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# A NOTE ON NEVANLINNA'S FIVE VALUE THEOREM

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Abstract. In the paper we prove a uniqueness theorem which improves and generalizes a number of uniqueness theorems for meromorphic functions related to Nevanlinna's five value theorem.

## 1. Introduction, definitions and results

In the paper, by meromorphic functions we always mean meromorphic functions in the open complex plane $\mathbb{C}$. Let $f$ be a non-constant meromorphic function. A meromorphic function $a = a(z)$ is said to be a small function of $f$ if either $a \equiv \infty$ or $T(r, a) = S(r, f)$. We denote by $S(f)$ the collection of all small functions of $f$. Clearly $\mathbb{C} \cup \{\infty\} \subset S(f)$ and $S(f)$ is a field over the set of complex numbers.

For a positive integer $p$ and $a \in S(f)$ we denote by $\overline{E}_{p)}(a; f)$ the set of those distinct zeros of $f - a$ whose multiplicities do not exceed $p$, where we mean by a zero of $f - \infty$ a pole of $f$. Also by $\overline{E}_{\infty)}(a; f)$ we denote the set of all distinct zeros of $f - a$.

For $A \subset \mathbb{C}$ we denote by $\overline{N}_A(r, a; f)$ the reduced counting function of those zeros of $f - a$ which belong to the set $A$, where $a \in S(f)$.

For a positive integer $p$ and $a \in S(f)$ we denote by $N_{p)}(r, a; f)$ $(\overline{N}_{p)}(r, a; f))$ the counting function (reduced counting function) of those zeros of $f - a$ whose multiplicities do not exceed $p$. Similarly we define $N_{(p}(r, a; f)$ and $\overline{N}_{(p}(r, a; f))$.

For standard definitions and notations of Nevanlinna theory we refer the reader to [4]. The modern theory of uniqueness of entire and meromorphic functions was initiated by R. Nevanlinna with his two famous theorems: The Five Value Theorem and The Four Value Theorem. The five value theorem of Nevanlinna may be stated as follows:

**Theorem A** ([4, p. 48]). *Let $f$ and $g$ be two non-constant meromorphic functions and $a_j \in \mathbb{C} \cup \{\infty\}$ be distinct for $j = 1, 2, \ldots, 5$. If $\overline{E}_{\infty)}(a_j; f) = \overline{E}_{\infty)}(a_j; g)$ for $j = 1, 2, \ldots, 5$, then $f \equiv g$.*

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In 1976 H. S. Gopalakrishna and S. S. Bhoosnurmath [3] improved Theorem A in the following manner.

**Theorem B** ([3]). *Let $f$, $g$ be distinct non-constant meromorphic functions. If there exist distinct elements $a_1, a_2 \dots, a_k$ of $\mathbb{C} \cup \{\infty\}$ such that $\overline{E}_{p_j)}(a_j; f) = \overline{E}_{p_j)}(a_j; g)$ for $j = 1, 2, \dots, k$, where $p_1, p_2, \dots, p_k$ are positive integers or $\infty$ with $p_1 \geq p_2 \geq \cdots \geq p_k$, then $\sum_{j=2}^{k} \frac{p_j}{1+p_j} \leq 2 + \frac{p_1}{1+p_1}$.*

As a consequence of Theorem B we obtain the following result, which is an improvement over Theorem A.

**Theorem C.** *Let $f$ and $g$ be two non-constant meromorphic functions. Suppose that there exist distinct elements $a_1, a_2, \dots, a_5$ in $\mathbb{C} \cup \{\infty\}$ such that $\overline{E}_{p_j)}(a_j; f) = \overline{E}_{p_j)}(a_j; g)$ for $j = 1, 2, \dots, 5$, where $p_1, p_2, \dots, p_5$ are positive integers or $\infty$ with $p_1 \geq p_2 \geq \cdots \geq p_5$. If $p_3 \geq 3$ and $p_5 \geq 2$, then $f \equiv g$.*

C. C. Yang [8, p. 157] improved Theorem A by considering partial sharing of values and proved the following theorem.

**Theorem D** ([8, p. 157]). *Let $f$ and $g$ be two non-constant meromorphic functions such that $\overline{E}_{\infty)}(a_j; f) \subset \overline{E}_{\infty)}(a_j; g)$ for five distinct elements $a_1, a_2, \dots, a_5$ of $\mathbb{C} \cup \{\infty\}$. If $\liminf_{r\to\infty} \frac{\sum_{j=1}^{5} \overline{N}(r,a_j;f)}{\sum_{j=1}^{5} \overline{N}(r,a_j;g)} > \frac{1}{2}$, then $f \equiv g$.*

In 2000 Y. Li and J. Qiao [5] improved Theorem A by considering shared small functions instead of shared values. Their result may be stated as follows:

**Theorem E.** *Let $f$, $g$ be non-constant meromorphic functions and $a_j \in S(f) \cap S(g)$ be distinct for $j = 1, 2, \dots, 5$. If $\overline{E}_{\infty)}(a_j; f) = \overline{E}_{\infty)}(a_j; g)$ for $j = 1, 2, \dots, 5$, then $f \equiv g$.*

In 2007 T. B. Cao and H. X. Yi [1] further improved Theorem E and also improved a result of D. D. Thai and T. V. Tan [6]. Following is the result of Cao and Yi.

**Theorem F** ([1]). *Let $f$ and $g$ be two non-constant meromorphic functions and $a_j \in S(f) \cap S(g)$ be distinct for $j = 1, 2, \dots, k$. Suppose further that $p_1, p_2, \dots, p_k$ be positive integers or $\infty$ such that $p_1 \geq p_2 \geq \cdots \geq p_k$ and $\overline{E}_{p_j)}(a_j; f) = \overline{E}_{p_j)}(a_j; g)$ for $j = 1, 2, \dots, k$. Then $f \equiv g$, if one of the following holds*: (i) *$k = 7$,* (ii) *$k = 6$ and $p_3 \geq 2$,* (iii) *$k = 5$, $p_3 \geq 3$ and $p_5 \geq 2$,* (iv) *$k = 5$ and $p_4 \geq 4$,* (v) *$k = 5$, $p_3 \geq 5$ and $p_4 \geq 3$,* (vi) *$k = 5$, $p_3 \geq 6$ and $p_4 \geq 2$.*

In the same year T. G. Chen, K. Y. Chen and Y. L. Tsai [2] improved Theorem D in the following manner.

**Theorem G** ([2]). *Let $f$ and $g$ be two non-constant meromorphic functions such that $\overline{E}_{\infty)}(a_j; f) \subset \overline{E}_{\infty)}(a_j; g)$ for distinct elements $a_1, a_2, \dots, a_k$ $(k \geq 5)$ of $S(f) \cap S(g)$. If $\liminf_{r\to\infty} \frac{\sum_{j=1}^{k} \overline{N}(r,a_j;f)}{\sum_{j=1}^{k} \overline{N}(r,a_j;g)} > \frac{1}{k-3}$, then $f \equiv g$.*

In the paper we prove the following theorem which includes all the above mentioned results.

**Theorem 1.1.** *Let $f$, $g$ be two non-constant meromorphic functions and $a_j = a_j(z) \in S(f) \cap S(g)$ be distinct for $j = 1, 2, \ldots, k$ $(k \geq 5)$. Suppose that $p_1 \geq p_2 \geq \cdots \geq p_k$ are positive integers or infinity and $\delta(\geq 0)$ is such that*

$$\frac{1}{p_1} + \left(1 + \frac{1}{p_1}\right) \sum_{j=2}^{k} \frac{1}{1 + p_j} + 1 + \delta < (k-2)\left(1 + \frac{1}{p_1}\right).$$

*Let $A_j = \overline{E}_{p_j)}(a_j; f) \backslash \overline{E}_{p_j)}(a_j; g)$ for $j = 1, 2, \ldots, k$. If $\sum_{j=1}^{k} \overline{N}_{A_j}(r, a_j; f) \leq \delta T(r, f)$ and*

$$\begin{aligned} &\liminf_{r\to\infty} \frac{\sum_{j=1}^{k} \overline{N}_{p_j)}(r, a_j; f)}{\sum_{j=1}^{k} \overline{N}_{p_j)}(r, a_j; g)} \\ &> \frac{p_1}{(1+p_1)(k-2) - p_1(1+\delta) - 1 - (1+p_1)\sum_{j=2}^{k} \frac{1}{1+p_j}}, \end{aligned}$$

*then $f \equiv g$.*

After the discovery of the second fundamental theorem for moving targets by K. Yamanoi [7], it becomes indispensable for proving "Five Value" type uniqueness theorems for shared small functions. So we mention below the result of Yamanoi.

**Lemma 1.1.** *Let $f$ be a non-constant meromorphic function and $a_j \in S(f)$ be distinct for $j = 1, 2, \ldots, k$. Then for any $\varepsilon(> 0)$*

$$(k - 2 - \varepsilon)T(r, f) \leq \sum_{j=1}^{k} \overline{N}(r, a_j; f) + S(r, f).$$

## 2. Proof of Theorem 1.1

*Proof.* Let $f \not\equiv g$. Then by Lemma 1.1 we get for $\varepsilon(> 0)$

$$\begin{aligned} (k-2-\varepsilon)T(r,f) &\leq \sum_{j=1}^{k} \overline{N}(r, a_J; f) + S(r, f) \\ &= \sum_{j=1}^{k} \left\{ \overline{N}_{p_j)}(r, a_j; f) + \overline{N}_{(p_j+1}(r, a_j; f) \right\} + S(r, f) \\ &\leq \sum_{j=1}^{k} \left\{ \overline{N}_{p_j)}(r, a_j; f) + \frac{1}{1+p_j} N_{(p_j+1}(r, a_j; f) \right\} + S(r, f) \\ &\leq \sum_{j=1}^{k} \left\{ \frac{p_j}{1+p_j} \overline{N}_{p_j)}(r, a_j; f) + \frac{1}{1+p_j} N(r, a_j; f) \right\} + S(r, f) \end{aligned}$$

$$\leq \sum_{j=1}^{k} \frac{p_j}{1+p_j} \overline{N}_{p_j)}(r, a_j; f) + \left( \sum_{j=1}^{k} \frac{1}{1+p_j} \right) T(r, f) + S(r, f)$$

i.e.,

$$\left\{ k - 2 - \sum_{j=1}^{k} \frac{1}{1+p_j} - \varepsilon + o(1) \right\} T(r, f) \leq \sum_{j=1}^{k} \frac{p_j}{1+p_j} \overline{N}_{p_j)}(r, a_j; f). \tag{2.1}$$

Similarly

$$\left\{ k - 2 - \sum_{j=1}^{k} \frac{1}{1+p_j} - \varepsilon + o(1) \right\} T(r, g) \leq \sum_{j=1}^{k} \frac{p_j}{1+p_j} \overline{N}_{p_j)}(r, a_j; g). \tag{2.2}$$

Let $B_j = \overline{E}_{p_j)}(a_j; f) \backslash A_j$ for $j = 1, 2, \ldots, k$. Now using (2.1) and (2.2) we get for a sequence of values of $r$ tending to $+\infty$

$$\begin{aligned} \sum_{j=1}^{k} \overline{N}_{p_j)}(r, a_j; f) &= \sum_{j=1}^{k} \overline{N}_{A_j}(r, a_j; f) + \sum_{j=1}^{k} \overline{N}_{B_j}(r, a_j; f) \\ &\leq \delta T(r, f) + N(r, 0; f - g) \\ &\leq (1+\delta) T(r, f) + T(r, g) + O(1) \end{aligned}$$

i.e.,

$$\left\{ k - 2 - \sum_{j=1}^{k} \frac{1}{1+p_j} - \varepsilon + o(1) \right\} \sum_{j=1}^{k} \overline{N}_{p_j)}(r, a_j; f)$$

$$\leq (1+\delta) \sum_{j=1}^{k} \frac{p_j}{1+p_j} N_{p_j)}(r, a_j; f) + \{1 + o(1)\} \sum_{j=1}^{k} \frac{p_j}{1+p_j} N_{p_j)}(r, a_j; g). \tag{2.3}$$

Since $1 \geq \frac{p_1}{1+p_1} \geq \frac{p_2}{1+p_2} \geq \cdots \geq \frac{p_k}{1+p_k} \geq \frac{1}{2}$, we get from (2.3) for a sequence of values of $r$ tending to $+\infty$

$$\left\{ k - 2 - \sum_{j=1}^{k} \frac{1}{1+p_j} - \varepsilon - \frac{(1+\delta)p_1}{1+p_1} + o(1) \right\} \sum_{j=1}^{k} \overline{N}_{p_j)}(r, a_j; f)$$

$$\leq \{1 + o(1)\} \frac{p_1}{1+p_1} \sum_{j=1}^{k} \overline{N}_{p_j)}(r, a_j; g).$$

Since $\varepsilon(>0)$ is arbitrary, this implies

$$\begin{aligned} &\liminf_{r \to \infty} \frac{\sum_{j=1}^{k} \overline{N}_{p_j)}(r, a_j; f)}{\sum_{j=1}^{k} \overline{N}_{p_j)}(r, a_j; g)} \\ &\leq \frac{p_1}{(1+p_1)(k-2) - p_1(1+\delta) - 1 - (1+p_1)\sum_{j=2}^{k} \frac{1}{1+p_j}}, \end{aligned}$$

which is a contradiction. Therefore $f \equiv g$. This proves the theorem. $\square$

## 3. Consequences of Theorem 1.1

In this section we discuss some consequences of Theorem 1.1.

**Corollary 3.1.** *Let* $p_k = \infty$ *and* $A = \liminf_{r\to\infty} \frac{\sum_{j=1}^{k} \overline{N}(r,a_j;f)}{\sum_{j=1}^{k} \overline{N}(r,a_j;g)} > \frac{1}{k-3}$. *If* $\sum_{j=1}^{k} \overline{N}_{A_j}(r,a_j;f) \le \delta T(r,f)$ *for some* $\delta$, $0 \le \delta < k-3-\frac{1}{A}$, *then* $f \equiv g$.

If we suppose $\overline{E}_{\infty)}(a_j;f) \subset \overline{E}_{\infty)}(a_j;f)$ for $j = 1,2,\ldots,k$, then $A_j = \emptyset$ for $j = 1,2,\ldots,k$ and so we can choose $\delta = 0$. Therefore Corollary 3.1 is an improvement over Theorem G.

**Corollary 3.2.** *Let* $f$, $g$ *be distinct non-constant meromorphic functions. If there exist distinct members* $a_1, a_2, \ldots, a_k$ *of* $S(f) \cap S(g)$ *such that* $\overline{E}_{p_j)}(a_j;f) \subset \overline{E}_{p_j)}(a_j;g)$ *for* $j = 1,2,\ldots,k$, *where* $p_1, p_2, \ldots, p_k$ *are positive integers or* $\infty$ *with* $p_1 \ge p_2 \ge \cdots \ge p_k$, *then* $\sum_{j=2}^{k} \frac{p_j}{1+p_j} \le \frac{p_1}{A(1+p_1)} + 2$, *where* $A = \liminf_{r\to\infty} \frac{\sum_{j=1}^{k} \overline{N}_{p_j)}(r,a_j;f)}{\sum_{j=1}^{k} \overline{N}_{p_j)}(r,a_j;g)}$.

*Proof.* Since $A_j = \emptyset$ for $j = 1,2,\ldots,k$, putting $\delta = 0$ we get from Theorem 1.1

$$\frac{p_1}{(1+p_1)(k-2) - p_1 - 1 - (1+p_1)\sum_{j=2}^{k} \frac{1}{1+p_j}} \ge A$$

and so $\sum_{j=2}^{k} \frac{p_j}{1+p_j} \le \frac{p_1}{A(1+p_1)} + 2$. This proves the corollary. $\square$

If we suppose that $\overline{E}_{p_j)}(a_j;f) = \overline{E}_{p_j)}(a_j;g)$, then $A = 1$ and so Corollary 3.2 improves Theorem B. If, further, we suppose that $k = 5$, $p_3 \ge 3$ and $p_5 \ge 2$, then $\sum_{j=1}^{5} \frac{p_j}{1+p_j} > \frac{p_1}{1+p_1} + 2$ and so by Corollary 3.2 we get $f \equiv g$. Hence Corollary 3.2 improves Theorem C. Also if we put $p_1 = p_2 = \cdots = p_5 = \infty$, then Theorem E follows from Corollary 3.2. Under the hypotheses of Theorem F we see that $A = 1$, $\sum_{j=2}^{k} \frac{p_j}{1+p_j} > \frac{p_1}{1+p_1} + 2$ and so by Corollary 3.2 we get $f \equiv g$. So Corollary 3.2 includes Theorem F.

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