Bull. Korean Math. Soc. **52** (2015), No. 2, pp. 351–361
http://dx.doi.org/10.4134/BKMS.2015.52.2.351

LIE IDEALS IN TRIDIAGONAL ALGEBRA ALG$\mathcal{L}_\infty$

Joo Ho Kang

Abstract. We give examples of Lie ideals in a tridiagonal algebra Alg$\mathcal{L}_\infty$ and study some properties of Lie ideals in Alg$\mathcal{L}_\infty$. We also investigate relationships between Lie ideals in Alg$\mathcal{L}_\infty$. Let k be a fixed natural number. Let $\mathcal{A}$ be a linear manifold in Alg$\mathcal{L}_\infty$ such that $T_{(2k-1,2k)} = 0$ for all $T \in \mathcal{A}$. Then $\mathcal{A}$ is a Lie ideal if and only if $T_{(2k-1,2k-1)} = T_{(2k,2k)}$ for all $T \in \mathcal{A}$.

1. Introduction

Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space with a fixed orthonormal base $\{e_1, e_2, \ldots\}$ and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded operators on $\mathcal{H}$. If $x_1, x_2, \ldots, x_k$ are vectors in $\mathcal{H}$, we denote by $[x_1, x_2, \ldots, x_k]$ the closed subspace spanned by the vectors $x_1, x_2, \ldots, x_k$. A subspace lattice $\mathcal{L}$ is a strongly closed lattice of orthogonal projections acting on $\mathcal{H}$. We denote by $\mathcal{L}_\infty$ the subspace lattice generated by the subspaces $[e_1]$, $[e_3], \ldots,$ $[e_{2n-1}], \ldots, [e_1, e_2, e_3]$, $[e_3, e_4, e_5], \ldots,$ $[e_{2n-3}, e_{2n-2}, e_{2n-1}], \ldots$. By Alg$\mathcal{L}_\infty$, we mean the algebra of bounded operators which leave invariant all of the subspaces in $\mathcal{L}_\infty$. It is easy to see that all such operators have the matrix form

$$\begin{pmatrix} * & * & & & & & \\ & * & & & & & \\ & * & * & * & & & \\ & & & * & & & \\ & & & * & * & * & \\ & & & & & & \ddots \\ & & & & & & \end{pmatrix}$$

where all non-starred entries are zero.

The algebra Alg$\mathcal{L}_\infty$ becomes a Lie algebra under the Lie product

$$[A, B] = AB - BA.$$

Received March 20, 2013; Revised April 30, 2014.
2010 *Mathematics Subject Classification.* 47L35.
Key words and phrases. linear manifold, Lie ideal, tridiagonal algebras.
This paper is supported by Daegu University Grant(2011).

Let $\mathcal{A}$ be a subalgebra of Alg$\mathcal{L}_\infty$. We say that $\mathcal{A}$ is *a left ideal* of Alg$\mathcal{L}_\infty$ if $AT \in \mathcal{A}$ for all A in Alg$\mathcal{L}_\infty$ and T in $\mathcal{A}$. $\mathcal{A}$ is called *a right ideal* of Alg$\mathcal{L}_\infty$ if $TA \in \mathcal{A}$ for all A in Alg$\mathcal{L}_\infty$ and T in $\mathcal{A}$. $\mathcal{A}$ is said to be *an ideal* of Alg$\mathcal{L}_\infty$ if $\mathcal{A}$ is a left ideal of Alg$\mathcal{L}_\infty$ and a right ideal of Alg$\mathcal{L}_\infty$. A linear manifold $\mathcal{A}$ in Alg$\mathcal{L}_\infty$ is called *a Lie ideal* in Alg$\mathcal{L}_\infty$ if $[A, X] \in \mathcal{A}$ for A in Alg$\mathcal{L}_\infty$ and $X \in \mathcal{A}$. In this paper, let I be the identity operator on $\mathcal{H}$. Let $\mathbb{C}$ be the set of all complex numbers and $\mathbb{N} = \{1, 2, \ldots\}$.

2. Examples of Lie ideals in Alg$\mathcal{L}_\infty$

If we know the following facts, then we can easily prove the following examples of Lie ideals in Alg$\mathcal{L}_\infty$.

Let $A = (a_{ij})$ and $T = (t_{ij})$ be operators in Alg$\mathcal{L}_\infty$. Then

(α) the (k, k)-entry of $AT - TA$ is 0 for all $k = 1, 2, \ldots$.

(β) the $(2k-1, 2k)$-entry of $AT - TA$ is $a_{2k-1\ 2k}(t_{2k\ 2k} - t_{2k-1\ 2k-1}) + t_{2k-1\ 2k}(a_{2k-1\ 2k-1} - a_{2k\ 2k})$ for all $k = 1, 2, \ldots$.

(γ) the $(2k+1, 2k)$-entry of $AT - TA$ is $a_{2k+1\ 2k}(t_{2k\ 2k} - t_{2k+1\ 2k+1}) + t_{2k+1\ 2k}(a_{2k+1\ 2k+1} - a_{2k\ 2k})$ for all $k = 1, 2, \ldots$.

We denote $T_{(i,j)}$ or $t_{i\ j}$ by the (i, j)-component of an operator T in Alg$\mathcal{L}_\infty$ and use the following notations in this paper:

Let n and l be fixed natural numbers ($n > 1$, $l > 1$). Let $\Gamma = \{k_1, k_2, \ldots, k_n, j_1, j_2, \ldots, j_l\}$, $\Gamma_n = \{k_1, k_2, \ldots, k_n\}$ and $\Upsilon_l = \{j_1, j_2, \ldots, j_l\}$ be finite subsets of $\mathbb{N}$. Let $\Omega = \{k_1, k_2, \ldots, j_1, j_2, \ldots\}$, $\Omega_1 = \{k_1, k_2, \ldots\}$ and $\Omega_2 = \{j_1, j_2, \ldots\}$ be infinite subsets of $\mathbb{N}$.

Example 1. Let $\mathcal{A}_0 = \{T \in \text{Alg}\mathcal{L}_\infty \,|\, T_{(k,k)} = 0,\ k \in \mathbb{N}\}$. Then $\mathcal{A}_0$ is a Lie ideal.

Example 2. Let I be the identity operator on $\mathcal{H}$ and let $\mathcal{A}_1 = \{\alpha I + T \,|\, \alpha \in \mathbb{C}, T \in \mathcal{A}_0\}$. Then $\mathcal{A}_1$ is a Lie ideal.

It is easy to show that an intersection of Lie ideals in Alg$\mathcal{L}_\infty$ is a Lie ideal in Alg$\mathcal{L}_\infty$.

Example 3. Let k and j be fixed natural numbers.

1) $\mathcal{A}_{0,2k-1} = \{T \in \mathcal{A}_0 \,|\, T_{(2k-1,2k)} = 0\}$. Then $\mathcal{A}_{0,2k-1}$ is a Lie ideal.

2) $\mathcal{A}_{0,2j+1} = \{T \in \mathcal{A}_0 \,|\, T_{(2j+1,2j)} = 0\}$. Then $\mathcal{A}_{0,2j+1}$ is a Lie ideal.

We denote Lie ideals that are obtained by intersections of Lie ideals $\mathcal{A}_{0,2k_i-1}$ and $\mathcal{A}_{0,2j_p+1}$ as follows:

$$\mathcal{A}_{0,\Gamma_n} = \cap_{i=1}^{n}\mathcal{A}_{0,2k_i-1},\ \mathcal{A}_{0,\Upsilon_l} = \cap_{p=1}^{l}\mathcal{A}_{0,2j_p+1},\ \mathcal{A}_{0,\Omega_1} = \cap_{i=1}^{\infty}\mathcal{A}_{0,2k_i-1},$$
$$\mathcal{A}_{0,\Omega_2} = \cap_{p=1}^{\infty}\mathcal{A}_{0,2j_p+1},\ \mathcal{A}_{0,\Gamma} = \mathcal{A}_{0,\Gamma_n} \cap \mathcal{A}_{0,\Upsilon_l},\ \text{ and } \mathcal{A}_{0,\Omega} = \mathcal{A}_{0,\Omega_1} \cap \mathcal{A}_{0,\Omega_2}.$$

Example 4. Let k and j be fixed natural numbers.

1) $\mathcal{A}_{1,2k-1} = \{T \in \mathcal{A}_1 \,|\, T_{(2k-1,2k)} = 0\}$. Then $\mathcal{A}_{1,2k-1}$ is a Lie ideal.

2) $\mathcal{A}_{1,2j+1} = \{T \in \mathcal{A}_1 \,|\, T_{(2j+1,2j)} = 0\}$. Then $\mathcal{A}_{1,2j+1}$ is a Lie ideal.

We denote Lie ideals that are obtained by intersections of Lie ideals $\mathcal{A}_{1,2k_i-1}$ and $\mathcal{A}_{1,2j_p+1}$ as follows:

$$\mathcal{A}_{1,\Gamma_n} = \cap_{i=1}^{n}\mathcal{A}_{1,2k_i-1},\ \mathcal{A}_{1,\Upsilon_l} = \cap_{p=1}^{l}\mathcal{A}_{1,2j_p+1},\ \mathcal{A}_{1,\Omega_1} = \cap_{i=1}^{\infty}\mathcal{A}_{1,2k_i-1},$$
$$\mathcal{A}_{1,\Omega_2} = \cap_{p=1}^{\infty}\mathcal{A}_{1,2j_p+1},\ \mathcal{A}_{1,\Gamma} = \mathcal{A}_{1,\Gamma_n} \cap \mathcal{A}_{1,\Upsilon_l},\ \text{ and } \mathcal{A}_{1,\Omega} = \mathcal{A}_{1,\Omega_1} \cap \mathcal{A}_{1,\Omega_2}.$$

Example 5. Let k and j be fixed natural numbers.

1) Let $\mathcal{A}_{2,2k-1} = \{T \in \mathrm{Alg}\mathcal{L}_\infty \mid T_{(2k-1,2k-1)} = T_{(2k,2k)}$ and $T_{(2k-1,2k)} = 0\}$. Then $\mathcal{A}_{2,2k-1}$ is a Lie ideal.
2) Let $\mathcal{A}_{2,2j+1} = \{T \in \mathrm{Alg}\mathcal{L}_\infty \mid T_{(2j,2j)} = T_{(2j+1,2j+1)}$ and $T_{(2j+1,2j)} = 0\}$. Then $\mathcal{A}_{2,2j+1}$ is a Lie ideal.

We denote $\mathcal{A}_{2,2k-1} \cap \mathcal{A}_{2,2j+1}$ by $\mathcal{A}_{2,2k-1,2j+1}$.

Proof. 1) Let A be an operator in $\mathrm{Alg}\mathcal{A}_\infty$ and T be an operator in $\mathcal{A}_{2,2k-1}$. Since the $(2k-1,2k)$-entry of AT and the $(2k-1,2k)$-entry of TA are

$$a_{2k-1\ 2k-1}t_{2k-1\ 2k} + a_{2k-1\ 2k}t_{2k\ 2k} = a_{2k-1\ 2k}t_{2k\ 2k} \text{ and}$$
$$t_{2k-1\ 2k-1}a_{2k-1\ 2k} + t_{2k-1\ 2k}a_{2k\ 2k} = t_{2k-1\ 2k-1}a_{2k-1\ 2k}.$$

So AT and TA are not in $\mathcal{A}_{2,2k-1}$. Hence $\mathcal{A}_{2,2k-1}$ is not a left ideal and not a right ideal. Since $T_{(2k-1,2k-1)} = T_{(2k,2k)}$, the $(2k-1,2k)$-component of $AT - TA$ is zero. Hence $AT - TA$ is in $\mathcal{A}_{2,2k-1}$. So $\mathcal{A}_{2,2k-1}$ is a Lie ideal.

2) Let A be an operator in $\mathrm{Alg}\mathcal{A}_\infty$ and T be an operator in $\mathcal{A}_{2,2j+1}$. Since the $(2j+1,2j)$-entry of AT and the $(2j+1,2j)$-entry of TA are

$$a_{2j+1\ 2j}t_{2j\ 2j} + a_{2j+1\ 2j+1}t_{2j+1\ 2j} = a_{2j+1\ 2j}t_{2j\ 2j} \text{ and}$$
$$t_{2j+1\ 2j}a_{2j\ 2j} + t_{2j+1\ 2j+1}a_{2j+1\ 2j} = t_{2j+1\ 2j+1}a_{2j+1\ 2j},$$

AT and TA are not in $\mathcal{A}_{2,2j+1}$. Hence $\mathcal{A}_{2,2j+1}$ is not a left ideal and not a right ideal. Since $T_{(2j+1,2j+1)} = T_{(2j,2j)}$, the $(2j+1,2j)$-component of $AT - TA$ is zero. Hence $AT - TA$ is in $\mathcal{A}_{2,2j+1}$. So $\mathcal{A}_{2,2j+1}$ is a Lie ideal. □

Example 6. Let k, j and l be natural numbers.

1) Let $\mathcal{B}_{k,l} = \{T \in \mathrm{Alg}\mathcal{L}_\infty \mid T_{(k,k)} = T_{(k+1,k+1)} = \cdots = T_{(k+l,k+l)}\}$. Then $\mathcal{B}_{k,l}$ is a Lie ideal.
2) Let $\mathcal{B}_{k,\infty} = \{T \in \mathrm{Alg}\mathcal{L}_\infty \mid T_{(k,k)} = T_{(k+1,k+1)} = \cdots\}$. Then $\mathcal{B}_{k,\infty}$ is a Lie ideal and $\mathcal{B}_{1,\infty} = \mathcal{A}_1$.
3) Let $\wedge = \{(k_i,l_i) \mid k_i + l_i < k_{i+1},\ i = 1,2,\ldots,n\}$. Then we denote $\cap_{i=1}^{n}\mathcal{B}_{k_i,l_i}$ by $\mathcal{B}_\wedge$ and $\mathcal{B}_\wedge \cap \mathcal{B}_{k,\infty}$ by $\mathcal{B}_{\wedge,k,\infty}$, where $k_n + l_n < k$.

Example 7. Let k and n be natural numbers ($n > 1$).

1) Let $\mathcal{A}_{n,2k-1} = \{T \in \mathcal{B}_{2k-1,n-1} \mid T_{(2k-1,2k)} = 0\}$. Then $\mathcal{A}_{n,2k-1}$ is a Lie ideal and $\mathcal{A}_{2,2k-1} = \{T \in \mathcal{B}_{2k-1,1} \mid T_{(2k-1,2k)} = 0\}$.
2) Let $\mathcal{A}_{n,2j+1} = \{T \in \mathcal{B}_{2j,n-1} \mid T_{(2j+1,2j)} = 0\}$. Then $\mathcal{A}_{n,2j+1}$ is a Lie ideal and $\mathcal{A}_{2,2j+1} = \{T \in \mathcal{B}_{2j,1} \mid T_{(2j+1,2j)} = 0\}$.

For $k > 1$,

3) Let $\mathcal{C}_{n,2k-1} = \{T \in \mathcal{B}_{2k-2,n-1} \,|\, T_{(2k-1,2k)} = 0\}$. Then $\mathcal{C}_{n,2k-1}$ is a Lie ideal.
4) Let $\mathcal{C}_{n,2j+1} = \{T \in \mathcal{B}_{2j-1,n-1} \,|\, T_{(2j+1,2j)} = 0\}$. Then $\mathcal{C}_{n,2j+1}$ is a Lie ideal.

Example 8. Let k and j be natural numbers.

1) Let $\mathcal{A}_{\infty,2k-1} = \{T \in \mathcal{B}_{2k-1,\infty} \,|\, T_{(2k-1,2k)} = 0\}$. Then $\mathcal{A}_{\infty,2k-1}$ is a Lie ideal.
2) Let $\mathcal{A}_{\infty,2j+1} = \{T \in \mathcal{B}_{2j,\infty} \,|\, T_{(2j+1,2j)} = 0\}$. Then $\mathcal{A}_{\infty,2j+1}$ is a Lie ideal.

For $k > 1$,

3) Let $\mathcal{C}_{\infty,2k-1} = \{T \in \mathcal{B}_{2k-2,\infty} \,|\, T_{(2k-1,2k)} = 0\}$. Then $\mathcal{C}_{\infty,2k-1}$ is a Lie ideal.
4) Let $\mathcal{C}_{\infty,2j+1} = \{T \in \mathcal{B}_{2j-1,\infty} \,|\, T_{(2j+1,2j)} = 0\}$. Then $\mathcal{C}_{\infty,2j+1}$ is a Lie ideal.

Example 9. Let Ω be a nonempty subset of $\mathbb{N}$ and let

$$\mathcal{A}_\Omega = \{T \in \mathrm{Alg}\mathcal{L}_\infty \,|\, T_{(k,k)} = 0 \ \text{ for all } k \in \Omega\}.$$

Then $\mathcal{A}_\Omega$ is a Lie ideal in $\mathrm{Alg}\mathcal{L}_\infty$.

Remark. Let k be a fixed natural number. Let $\Gamma_n(\Upsilon_l)$ be a set of finite natural numbers containing $k(j)$ and $\Omega_1(\Omega_2)$ be a set of infinite natural numbers containing $\Gamma_n(\Upsilon_l)$ respectively. Then the following diagram 1 shows the relationships between Lie ideals introduced on the above examples. We will prove them in Section 4. Each arrow between Lie ideals means set inclusion relationship. The diagram holds for $2j+1$, Υ_l and Ω_2 instead of $2k-1$, Γ_n and Ω_1, respectively.

Let $(\Gamma_n) = \{(2k_i - 1, 1) \,|\, k_i \in \Gamma_n\}$, $(\Upsilon_l) = \{(2j_p, 1) \,|\, j_p \in \Upsilon_l\}$, $(\Omega_1) = \{(2k_i - 1, 1) \,|\, k_i \in \Omega_1\}$, $(\Omega_2) = \{(2j_p, 1) \,|\, j_p \in \Omega_2\}$. Then we have the following diagram:

$$\begin{tikzcd}
 & \mathcal{B}_{(\Omega)} \arrow[r] & \mathcal{B}_{(\Omega_1)} \arrow[r] & \mathcal{B}_{(\Gamma_n)} \arrow[r] & \mathcal{B}_{2k-1,1} \arrow[r, Rightarrow] & \mathrm{Alg}\mathcal{L}_\infty \\
 & \mathcal{A}_{2,\Omega} \arrow[r] \arrow[u] & \mathcal{A}_{2,\Omega_1} \arrow[r] \arrow[u] & \mathcal{A}_{2,\Gamma_n} \arrow[r] \arrow[u] & \mathcal{A}_{2,2k-1} \arrow[r] \arrow[u, Rightarrow] & \mathcal{B}_{k,\infty} \arrow[u] \\
 & \mathcal{A}_{1,\Omega} \arrow[r] \arrow[u] & \mathcal{A}_{1,\Omega_1} \arrow[r] \arrow[u] & \mathcal{A}_{1,\Gamma_n} \arrow[r, Rightarrow] \arrow[u] & \mathcal{A}_{1,2k-1} \arrow[r, Rightarrow] \arrow[u] & \mathcal{A}_1 \arrow[u] \\
\{\mathbf{0}\} \arrow[r] & \mathcal{A}_{0,\Omega} \arrow[r] \arrow[u] & \mathcal{A}_{0,\Omega_1} \arrow[r] \arrow[u] & \mathcal{A}_{0,\Gamma_n} \arrow[r, Rightarrow] \arrow[u] & \mathcal{A}_{0,2k-1} \arrow[r, Rightarrow] \arrow[u, Rightarrow] & \mathcal{A}_0 \arrow[u, Rightarrow]
\end{tikzcd}$$

Diagram 1.

3. Properties of Lie ideals in $\mathrm{Alg}\mathcal{L}_\infty$

In this section we investigate some properties of Lie ideals in $\mathrm{Alg}\mathcal{L}_\infty$. The following theorems show necessary and sufficient conditions in which a linear manifold can be a Lie ideal in $\mathrm{Alg}\mathcal{L}_\infty$.

Theorem 1. *Let k be a fixed natural number. Let $\mathcal{A}$ be a linear manifold in* $\mathrm{Alg}\mathcal{L}_\infty$ *such that $T_{(2k-1,2k)} = 0$ for all T in $\mathcal{A}$. Then $\mathcal{A}$ is a Lie ideal if and only if $T_{(2k-1,2k-1)} = T_{(2k,2k)}$ for all $T \in \mathcal{A}$.*

Proof. Let $T = (t_{ij})$ be an operator in $\mathcal{A}$ and $A = (a_{ij})$ be in $\mathrm{Alg}\mathcal{L}_\infty$. Since the $(2k-1, 2k)$-component of $AT - TA$ is

$$a_{2k-1\ 2k}(t_{2k\ 2k} - t_{2k-1\ 2k-1}) + t_{2k-1\ 2k}(a_{2k-1\ 2k-1} - a_{2k\ 2k}),$$

$t_{2k-1\ 2k} = 0$ and $\mathcal{A}$ is a Lie ideal,

$$(*) \qquad a_{2k-1\ 2k}(t_{2k\ 2k} - t_{2k-1\ 2k-1}) = 0 \quad \text{for all } A \in \mathrm{Alg}\mathcal{L}_\infty.$$

Since $(*)$ holds for all A in $\mathrm{Alg}\mathcal{L}_\infty$,

$$t_{2k-1\ 2k-1} = t_{2k\ 2k}.$$

The converse is just 1) of Example 5. □

Theorem 2. *Let j be a fixed natural number. Let $\mathcal{A}$ be a linear manifold in* $\mathrm{Alg}\mathcal{L}_\infty$ *such that $T_{(2j+1,2j)} = 0$ for all T in $\mathcal{A}$. Then $\mathcal{A}$ is a Lie ideal if and only if $T_{(2j,2j)} = T_{(2j+1,2j+1)}$ for $T \in \mathcal{A}$.*

Proof. Suppose $\mathcal{A}$ is a Lie ideal. Let A be an operator in $\mathrm{Alg}\mathcal{L}_\infty$ and let T be an operator in $\mathcal{A}$. Since the $(2j+1, 2j)$-component of $AT - TA$ is

$$a_{2j+1\ 2j}(t_{2j\ 2j} - t_{2j+1\ 2j+1}) + t_{2j+1\ 2j}(a_{2j+1\ 2j+1} - a_{2j\ 2j}),$$

$t_{2j+1\ 2j} = 0$ and $\mathcal{A}$ is a Lie ideal,

$$(**) \qquad a_{2j+1\ 2j}(t_{2j\ 2j} - t_{2j+1\ 2j+1}) = 0 \quad \text{for all } A \in \mathrm{Alg}\mathcal{L}_\infty.$$

Since $(**)$ holds for all A in $\mathrm{Alg}\mathcal{L}_\infty$,

$$t_{2j\ 2j} = t_{2j+1\ 2j+1}.$$

The converse is just 2) of Example 5. □

We can derive the following theorems in a similar way as the above theorem.

Theorem 3. *Let $k_1, k_2, \ldots, k_n$ and $j_1, j_2, \ldots, j_l$ be different natural numbers.*

1) *Let $\mathcal{A}$ be a linear manifold in $\mathrm{Alg}\mathcal{L}_\infty$ such that $T_{(2k_i-1,2k_i)} = 0$ $(i = 1, 2, \ldots, n)$ for all $T \in \mathcal{A}$. Then $\mathcal{A}$ is a Lie ideal if and only if*

$$T_{(2k_i-1,2k_i-1)} = T_{(2k_i,2k_i)} \ (i = 1, 2, \ldots, n)$$

for all T in $\mathcal{A}$.

2) *Let $\mathcal{A}$ be a linear manifold in $\mathrm{Alg}\mathcal{L}_\infty$ such that $T_{(2j_p+1,2j_p)} = 0$ $(p = 1, 2, \ldots, l)$ for all $T \in \mathcal{A}$. Then $\mathcal{A}$ is a Lie ideal if and only if*

$$T_{(2j_p,2j_p)} = T_{(2j_p+1,2j_p+1)} \ (p = 1, 2, \ldots, l)$$

for all T in $\mathcal{A}$.

3) *Let $\mathcal{A}$ be a linear manifold in $\mathrm{Alg}\mathcal{L}_\infty$ such that $T_{(2k_i-1,2k_i)} = 0$ $(i = 1, 2, \ldots, n)$ and $T_{(2j_p+1,2j_p)} = 0$ $(p = 1, 2, \ldots, l)$ for all T in $\mathcal{A}$. Then $\mathcal{A}$ is a Lie ideal if and only if $T_{(2k_i-1,2k_i-1)} = T_{(2k_i,2k_i)}$ $(i = 1, 2, \ldots, n)$ and $T_{(2j_p,2j_p)} = T_{(2j_p+1,2j_p+1)}$ $(p = 1, 2, \ldots, l)$ for all T in $\mathcal{A}$.*

Theorem 4. *Let $k_1, k_2, \ldots$ and $j_1, j_2, \ldots$ be different natural numbers.*

1) *Let $\mathcal{A}$ be a linear manifold in $\mathrm{Alg}\mathcal{L}_\infty$ such that $T_{(2k_i-1,2k_i)} = 0$ ($i = 1, 2, \ldots$) for all T in $\mathcal{A}$. Then $\mathcal{A}$ is a Lie ideal if and only if*
$$T_{(2k_i-1,2k_i-1)} = T_{(2k_i,2k_i)} \ (i = 1, 2, \ldots)$$
for all T in $\mathcal{A}$.
2) *Let $\mathcal{A}$ be a linear manifold in $\mathrm{Alg}\mathcal{L}_\infty$ such that $T_{(2j_p+1,2j_p)} = 0$ ($p = 1, 2, \ldots$) for all T in $\mathcal{A}$. Then $\mathcal{A}$ is a Lie ideal if and only if*
$$T_{(2j_p,2j_p)} = T_{(2j_p+1,2j_p+1)} \ (p = 1, 2, \ldots)$$
for all T in $\mathcal{A}$.
3) *Let $\mathcal{A}$ be a linear manifold in $\mathrm{Alg}\mathcal{L}_\infty$ such that $T_{(2k_i-1,2k_i)} = 0$ and $T_{(2j_p+1,2j_p)} = 0$ ($i, p = 1, 2, \ldots$) for all T in $\mathcal{A}$. Then $\mathcal{A}$ is a Lie ideal if and only if $T_{(2k_i-1,2k_i-1)} = T_{(2k_i,2k_i)}$ and $T_{(2j_p,2j_p)} = T_{(2j_p+1,2j_p+1)}$ ($i, p = 1, 2, \ldots$) for all T in $\mathcal{A}$.*

4. Relationships between Lie ideals in $\mathrm{Alg}\mathcal{L}_\infty$

In this section we will investigate relationships between Lie ideals introduced in the examples in Section 2. Each Lie ideal (showed in Section 2) is weakly closed and each arrow on Diagram 1 in the previous section means an inclusion relationship. Some arrows, indicated by " $\Rightarrow$ ", between the two Lie ideals mean that there is no Lie ideal between two Lie ideals connected by the arrow. Every theorem in this section holds for $2j+1$, Υ_l and Ω_2 instead of $2k-1$, Γ_n and Ω_1 respectively.

Theorem 5. *Let k be a fixed natural number. Let $\mathcal{A}$ be a linear manifold in $\mathrm{Alg}\mathcal{L}_\infty$ and let $\mathcal{A}$ be a Lie ideal such that $\mathcal{A}_{0,2k-1} \subset \mathcal{A} \subset \mathcal{A}_0$. Then $\mathcal{A} = \mathcal{A}_{0,2k-1}$ or $\mathcal{A} = \mathcal{A}_0$.*

Proof. It is sufficient to show for the case $k = 1$, i.e., $\mathcal{A}_{0,1} \subset \mathcal{A} \subset \mathcal{A}_0 \Rightarrow \mathcal{A} = \mathcal{A}_{0,1}$ or $\mathcal{A} = \mathcal{A}_0$.

Assume that $\mathcal{A} \neq \mathcal{A}_{0,1}$. Then there exists $T = (t_{ij}) \in \mathcal{A}$ such that $T \notin \mathcal{A}_{0,1}$. Then $t_{ii} = 0$ for all $i \in \mathbb{N}$ and $t_{12} \neq 0$. Let $A = (a_{ij}) \in \mathcal{A}_0$. If $a_{12} = 0$, then $A \in \mathcal{A}_{0,1} \subset \mathcal{A}$. If $a_{12} \neq 0$, let A_1 be defined by
$$\begin{cases} A_{1(1,2)} = 0, \\ A_{1(i,j)} = a_{ij}, \ \text{otherwise}. \end{cases}$$
Then $A_1 \in \mathcal{A}_{0,1} \subset \mathcal{A}$. Let T_1 be defined by
$$\begin{cases} T_{1(1,2)} = 0, \\ T_{1(i,j)} = -t_{ij}, \ \text{otherwise}. \end{cases}$$
Then $T_1 \in \mathcal{A}_{0,1} \subset \mathcal{A}$. Let $T_2 = T + T_1$. Then $T_2 \in \mathcal{A}$ and
$$\begin{cases} T_{2(1,2)} = t_{12}, \\ T_{2(i,j)} = 0, \ \text{otherwise}. \end{cases}$$

Let $x = \frac{a_{12}}{t_{12}}$. Then $xT_2 + A_1 = A \in \mathcal{A}$. Hence $\mathcal{A} = \mathcal{A}_0$. □

The following two theorems are proved in a similar way as Theorem 5.

Theorem 6. *Let k be a fixed natural number and $\Gamma_2 = \{k = k_1, k_2\}$.*

1) *If $\mathcal{A}$ is a Lie ideal such that $\mathcal{A}_{1,2k-1} \subset \mathcal{A} \subset \mathcal{A}_1$, then $\mathcal{A} = \mathcal{A}_{1,2k-1}$ or $\mathcal{A} = \mathcal{A}_1$.*
2) *If $\mathcal{A}$ is a Lie ideal such that $\mathcal{A}_{0,\Gamma_2} \subset \mathcal{A} \subset \mathcal{A}_{0,2k-1}$, then $\mathcal{A} = \mathcal{A}_{0,\Gamma_2}$ or $\mathcal{A} = \mathcal{A}_{0,2k-1}$.*
3) *If $\mathcal{A}$ is a Lie ideal such that $\mathcal{A}_{1,\Gamma_2} \subset \mathcal{A} \subset \mathcal{A}_{1,2k-1}$, then $\mathcal{A} = \mathcal{A}_{1,\Gamma_2}$ or $\mathcal{A} = \mathcal{A}_{1,2k-1}$.*

Theorem 7. *Let $k = k_1 < k_2 < \cdots < k_n$. Let $\Gamma_1 = \{k_1\}$, $\Gamma_2 = \{k_1, k_2\}$, $\Gamma_3 = \{k_1, k_2, k_3\}$, ..., $\Gamma_n = \{k_1, \ldots, k_n\}$ and $\Omega_1 = \{k_1, k_2, \ldots\}$. Then*

$$\mathcal{A}_{0,\Omega_1} \subset \cdots \subset \mathcal{A}_{0,\Gamma_n} \subset \mathcal{A}_{0,\Gamma_{n-1}} \subset \cdots \subset \mathcal{A}_{0,\Gamma_2} \subset \mathcal{A}_{0,2k-1} = \mathcal{A}_{0,\Gamma_1},$$
$$\mathcal{A}_{1,\Omega_1} \subset \cdots \subset \mathcal{A}_{1,\Gamma_n} \subset \mathcal{A}_{1,\Gamma_{n-1}} \subset \cdots \subset \mathcal{A}_{1,\Gamma_2} \subset \mathcal{A}_{1,2k-1} = \mathcal{A}_{1,\Gamma_1}.$$

And there is no Lie ideal between the above two adjacent Lie ideals.

Proof. The last inclusion is proved by Theorem 5. The other inclusions are proved in the same way. □

Theorem 8. *Let k be a fixed natural number. Let $\mathcal{A}$ be a linear manifold in* Alg$\mathcal{L}_\infty$ *and let $\mathcal{A}$ be a Lie ideal such that $\mathcal{B}_{2k-1,1} \subset \mathcal{A} \subset \mathrm{Alg}\mathcal{L}_\infty$. Then $\mathcal{A} = \mathcal{B}_{2k-1,1}$ or $\mathcal{A} = \mathrm{Alg}\mathcal{L}_\infty$.*

Proof. It is enough to show for the case $k = 1$, i.e., $\mathcal{B}_{1,1} \subset \mathcal{A} \subset \mathrm{Alg}\mathcal{L}_\infty \Rightarrow \mathcal{A} = \mathcal{B}_{1,1}$ or $\mathcal{A} = \mathrm{Alg}\mathcal{L}_\infty$. If $\mathcal{A} \neq \mathcal{B}_{1,1}$, then there exists $T = (t_{ij}) \in \mathcal{A}$ such that $T \notin \mathcal{B}_{1,1}$. Then $t_{11} \neq t_{22}$. Let $A = (a_{ij}) \in \mathrm{Alg}\mathcal{L}_\infty$. If $a_{11} = a_{22}$, then $A \in \mathcal{B}_{1,1} \subset \mathcal{A}$. If $a_{11} \neq a_{12}$, let T_1 be defined by

$$\begin{cases} T_{1(1,1)} = T_{1(2,2)} = 0, \\ T_{1(i,j)} = -t_{ij}, \text{ otherwise.} \end{cases}$$

Let $T_2 = T + T_1$. Then $T_2 \in \mathcal{A}$.

Let A_1 and A_2 be defined by

$$\begin{cases} A_{1(1,1)} = A_{1(2,2)} = 1, \\ A_{1(i,j)} = 0, \text{ otherwise} \end{cases} \quad \text{and} \quad \begin{cases} A_{2(1,1)} = A_{2(2,2)} = 0, \\ A_{2(i,j)} = a_{ij}, \text{ otherwise.} \end{cases}$$

Then A_1 and A_2 are in $\mathcal{B}_{1,1}$. So $A_1, A_2 \in \mathcal{A}$. Let $x = \frac{a_{11} - a_{22}}{t_{11} - t_{22}}$ and $y = \frac{t_{11}a_{22} - t_{22}a_{11}}{t_{11} - t_{22}}$. Then $xT_2 + yA_1 + A_2 = A \in \mathcal{A}$. Hence $\mathcal{A} = \mathrm{Alg}\mathcal{L}_\infty$. □

Theorem 9. *Let $\mathcal{A}$ be a linear manifold in* Alg$\mathcal{L}_\infty$ *and let $\mathcal{A}$ be a Lie ideal such that $\mathcal{A}_0 \subset \mathcal{A} \subset \mathcal{A}_1$. Then $\mathcal{A} = \mathcal{A}_0$ or $\mathcal{A} = \mathcal{A}_1$.*

Proof. If $\mathcal{A} \neq \mathcal{A}_0$, then there exists $T = (t_{ij}) \in \mathcal{A}$ such that $T \notin \mathcal{A}_0$, i.e., $t_{ii} = t_{jj} \neq 0$ for all $i \neq j$. Let $A = (a_{ij}) \in \mathcal{A}_1$. We define T_1 and A_1 as follows:

$$\begin{cases} T_{1(i,i)} = 0 & \text{for all } i \in \mathbb{N}, \\ T_{1(i,j)} = -t_{ij} & \text{for } i \neq j \end{cases} \quad \text{and} \quad \begin{cases} A_{1(i,i)} = 0 & \text{for all } i \in \mathbb{N}, \\ A_{1(i,j)} = a_{ij} & \text{for } i \neq j. \end{cases}$$

Then $T_1, A_1 \in \mathcal{A}_0 \subset \mathcal{A}$. Let $T_2 = T + T_1$. Then $T_2 \in \mathcal{A}$ and

$$\begin{cases} T_{2(i,i)} = t_{11} & \text{for all } i \in \mathbb{N}, \\ T_{2(i,j)} = 0 & \text{for } i \neq j. \end{cases}$$

Let $x = \frac{a_{11}}{t_{11}}$. Then $xT_2 + A_1 = A \in \mathcal{A}$. Hence $\mathcal{A} = \mathcal{A}_1$. □

The following theorem is proved in the same way as Theorem 9.

Theorem 10. *Let k be a fixed natural number. Let $\mathcal{A}$ be a Lie ideal in* $\mathrm{Alg}\mathcal{L}_\infty$ *such that $\mathcal{A}_{0,2k-1} \subset \mathcal{A} \subset \mathcal{A}_{1,2k-1}$. Then $\mathcal{A} = \mathcal{A}_{0,2k-1}$ or $\mathcal{A} = \mathcal{A}_{1,2k-1}$.*

Theorem 11. *Let $\Gamma_n = \{k_1, \ldots, k_n\}$. Then*

1) *Let $\mathcal{A}$ be a linear manifold in* $\mathrm{Alg}\mathcal{L}_\infty$ *such that $\mathcal{A}_{0,\Gamma_n} \subset \mathcal{A} \subset \mathcal{A}_{1,\Gamma_n}$. Then $\mathcal{A}$ is a Lie ideal.*
2) *Let $\mathcal{A}$ be a linear manifold in* $\mathrm{Alg}\mathcal{L}_\infty$ *such that $\mathcal{A}_{1,\Gamma_n} \subset \mathcal{A} \subset \mathcal{A}_{2,\Gamma_n}$. Then $\mathcal{A}$ is a Lie ideal.*

Proof. 1) Let $A = (a_{ij}) \in \mathrm{Alg}\mathcal{L}_\infty$ and $T = (t_{ij}) \in \mathcal{A}$. Then $(AT-TA)_{(2k_i-1,2k_i)} = a_{2k_i-12k_i}(t_{2k_i2k_i} - t_{2k_i-12k_i-1}) + t_{2k_i-12k_i}(a_{2k_i-12k_i-1} - t_{2k_i2k_i}) = 0$ for all $i = 1, 2, \ldots, n$ and $(AT-TA)_{(k,k)} = 0$ for all $k \in \mathbb{N}$. So $AT-TA \in \mathcal{A}_{1,\Gamma_n} \subset \mathcal{A}$. Hence $\mathcal{A}$ is a Lie ideal.

2) The proof is the same as 1). □

Theorem 12. *Let k be a fixed natural number.*

1) *$\mathcal{A}_{1,2k-1} \subset \mathcal{A}_{3,2k-1} \subset \mathcal{A}_{2,2k-1}$ and $\mathcal{A}_{1,2k-1} \subset \mathcal{C}_{3,2k-1} \subset \mathcal{A}_{2,2k-1}$ (when $k \neq 1$).*
2) *Let $\mathcal{A}$ be a Lie ideal such that $\mathcal{A}_{3,2k-1} \subset \mathcal{A} \subset \mathcal{A}_{2,2k-1}$. Then $\mathcal{A} = \mathcal{A}_{3,2k-1}$ or $\mathcal{A} = \mathcal{A}_{2,2k-1}$.*
3) *Let $\mathcal{A}$ be a Lie ideal such that $\mathcal{C}_{3,2k-1} \subset \mathcal{A} \subset \mathcal{A}_{2,2k-1}$ (when $k \neq 1$). Then $\mathcal{A} = \mathcal{C}_{3,2k-1}$ or $\mathcal{A} = \mathcal{A}_{2,2k-1}$.*
4) *$\mathcal{A}_{1,2k-1} \subset \mathcal{A}_{\infty,2k-1} \subset \cdots \subset \mathcal{A}_{4,2k-1} \subset \mathcal{A}_{3,2k-1}$ and $\mathcal{A}_{1,2k-1} \subset \mathcal{C}_{\infty,2k-1} \subset \cdots \subset \mathcal{C}_{4,2k-1} \subset \mathcal{C}_{3,2k-1}$.*

Proof. 1) and 4) are obvious and we prove only 2).

2) It is enough to prove this for the case $k = 1$. Let $\mathcal{A} \neq \mathcal{A}_{3,1}$ and let $T = (t_{ij}) \in \mathcal{A}$ such that $T \notin \mathcal{A}_{3,1}$. Since $T \in \mathcal{A} \subset \mathcal{A}_{2,1}$, $t_{11} = t_{22} \neq t_{33}$. Let T_1 be defined by

$$\begin{cases} T_{1(1,1)} = T_{1(2,2)} = T_{1(3,3)} = -t_{11}, \\ T_{1(i,j)} = -t_{ij}, \text{ otherwise.} \end{cases}$$

Then $T_1 \in \mathcal{A}_{3,1} \subset \mathcal{A}$. Let $T_2 = T + T_1$. Then $T_2 \in \mathcal{A}$ and

$$\begin{cases} T_{2(3,3)} = t_{33} - t_{11}, \\ T_{2(i,j)} = 0, \text{ otherwise.} \end{cases}$$

Let $A = (a_{ij}) \in \mathcal{A}_{2,1}$. If $a_{11} = a_{22} = a_{33}$, then $A \in \mathcal{A}_{3,1} \subset \mathcal{A}$. If $a_{11} = a_{22} \neq a_{33}$, let $x = \frac{a_{33}-a_{11}}{t_{33}-t_{11}}$. Then $xT_2 \in \mathcal{A}$ and

$$\begin{cases} xT_{2(3,3)} = a_{33} - a_{11}, \\ xT_{2(i,j)} = 0, \text{ otherwise.} \end{cases}$$

Let A_1 be defined by

$$\begin{cases} A_{1(1,1)} = A_{1(2,2)} = A_{1(3,3)} = a_{11}, \\ A_{1(i,j)} = a_{ij}, \text{ otherwise.} \end{cases}$$

Then $A_1 \in \mathcal{A}_{3,1} \subset \mathcal{A}$ and $A = A_1 + xT_2 \in \mathcal{A}$. Hence $\mathcal{A} = \mathcal{A}_{2,1}$.

3) It is proved in a similar way as 2). □

Theorem 13. *Let k be a fixed natural number. Let $\mathcal{A}$ be a Lie ideal in* Alg$\mathcal{L}_\infty$ *such that $\mathcal{A}_{2,2k-1} \subset \mathcal{A} \subset \mathcal{B}_{2k-1,1}$. Then $\mathcal{A} = \mathcal{A}_{2,2k-1}$ or $\mathcal{A} = \mathcal{B}_{2k-1,1}$.*

Proof. It is sufficient to show for the case $k = 1$, i.e., $\mathcal{A}_{2,1} \subset \mathcal{A} \subset \mathcal{B}_{1,1} \Rightarrow \mathcal{A} = \mathcal{A}_{2,1}$ or $\mathcal{A} = \mathcal{B}_{1,1}$. Suppose that $\mathcal{A}_{2,1} \neq \mathcal{A}$. Let $T = (t_{ij}) \in \mathcal{A}$ and $T \notin \mathcal{A}_{2,1}$. Then $t_{12} \neq 0$, $t_{11} = t_{22}$. Let $A = (a_{ij}) \in \mathcal{B}_{1,1}$. Then $a_{11} = a_{22}$.
If $a_{12} = 0$, then $A \in \mathcal{A}_{2,1} \subset \mathcal{A}$. If $_{12} \neq 0$, let we define T_1 by

$$\begin{cases} T_{1(1,1)} = T_{1(2,2)} = -t_{11}, \\ T_{1(1,2)} = 0, \\ T_{1(i,j)} = -t_{ij}, \text{ otherwise.} \end{cases}$$

Then $T_1 \in \mathcal{A}_{2,1} \subset \mathcal{A}$. Let $T_2 = T + T_1$. Then$T_2 \in \mathcal{A}$ and

$$\begin{cases} T_{2(1,2)} = t_{12}, \\ T_{2(i,j)} = 0 \text{ for } (i,j) \neq (1,2). \end{cases}$$

Let $x = \frac{a_{12}}{t_{12}}$ and let A_1 be defined by

$$\begin{cases} A_{1(1,1)} = A_{1(2,2)} = a_{11}, \\ A_{1(1,2)} = 0, \\ A_{1(i,j)} = a_{ij}, \text{ otherwise.} \end{cases}$$

Then $A_1 \in \mathcal{A}_{2,1}$ and $A = A_1 + xT_2 \in \mathcal{A}$. So $\mathcal{A} = \mathcal{C}_{1,1}$. □

Theorem 14. *Let $\mathcal{B} = \mathcal{A}_{2,1} \cap \mathcal{B}_{3,1}$ and let $\Gamma_2 = \{1,2\}$.*

1) *$\mathcal{A}_{2,\Gamma_2} \subset \mathcal{B} \subset \mathcal{A}_{2,1}$.*

2) *Let $\mathcal{A}$ be a Lie ideal such that $\mathcal{B} \subset \mathcal{A} \subset \mathcal{A}_{2,1}$. Then $\mathcal{A} = \mathcal{B}$ or $\mathcal{A} = \mathcal{A}_{2,1}$.*

3) *Let $\mathcal{A}$ be a Lie ideal such that $\mathcal{A}_{2,\Gamma_2} \subset \mathcal{A} \subset \mathcal{B}$. Then $\mathcal{A} = \mathcal{A}_{2,\Gamma_2}$ or $\mathcal{A} = \mathcal{B}$.*

Proof. 2) Assume that $\mathcal{B} \neq \mathcal{A}$. Then there exists $T = (t_{ij}) \in \mathcal{A}$ such that $T \notin \mathcal{B}$. Then $t_{12} = 0$, $t_{11} = t_{22}$ and $t_{33} \neq t_{44}$. Let $A = (a_{ij}) \in \mathcal{A}_{2,1}$. Then $a_{11} = a_{22}$ and $a_{12} = 0$.

Case 1. If $a_{33} = a_{44}$, then $A \in \mathcal{B} \subset \mathcal{A}$.

Case 2. Assume that $a_{33} \neq a_{44}$. Then let A_1 be defined by

$$\begin{cases} A_{1(3,3)} = A_{1(4,4)} = -t_{44}, \\ A_{1(i,j)} = a_{ij}, \text{ otherwise.} \end{cases}$$

Then $A_1 \in \mathcal{B} \subset \mathcal{A}$. Let T_1 and T_1' be defined by

$$\begin{cases} T_{1(3,3)} = T_{1(4,4)} = -t_{44}, \\ T_{1(i,j)} = -t_{ij}, \text{ otherwise} \end{cases} \quad \text{and} \quad \begin{cases} T'_{1(3,3)} = T'_{1(4,4)} = -t_{33}, \\ T'_{1(i,j)} = -t_{ij}, \text{ otherwise,} \end{cases}$$

and $T_2 = T + T_1$ and $T_3 = T + T_1'$. Then $T_2, T_3 \in \mathcal{A}$. So $A_2 = \frac{a_{33}}{t_{33}-t_{44}} T_2 \in \mathcal{A}$ and $A_3 = \frac{a_{44}}{t_{44}-t_{33}} T_3 \in \mathcal{A}$. Hence $A = A_1 + A_2 + A_3 \in \mathcal{A}$. Therefore $\mathcal{A} = \mathcal{A}_{2,1}$.

3) Assume that $\mathcal{A}_{2,\Gamma_2} \neq \mathcal{A}$. Then there exists $T = (t_{ij}) \in \mathcal{A}$ such that $T \notin \mathcal{A}_{2,\Gamma_2}$. Then $t_{11} = t_{22}$, $t_{12} = 0$, $t_{33} = t_{44}$ and $t_{34} \neq 0$. Let $A = (a_{ij}) \in \mathcal{B}$. Then $a_{11} = 0$, $a_{11} = a_{22}$ and $a_{33} = a_{44}$. Let A_1 be defined by

$$\begin{cases} A_{1(3,4)} = 0, \\ A_{1(i,j)} = a_{ij}, \text{ otherwise.} \end{cases}$$

Then $A_1 \in \mathcal{A}_{2,\Gamma_2} \subset \mathcal{A}$. Let T_1 be defined by

$$\begin{cases} T_{1(3,4)} = 0, \\ T_{1(i,j)} = -t_{ij}, \text{ otherwise.} \end{cases}$$

Then $T_1 \in \mathcal{A}_{2,\Gamma_2} \subset \mathcal{A}$. Let $T_2 = T + T_1$. Then $T_2 \in \mathcal{A}$. Since $t_{34} \neq 0$, let $A_2 = \frac{a_{34}}{t_{34}} T_2$. Then $A_2 \in \mathcal{A}$ and $A = A_1 + A_2 \in \mathcal{A}$. Therefore $\mathcal{B} = \mathcal{A}$. $\square$

The above theorem holds for any $\{k_1, k_2\}$ instead of $\{1, 2\}$ and can be generalized as the following theorem.

Theorem 15. *Let* $k = k_1 < k_2 < \cdots < k_n$. *Let* $\Gamma_1 = \{k = k_1\}$, $\Gamma_2 = \{k_1, k_2\}$, $\Gamma_3 = \{k_1, k_2, k_3\}, \ldots, \Gamma_n = \{k_1, \ldots, k_n\}$ *and* $\Omega_1 = \{k_1, k_2, \ldots\}$. *Let* $\mathcal{B} = \mathcal{A}_{2,2k_i-1} \cap \mathcal{B}_{2k_{i+1}-1,1}$. *Then*

1) $\mathcal{A}_{2,\Omega_1} \subset \cdots \subset \mathcal{A}_{2,\Gamma_n} \subset \mathcal{A}_{2,\Gamma_{n-1}} \subset \cdots \subset \mathcal{A}_{2,\Gamma_2} \subset \mathcal{A}_{2,2k-1} = \mathcal{A}_{2,\Gamma_1}$.
2) *Let* $\mathcal{A}$ *be a Lie ideal and* $\mathcal{B} \subset \mathcal{A} \subset \mathcal{A}_{2,\Gamma_{i-1}}$. *Then* $\mathcal{A} = \mathcal{B}$ *or* $\mathcal{A} = \mathcal{A}_{2,\Gamma_{i-1}}$.
3) *Let* $\mathcal{A}$ *be a Lie ideal and* $\mathcal{A}_{2,\Gamma_i} \subset \mathcal{A} \subset \mathcal{B}$. *Then* $\mathcal{A} = \mathcal{A}_{2,\Gamma_i}$ *or* $\mathcal{A} = \mathcal{B}$.

References

[1] C. K. Fong, C. R. Miers, and A. R. Sourour, *Lie and Jordan ideals of operators on Hilbert spaces*, Proc. Amer. Math. Soc. **84** (1982), no. 4, 516–520.

[2] A. Hopenwasser and V. Paulsen, *Lie ideal in operator algebras*, J. Operator Theory **52** (2004), no. 2, 325–340.

[3] T. D. Hudson, L. W. Marcoux, and A. R. Sourour, *Lie ideal in triangular operator algebras*, Trans. Amer. Math. Soc. **350** (1998), no. 8, 3321–3339.
[4] Y. S. Jo, *Isometries of tridiagonal algebras*, Pacific J. Math. **140** (1989), no. 1, 97–115.
[5] Y. S. Jo and T. Y. Choi, *Isomorphisms of Alg$\mathcal{L}_n$ and Alg$\mathcal{L}_\infty$*, Michigan Math. J. **37** (1990), no. 2, 305–314.

Department of Mathematics
Daegu University
Daegu 712-714, Korea
E-mail address: jhkang@daegu.ac.kr