then G has no cyclic normal subgroups of order p^2.*

Lemma 2.9 ([1, Section 9, Exercise 10])**.** *Let G be a 3-group of maximal class. Then the fundamental subgroup G_1 of G is abelian or metacyclic minimal nonabelian.*

3. Some properties of $\mathcal{P}$-groups

Lemma 3.1. *If G is a $\mathcal{P}$-group, then $r(G) \le 2$.*

Proof. If not, then there exists $A \le G$ and $A \cong C_p^3$. If $A \not\le Z(G)$, then there exists $x \in A \setminus Z(G)$. Since A is abelian, $A \le C_G(x)$. Since $A/\langle x\rangle \cong C_p^2$, $C_G(x)/\langle x\rangle$ is not cyclic. This contradicts hypothesis. If $A \le Z(G)$, then $A\langle x\rangle/\langle x\rangle \le C_G(x)/\langle x\rangle$ for all $x \in G \setminus Z(G)$. Since $A\langle x\rangle/\langle x\rangle \cong A/A\cap\langle x\rangle \cong C_p^2$ or C_p^3, $C_G(x)/\langle x\rangle$ is not cyclic. This contradicts hypothesis again. □

Lemma 3.2. *Assume G is a metacyclic nonabelian p-group and $p > 2$. Then G is a $\mathcal{P}$-group if and only if G is minimal nonabelian.*

Proof. $\Longleftarrow$: Let $x \in G \setminus Z(G)$. By Lemma 2.1(3), $Z(G) = \Phi(G)$. Since $\Phi(C_G(x)) \le \Phi(G)$, $x \notin \Phi(C_G(x))$. Since G is metacyclic, $C_G(x)$ is metacyclic. So $d(C_G(x)) \le 2$. It follows that there exists $y \in G$ such that $C_G(x) = \langle x, y\rangle$. Hence $C_G(x)/\langle x\rangle$ is cyclic. That is, G is a $\mathcal{P}$-group.

$\Longrightarrow$: Since $p > 2$ and G is metacyclic, $\Omega_1(G) \cong C_p^2$. Let $G = \langle a, b\rangle$ and $H = \langle a\rangle\Omega_1(G)$, where $\langle a\rangle \lhd G$. Then $H' \le \langle a\rangle \cap \Omega_1(G)$. In particular, $|H'| \le p$. Thus $\langle a^p\rangle\Omega_1(G)$ is abelian. Hence $H \le C_G(a^p)$. Since G is a $\mathcal{P}$-group and $H/\langle a^p\rangle = H/\mho_1(H) \cong C_p^2$, we get $a^p \in Z(G)$. Since $p > 2$ and G is metacyclic, G is regular. Hence $[a, b]^p = 1$ is equivalent to $[a^p, b] = 1$. It follows that $|G'| = p$. By Lemma 2.1(2), G is minimal nonabelian. □

It is easy to see that the argument in Lemma 3.2 is true for ordinary metacyclic 2-groups. Thus we have:

Corollary 3.3. *Assume G is an ordinary metacyclic 2-group. Then G is a $\mathcal{P}$-group if and only if G is minimal nonabelian.*

Lemma 3.4. *Assume G is a $\mathcal{P}$-group and H is a nonabelian subgroup of G. Then H is a $\mathcal{P}$-group.*

Proof. $\forall x \in H \setminus Z(H)$, we have $x \notin Z(G)$. If not, $x \in Z(G) \cap H \le Z(H)$, a contradiction. Thus $x \in G \setminus Z(G)$. Since G is a $\mathcal{P}$-group, $C_G(x)/\langle x\rangle$ is cyclic. Since $C_H(x) \le C_G(x)$, $C_H(x)/\langle x\rangle$ is cyclic. □

4. $\mathcal{P}$-groups of order odd

Theorem 4.1. *Let p be an odd prime. Then G is a $\mathcal{P}$-group if and only if G is one of the following pairwise non-isomorphic groups.*

(1) *metacyclic minimal nonabelian p-groups of order p^n, where $n > 3$;*
(2) $\mathrm{M}_p(1,1,1)$;
(3) $G = \langle a, b, c \mid a^9 = c^3 = 1, b^3 = a^3, [a,b] = c, [c,a] = 1, [c,b] = a^{-3}\rangle$;
(4) $G \cong \mathrm{M}_p(1,1,1) * C_{p^{n-2}}$, *where $n > 2$;*
(5) $G = \langle a, x, y \mid a^{p^{n-2}} = 1, x^p = y^p = 1, [a,x] = y, [x,y] = a^{ip^{n-3}}, [y,a] = 1\rangle$, $i = 1$ *or σ, where σ is a fixed square non-residue modulo p.*

Proof. If $|G| \leq p^4$, then, the conclusion holds by checking the list of groups of order p^3 and p^4. Assume $|G| \geq p^5$. By Lemma 3.1, $r(G) \leq 2$. Thus $r_n(G) \leq 2$. If $r_n(G) = 1$, then G is cyclic, a contradiction. So $r_n(G) = 2$. Thus G is one of the groups listed in Lemma 2.6. We discuss case by case.

If G is the group (1) in Lemma 2.6, then, by Lemma 3.2, G is the group (1).

If G is the group (2) in Lemma 2.6, then $Z(G)$ is a cyclic subgroup of index p^2. Let $g \in G \setminus Z(G)$. Then $C_G(g) < G$. Obviously, $C_G(g) \geq \langle g\rangle Z(G)$. Thus $C_G(g) = \langle g\rangle Z(G)$. Hence $C_G(g)/\langle g\rangle \cong Z(G)/Z(G) \cap \langle g\rangle$ is cyclic. That is, G is a $\mathcal{P}$-group. This is the group (4).

If G is the group (3) in Lemma 2.6, then, by Lemma 2.9, G_1 is abelian or metacyclic non-abelian. Since $|G| \geq 3^5$, $|G_1| \geq 3^4$. It follows that there exists a subgroup $H \trianglelefteq G$ such that $|H| = 3^4$ with $d(H) \geq 2$ and H is abelian or metacyclic non-abelian. By Lemma 3.1, $d(H) = 2$. By Lemma 2.8, $\mho_1(H) \ncong C_9$. Hence $H \cong C_9^2$ or $\mathrm{M}_3(2,2)$. Thus $\mho_1(H) \leq Z(H)$ and $\mho_1(H) \cong C_3^2$. Since G is of maximal class, $|Z(G)| = 3$. So there exists $x \in \mho_1(H) \setminus Z(G)$. Since H is not cyclic, $H/\langle x\rangle$ is not cyclic. It follows from $H \leq C_G(x)$ that $C_G(x)/\langle x\rangle$ is not cyclic. That means G is not a $\mathcal{P}$-group.

If G is the group (4) in Lemma 2.6, then, by calculations, we get $Z(G) = \langle a^p\rangle$ is a subgroup of index p^3 and $\langle a, y\rangle$ is a abelian maximal subgroup. Since $|G'| = p^2$, G has a unique abelian maximal subgroup by Lemma 2.4. Let $g \in G \setminus Z(G)$. Then $\langle a^p, g\rangle \leq C_G(g) < G$. If $C_G(g) = \langle a^p, g\rangle$, then $C_G(g)/\langle g\rangle$ is cyclic. If $C_G(g) > \langle a^p, g\rangle$, then $C_G(g) \lessdot G$. Since $\langle a^p, g\rangle \leq Z(C_G(g))$ and $|C_G(g)/\langle a^p, g\rangle| = p$, $C_G(g)$ is abelian. Thus $C_G(g) = \langle a, y\rangle$ and $g = a^i y^j$, where $(i,p) = 1$ or $(j,p) = 1$. It follows that $C_G(g)/\langle g\rangle$ is cyclic. Hence G is a $\mathcal{P}$-group. This is the group (5). □

5. $\mathcal{P}$-groups of order even

Lemma 5.1. *Assume G is a $\mathcal{P}$-group of order 2^n and $n \geq 5$, M is a maximal subgroup of G. If M is of maximal class, then G is of maximal class.*

Proof. Otherwise, by Lemma 2.7, $G/G' \cong C_p^3$. It follows that

$$\mho_1(G) = G' = M' = \mho_1(M)$$

and hence $\exp(G) = \exp(M)$. Since M is of maximal class, M has a maximal subgroup H which is cyclic by the classification of 2-groups of maximal class. Let $K = \Omega_2(H)$. Then $K \cong C_4$ and K char H char $M \trianglelefteq G$. Hence $K \trianglelefteq G$. By N/C theorem,

$$G/C_G(K) \lesssim \mathrm{Aut}\,(K) \cong C_2.$$

Since M is of maximal class, $|Z(M)| = 2$. Hence $K \nleq Z(M)$. Thus $G/C_G(K) \cong C_2$. Since G is not of maximal class, $C_G(K)$ is not cyclic. Since $n \geq 5$,

$$K \leq \mho_1(M') \leq \Phi(C_G(K)).$$

It follows that $C_G(K)/K$ is not cyclic. This contradicts that G is a $\mathcal{P}$-group. □

Lemma 5.2. *Assume G is a $\mathcal{P}$-group of order 2^n and $n \geq 6$, M is a maximal subgroup of G and $M = \langle a, b, c \mid a^{2^{n-4}} = b^{2^2} = 1, c^2 = a^2b^2, [a,b] = b^2, [c,a] = [c,b] = 1\rangle$. Then*

(1) $Z(M) = \langle a^2, c\rangle = \langle b^2, c\rangle \cong C_2 \times C_{2^{n-4}}$ *and* $\Omega_1(M) = \langle a^{2^{n-5}}, b^2\rangle \cong C_2^2$;
(2) $\Phi(G) = \mho_1(G) \leq Z(M) \leq Z(G)$;
(3) $G' \leq \Omega_1(Z(M)) = \Omega_1(M)$;
(4) $M = \Omega_{n-4}(G)$, *in particular,* $o(x) = 2^{n-3}$ *for all* $x \in G \setminus M$.

Proof. (1) Obviously.

(2) Firstly, we prove $Z(M) \leq Z(G)$. If $a^2 \notin Z(G)$, then $C_G(a^2)/\langle a^2\rangle$ is cyclic since G is a $\mathcal{P}$-group. Since $C_G(a^2) \geq M$, $M/\langle a^2\rangle$ is cyclic. This is a contradiction. Thus $a^2 \in Z(G)$. Similarly, $c \in Z(G)$.

Secondly, we prove $x^2 \in Z(M)$ for all $x \in G$. Since $x^2 \in M$, it suffices to prove $x^2 \in Z(G)$. If not, then, since $C_G(x^2)/\langle x^2\rangle \geq Z(M)\langle x\rangle/\langle x^2\rangle$ and G is a $\mathcal{P}$-group, $Z(M)\langle x\rangle/\langle x^2\rangle$ is cyclic. Since $\langle x^2\rangle \leq \Phi(Z(M)\langle x\rangle)$, $Z(M)\langle x\rangle$ is cyclic. In particular, $Z(M)$ is cyclic. This is a contradiction.

(3) By (2), $G/Z(G)$ is elementary abelian. By Lemma 2.3, G' is elementary abelian. In particular, $G' \leq \Omega_1(Z(M)) = \Omega_1(M)$.

(4) If not, then $\Omega_{n-4}(G) = G$. By (2) and (3), G is 4-abelian. Since $n-4 \geq 2$, $\exp(G) = \exp(\Omega_{n-4}(G)) = 2^{n-4}$. Hence $\exp(\mho_1(G)) = 2^{n-5}$. It follows that

$$\Phi(G) = \mho_1(G) \leq \Omega_{n-5}(Z(M)) = \langle b^2, c^2\rangle = \Phi(M).$$

Thus $d(G) = d(M) + 1 = 4$.

Take $x \in G \setminus M$. Then $\langle a, b, c, x\rangle = G$. Since

$$G' \leq \Omega_1(M) = \langle a^{2^{n-5}}, b^2\rangle,$$

$[a,x] \in \langle a^{2^{n-5}}\rangle$ or $[a,bx] \in \langle a^{2^{n-5}}\rangle$. Without loss of generality, we can assume $[a,x] \in \langle a^{2^{n-5}}\rangle$. If $[a,x] = 1$, then

$$C_G(a)/(\langle a\rangle\Phi(G)) \geq \langle \bar{c}, \bar{x}\rangle \cong C_2^2.$$

Hence $C_G(a)/\langle a\rangle$ is not cyclic. This is a contradiction. So $[a,x] = a^{2^{n-5}}$.

Note that $c \in Z(G)$ and $[a,x] = [a, xc^i]$, where i is an integer. Since

$$x^2 \in \Phi(G) = \langle b^2, c^2\rangle,$$

$(xc^i)^2 = c^2$ or $(xc^i)^2 = c^2b^2 = a^2$ for a suitable i. Without loss of generality, we assume $x^2 = c^2$ or a^2. If $x^2 = c^2$, then $o(xc^{-1}) = 2$. If $x^2 = a^2$, then $o(xa^{-1+2^{n-6}}) = 2$. In either case, there is an involution $y \in G \setminus M$. Hence $\langle y\rangle\Omega_1(M) \cong C_2^3$, This contradicts Lemma 3.1. □

Lemma 5.3. *Assume G and M are the same as Lemma 5.2. Then G is isomorphic to one of the following non-isomorphic groups*:

(1) $\langle a, b, c \mid a^{2^{n-3}} = b^{2^2} = 1, c^2 = a^2b^2, [a, b] = b^2, [c, a] = [c, b] = 1\rangle$;

(2) $\langle a, b, c \mid a^{2^{n-3}} = b^{2^2} = 1, c^2 = a^2b^2, [a, b] = b^2, [c, a] = a^{2^{n-4}}, [c, b] = 1\rangle$.

Proof. By Lemma 5.2(4), $\exp(G) = 2^{n-3}$ and $o(x) = 2^{n-3}$ for all $x \in G \setminus M$. By Lemma 5.2(2), $x^2 \in Z(M) = \langle b^2, c\rangle$. We can assume $x^2 = c$ or $x^2 = cb^2$. By Lemma 5.2(3), $G' \cong C_2$ or $G' \cong C_2^2$.

If $G' \cong C_2$, then $G' = M' = \langle b^2\rangle$. If $[b, x] = 1$ and $[a, x] = 1$, then let $a_1 = ax^{-1}$. Thus

$$a_1^{2^{n-3}} = 1, [a_1, b] = [ax^{-1}, b] = [a, b] = b^2 \text{ and } a_1^2b^2 = x^2.$$

Hence

$$G = \langle a_1, b, x \mid a_1^{2^{n-3}} = b^{2^2} = 1, x^2 = a_1^2b^2, [a_1, b] = b^2, [x, a_1] = [x, b] = 1\rangle.$$

Here G is isomorphic to the group (1). If $[b, x] = 1$ and $[a, x] = b^2$, then let $x_1 = bx$. Thus $[b, x_1] = 1$ and $[a, x_1] = 1$. If $[b, x] = b^2$, then let $x_1 = ax$. Thus $[b, x_1] = 1$. In the two cases, we also get G is isomorphic to the group (1).

If $G' \cong C_2^2$, then $G' = \Omega_1(M)$. We consider the possible cases of $[b, x]$.

Case 1. $[b, x] = 1$

Then $[a, x] = a^{2^{n-5}}$ or $[a, x] = a^{2^{n-5}}b^2$.

If $[a, x] = a^{2^{n-5}}$, then let $a_1 = ax^{-1+2^{n-5}}$. Thus

$$a_1^{2^{n-3}} = 1, x^2 = a_1^2b^2, [a_1, b] = [ax^{-1+2^{n-5}}, b] = [a, b] = b^2,$$
$$[a_1, x] = [ax^{-1+2^{n-5}}, x] = [a, x] = a^{2^{n-5}} = a_1^{2^{n-4}}.$$

Thus $G = \langle a, b, x\rangle = \langle a_1, b, x\rangle$ with defining relations as above. Here G is isomorphic to the group (2).

If $[a, x] = a^{2^{n-5}}b^2$, then let $x_1 = bx$. Thus $[b, x_1] = 1$ and $[a, x_1] = a^{2^{n-5}}$. We get the group (2) as that of $[a, x] = a^{2^{n-5}}$.

Case 2. $[b, x] = b^2$

Let $x_1 = ax$. Then $[b, x_1] = 1$. This is reduced to Case 1.

Case 3. $[b, x] = a^{2^{n-5}}$

If $[a, x] = 1$, then let $a_1 = ax^{-1}$, $b_1 = bx^{2^{n-5}}$ and $x_1 = x^{1+2^{n-5}}$. By calculations, we get

$$o(a_1) = 2^{n-3}, o(b_1) = 2^2, [a_1, b_1] = b^2x^{2^{n-4}} = b_1^2,$$
$$[a_1, x_1] = 1, [b_1, x_1] = x_1^{2^{n-4}}, a_1^2b_1^2 = x_1^2.$$

Thus $G=\langle a,b,x\rangle=\langle a_1,b_1,x_1\rangle$ with defining relations as above. By a simple checking we get G is isomorphic to the group (2).

If $[a,x]=b^2$, then let $a_1=a$ and $x_1=bx$. If $[a,x]=a^{2^{n-5}}$, then let $a_1=ab$ and $x_1=x$. If $[a,x]=a^{2^{n-5}}b^2$, then let $a_1=ab$ and $x_1=bx$. In this three cases, we get $[a_1,x_1]=1$ and $[b,x_1]=a_1^{2^{n-5}}$. This is reduced to the case of $[a,x]=1$.

Case 4. $[b,x]=a^{2^{n-5}}b^2$

Let $x_1=ax$. Then $[b,x_1]=a^{2^{n-5}}$. This is reduced to Case 3. □

Lemma 5.4. *Assume G is a $\mathcal{P}$-group of order 2^n and $n\geq 6$, M is a maximal subgroup of G and $M=\langle a,b,c \mid a^{2^{n-4}}=b^{2^2}=1, c^2=a^2b^2, [a,b]=b^2, [c,a]=a^{2^{n-5}}, [c,b]=1\rangle$. Then $n=6$ and $G\cong\langle a,b,c,d \mid a^4=b^4=1, c^2=a^2b^2, b^2=d^2, [a,b]=b^2, [a,c]=a^2, [a,d]=1, [b,c]=1, [b,d]=a^2, [c,d]=c^2\rangle$.*

Proof. By a similar argument as that of Lemma 5.2, we get

(1) $\Phi(M)=Z(M)=\langle a^2,b^2\rangle\cong C_2\times C_{2^{n-5}}$ and $\Omega_1(M)=M'=\langle a^{2^{n-5}},b^2\rangle\cong C_2^2$;
(2) $\Phi(G)=\Phi(M)=Z(M)\leq Z(G)$, in particular, $d(G)=4$ and $\exp(G)=2^{n-4}$;
(3) $G'=\Omega_1(Z(M))=\Omega_1(M)=M'$;
(4) $G\setminus M$ has no element of order 2.

Noting $[a,M]=M'=G'$, we can take a suitable $d\in G\setminus M$ such that $[a,d]=1$. Then $G=\langle a,b,c,d\rangle$. Assume $d^2=a^{2i}b^{2j}$, where i and j are integers. Replacing d by da^{-i}, we can assume $d^2=b^{2j}$. By (4), $j\neq 0$. Hence $d^2=b^2$.

If $[b,d]\in\langle b^2\rangle$, then $[b,d]=1$ or $[b,ad]=1$. Hence $|C_G(b)/(\langle b\rangle\Phi(G))|\geq 4$. This contradicts that G is a $\mathcal{P}$-group. Thus $[b,d]=a^{2^{n-5}}$ or $a^{2^{n-5}}b^2$. Similarly, $[c,d]=b^2$ or $b^2a^{2^{n-5}}$.

If $n\geq 7$, then $a^{2^{n-6}}\in\mho_1(G)\leq Z(G)$. Hence $(bda^{2^{n-6}})^2=(bd)^2a^{2^{n-5}}=[b,d]a^{2^{n-5}}$. By (4), $[b,d]a^{2^{n-5}}\neq 1$. Thus $[b,d]=b^2a^{2^{n-5}}$. It follows that $(abcd)^2=b^2[c,d]$. By (4), $b^2[c,d]\neq 1$. Thus $[c,d]=b^2a^{2^{n-5}}=[b,d]$. So $[bc,d]=1$. It follows that $|C_G(d)/(\langle d\rangle\Phi(G))|\geq 4$. This contradicts that G is a $\mathcal{P}$-group. Hence $n=6$.

By (4), $1\neq(abd)^2=a^2b^2[b,d]$. Hence $[b,d]=a^2$. By (4) again, $1\neq(bcd)^2=b^2[c,d]$. Hence $[c,d]=b^2a^2=c^2$, and we get the desired group G. □

Lemma 5.5. *Assume G is a $\mathcal{P}$-group of order 2^n and $n\geq 5$, M is a maximal subgroup of G and $M=\langle a,b,c \mid a^{2^{n-3}}=b^2=c^2=1, [c,b]=a^{2^{n-4}}, [b,a]=[c,a]=1\rangle\cong D_8*C_{2^{n-3}}$. Then $G\cong D_8*C_{2^{n-2}}$.*

Proof. By a similar and more simple argument as that of Lemma 5.2, we get

(1) $Z(M)=\langle a\rangle\cong C_{2^{n-3}}$ and $\Omega_1(Z(M))=\langle a^{2^{n-4}}\rangle$;
(2) $\Phi(G)=\mho_1(G)\leq Z(M)\leq Z(G)$, in particular, $\exp(G)\leq 2^{n-2}$;

(3) $G' = \Omega_1(Z(M)) = M'$;

(4) $M = \Omega_{n-3}(G)$, in particular, $o(x) = 2^{n-2}$ for all $x \in G \setminus M$.

By (4), we can assume $x^2 = a$. We consider $[b, x]$ and $[c, x]$. If $[b, x] = 1$ and $[c, x] = 1$, then $G = \langle b, c\rangle * \langle x\rangle \cong D_8 * C_{2^{n-2}}$. If $[b, x] = 1$ and $[c, x] = a^{2^{n-4}}$, then, by letting $x_1 = bx$, we get $[b, x_1] = 1$ and $[c, x_1] = [c, bx] = 1$. Thus $G = \langle b, c\rangle * \langle x_1\rangle \cong D_8 * C_{2^{n-2}}$. If $[b, x] = a^{2^{n-4}}$, then, by letting $x_1 = cx$, we get $[b, x_1] = 1$. This is reduced to that of $[b, x] = 1$. □

Lemma 5.6. *Assume G is a $\mathcal{P}$-group of order 2^6. Then G has no subgroup $M \cong \langle a, b, c \mid a^4 = c^4 = 1, a^2 = b^2, [a, b] = a^2, [c, a] = c^2, [c, b] = 1\rangle$.*

Proof. Otherwise, by a similar argument as that of Lemma 5.2, we get

(1) $\Phi(M) = Z(M) = \Omega_1(M) = M' = \langle a^2, c^2\rangle \cong C_2^2$;

(2) $\Phi(G) = \Phi(M) = Z(M) \le Z(G)$, in particular, $d(G) = 4$ and $\exp(G) = 4$;

(3) $G' = \Omega_1(Z(M)) = \Omega_1(M) = M'$;

(4) $G \setminus M$ has no element of order 2.

Notice that $[a, M] = M' = G'$. We can take a suitable $x \in G \setminus M$ such that $[a, x] = 1$. Without loss of generality, we can assume $x^2 = c^2$. By a similar argument as that of Lemma 5.4, we have $[b, x] = c^2$ or a^2c^2 and $[c, x] = a^2$ or a^2c^2.

By (4), $1 \ne (abx)^2 = a^2c^2[b, x]$ and $1 \ne (acx)^2 = a^2c^2[c, x]$. Hence $[b, x] = c^2$ and $[c, x] = a^2$. It follows that $(abcx)^2 = 1$. This contradicts (4). □

Lemma 5.7. *Assume G is a $\mathcal{P}$-group of order 2^7. Then G has no subgroup $M \cong \langle a, b, c, d \mid a^4 = b^4 = 1, c^2 = a^2b^2, b^2 = d^2, [a, b] = b^2, [a, c] = a^2, [a, d] = 1, [b, c] = 1, [b, d] = a^2, [c, d] = c^2\rangle$.*

Proof. Otherwise, by a similar argument as that of Lemma 5.2, we get

(1) $\Phi(M) = Z(M) = \Omega_1(M) = M' = \langle a^2, b^2\rangle \cong C_2^2$;

(2) $\Phi(G) = \Phi(M) = Z(M) \le Z(G)$, in particular, $d(G) = 5$ and $\exp(G) = 4$;

(3) $G' = \Omega_1(Z(M)) = \Omega_1(M) = M'$;

Notice that $[a, M] = M' = G'$. We can take a suitable $x \in G \setminus M$ such that $[a, x] = 1$. Hence $C_G(a)/(\langle a\rangle\Phi(G)) \ge \langle \bar{d}, \bar{x}\rangle \cong C_2^2$. This contradicts that G is a $\mathcal{P}$-group. □

Theorem 5.8. *Assume G is a group of order 2^n. Then G is a $\mathcal{P}$-group if and only if G is one of the following pairwise non-isomorphic groups.*

(1) *metacyclic minimal nonabelian p-groups*;

(2) 2-*groups of maximal class*;

(3) $D_8 * C_{2^{n-2}}$;

(4) $\langle a, b, c \mid a^{2^{n-3}} = b^{2^2} = 1, c^2 = a^2b^2, [a, b] = b^2, [c, a] = [c, b] = 1\rangle$;

(5) $\langle a, b, c \mid a^{2^{n-3}} = b^{2^2} = 1, c^2 = a^2b^2, [a, b] = b^2, [c, a] = a^{2^{n-4}}, [c, b] = 1\rangle$;

(6) $Q_8 \times C_2$;

(7) $\langle a, b, c \mid a^4 = c^4 = 1, a^2 = b^2, [a, b] = a^2, [c, a] = c^2, [c, b] = 1\rangle$;

(8) $\langle a,b,c,d \mid a^4=b^4=1, c^2=a^2b^2, b^2=d^2, [a,b]=b^2, [a,c]=a^2, [a,d]=1, [b,c]=1, [b,d]=a^2, [c,d]=c^2\rangle$.

Proof. If $n\le 5$, then, by classification of 2-groups of order $\le 2^5$, the conclusion holds. In following assume $n\ge 6$ and G is a $\mathcal{P}$-group.

By induction hypothesis, each maximal subgroup of G is abelian or isomorphic to one of the groups (1)–(5), (7) and (8). If G has a maximal subgroup which is isomorphic to one of the groups (2)–(5), (7) and (8), then G is isomorphic to one of the groups (2)–(5) and (8) by Lemma 5.1, Lemmas 5.3, 5.4, 5.5, 5.6 and 5.7.

Assume every maximal subgroup of G is abelian or metacyclic minimal non-abelian. By Lemma 3.1, every maximal subgroup of G is metacyclic. If G is not metacylcic, then G is minimal non-metacyclic. It follows that $|G|\le 2^5$ by Lemma 2.5. This contradicts $|G|\ge 2^6$. Thus G is metacyclic.

If G is minimal non-abelian, then we get the group (1).

If G is not minimal nonabelian, then G is a metacyclic $\mathcal{A}_2$-group. By Lemma 2.2, $|G'|=4$. Assume $G=\langle a,b\rangle$, where $G'<\langle a\rangle$. Then $o(a)\ge 8$ and $a^t\in Z(G)$ if and only if $4|t$. Hence $a^2\notin Z(G)$. Since $|G|\ge 2^6$ and $|G'|=4$, G has no cyclic maximal subgroup. It follows that $C_G(a^2)=\langle a,b^2\rangle$ is not cyclic. Notice that $\langle a^2\rangle\le \mho_1(C_G(a^2))$. $C_G(a^2)/\langle a^2\rangle$ is not cyclic. This contradicts G is a $\mathcal{P}$-group.

It is easy to see that those groups in the theorem are pairwise non-isomorphic. In following we prove those groups in the theorem are $\mathcal{P}$-groups.

If G is the group (1), then G is a $\mathcal{P}$-group by Corollary 3.3.

If G is the group (2), then G has a cyclic subgroup of index 2 and G is metacyclic by the classification of 2-groups of maximal class. Let $\langle a\rangle$ be a cyclic subgroup of index 2 of G. Then $\Phi(G)=\langle a^2\rangle$ and $Z(G)=\langle a^{2^{n-2}}\rangle$. Let $x\in G\setminus Z(G)$. If $x\notin\Phi(G)$, then $x\notin\Phi(C_G(x))$. Since G is metacyclic, $C_G(x)$ is metacyclic. Hence $d(C_G(x))\le 2$. Thus there exists $y\in G$ such that $C_G(x)=\langle x,y\rangle$. It follows that $C_G(x)/\langle x\rangle=\langle\bar{y}\rangle$ is cyclic. If $x\in\Phi(G)\setminus Z(G)$, then $C_G(x)=\langle a\rangle$. Obviously, $C_G(x)/\langle x\rangle$ is cyclic. So G is a $\mathcal{P}$-group.

If G is one of the groups (3)–(7), then $|G:Z(G)|\le 8$. It follows that $|G:\langle x,Z(G)\rangle|\le 4$ for all $x\in G\setminus Z(G)$. Notice that $\langle x,Z(G)\rangle\le Z(C_G(x))$ and $C_G(x)<G$. We have $|C_G(x)/Z(C_G(x))|\le 2$. Hence $C_G(x)$ is abelian. It is easy to check $r(G)=2$. Hence $d(C_G(x))\le 2$. Thus there exists $y\in G$ such that $C_G(x)=\langle x,y\rangle$. It follows that $C_G(x)/\langle x\rangle=\langle\bar{y}\rangle$ is cyclic.

If G is the group (8), then $Z(G)=\Phi(G)=\Omega_1(G)\cong C_2^2$. It is easy to check $Z(M)=Z(G)$ for all subgroups M of order 32. Since $Z(C_G(x))\ge\langle x,Z(G)\rangle$ for all $x\in G\setminus M$, $Z(C_G(x))>Z(G)$. Thus $|C_G(x)|\le 16$. It follows that $C_G(x)$ is abelian and hence $d(C_G(x))\le 2$. Thus there exists $y\in G$ such that $C_G(x)=\langle x,y\rangle$. It follows that $C_G(x)/\langle x\rangle=\langle\bar{y}\rangle$ is cyclic. □

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