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Estimation for scale parameter of type-I extreme value distribution

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Abstract

In a various range of applications including hydrology, the type-I extreme value distribution has been extensively used as a probabilistic model for analyzing extreme events. In this paper, we introduce methods for estimating the scale parameter of the type-I extreme value distribution. A simulation study is performed to compare the estimators in terms of mean-squared error and bias, and the obtained results are provided.

Keywords: Bias, generalized probability weighted moments, maximum entropy, maximum likelihood, mean-squared error, probability weighted moments, type-I extreme value distribution.

1. Introduction

Models in extreme value theory are concerned for the statistical behavior of M_n, where M_n is a sequence of independent random variables $X_1, X_2, \ldots, X_n$ with a common distribution function F. In applications, X_i's usually represent values of a process measured on a regular time-scale or daily mean temperature so that M_n represents the maximum of the process over n time units of observations. For all values of n, the distribution of M_n can be derived exactly as follows: $P\left(M_n \le z\right) = P\left(X_1 \le z, \ldots, X_n \le z\right) = \left\{F\left(z\right)\right\}^n$. However, this is not helpful in practice since F is unknown in many cases. An alternative approach is to find limit distributions for M_n^* rather than M_n, considering the normalized variable $M_n^* = \left(M_n - a_n\right)/b_n$ for sequences of constants $\{a_n > 0\}$ and $\{b_n > 0\}$. If there exist sequences $\{a_n\}$ and $\{b_n\}$ such that $P\left(M_n \le z\right) \to G\left(z\right)$ as $n \to \infty$, where G is a non-degenerate distribution function, G belongs to one of the extreme value distributions with type-I, type-II, and type-III, regardless of F.

The type-I extreme value (EV1) distribution known as the Gumbel distribution has been extensively used in various research fields such as life testing, water resource management and hydrology for statistically modeling extreme values. See Hershfield and Kohler (1960), Stol (1971), Lambert and Duan (1994), and for a comprehensive review of applications, refer to Johnson *et al.* (1995). The distribution function of the EV1 distribution is defined by

$$F\left(x;\mu,\beta\right) = \exp\left\{-\exp\left(-\frac{x-\mu}{\beta}\right)\right\},\ -\infty < x < \infty,\ -\infty < \mu < \infty,\ \beta > 0, \qquad (1.1)$$

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where μ and β are location and scale parameters, respectively. Differentiating equation (1.1) with respect to x gives the probability density function

$$f(x;\mu,\beta) = \frac{1}{\beta}\exp\left(-\frac{x-\mu}{\beta}\right)\exp\left\{-\exp\left(-\frac{x-\mu}{\beta}\right)\right\}, \tag{1.2}$$

where $-\infty < x < \infty$, $-\infty < \mu < \infty$ and $\beta > 0$.

In applications, one may be interested in estimating the scale parameter of the EV1 distribution. For instance, the entropy introduced by Shannon (1948) as a measure of uncertainty is commonly used in information-theoretic statistics to develop a diagnostic tool for model building or model testing. The entropy of the EV1 distribution is given by $H(f) = \log\beta + \gamma + 1$, where γ is Euler's constant. It relies on having an acceptable statistical estimator of the scale parameter whether a developed statistical procedure based on $H(f)$ for model diagnostic is successful or not, and therefore the scale parameter needs to be estimated precisely, accurately and efficiently. Various methods have been proposed for estimating the scale parameter and can be found in the literature. See Gumbel (1958) for the method of moments, Kimball (1949) for the method of maximum likelihood, Landwehr *et al.* (1979) for the method of probability weighted moments and Jowitt (1979) for the method based on the principle of maximum entropy. Nam and Kang (2014) discussed on estimation for the extreme value distribution under progressive type-I interval censoring.

Recently, Fiorentino and Gabriele (1984) pointed out that the maximum likelihood estimator of the scale parameter is biased, and discussed how to reduce its bias. Through a small scale of simulation, they showed that the suggested bias-corrected maximum likelihood estimator is nearly unbiased and that its mean-squared error is close to that of the maximum likelihood estimator. They also compared the mean-squared error performance of the suggested estimator with the moment estimator, the maximum entropy estimator and the classical probability weighted moments estimator based on Greenwood *et al.* (1979)'s estimator. The simulation results reported that the bias-corrected maximum likelihood estimator has better performance than its competitors. Rasmussen and Gautam (2003) introduced an estimation method for the location and scale parameters of the EV1 distribution, based on the generalized probability weighted moments and used them to get an quantile estimator of the EV1 distribution. They performed a small scale of simulation to compare the performance of the suggested quantile estimator with that of some quantile estimators, which are based on the moment estimators, the maximum likelihood estimators and the probability weighted moments estimators obtained by applying Gringorten (1963)'s formula for the parameters of the EV1 distribution. The simulated results reported that Rasmussen and Gautam (2003)'s quantile estimator has slightly better performance than the quantile estimator based on the probability weighted moments estimators obtained by applying Gringorten (1963)'s formula, but that it is slightly inferior to its competitor based on the maximum likelihood estimators. However, Rasmussen and Gautam (2003) did not make a trial of comparing the performance of the generalized probability weighted moments estimator of the individual EV1 parameter with that of all available estimators.

In this paper, we consider all estimators available for the scale parameter of the EV1 distribution and investigate their performance in terms of mean-squared error and bias by means of Monte Carlo simulation. This paper is organized as follows. In Section 2, the estimators for the scale parameter of the type-I extreme distribution are introduced. In

Section 3, Monte Carlo simulations are performed to compare the estimators in terms of mean-squared error and bias. In Section 4, some brief conclusions are provided.

2. Estimation for scale parameter

2.1. Method of moments

Let X be a random variable following the EV1 distribution. The moments of orders 1 and 2, μ_1 and μ_2, of X are used to get an estimator of β. The moment-generating function of X is given by $M_X(t) = \Gamma(1-\beta t)\, e^{\mu t}$. From this, the moment of order 1 is easily obtained as

$$\mu_1 = M'_X(0) = \mu - \beta\ \Gamma'(1) = \mu + \gamma\beta, \tag{2.1}$$

where $\Gamma'(1) = \psi(1) = -\gamma$ and ψ is the digamma function given by $\psi(n) = \Gamma(n)'/\Gamma(n)$. By noting that

$$M''_X(t) = \beta^2\Gamma''(1-\beta t)\, e^{\mu t} - 2\mu\beta\Gamma'(1-\beta t)\, e^{\mu t} + \mu^2\Gamma(1-\beta t)\, e^{\mu t} \tag{2.2}$$

and using the trigamma function defined by $\psi'(n) = d\psi(n)/dn$, the moment of order 2 is also obtained as

$$\begin{aligned}\mu_2 = M''_X(0) &= \beta^2\Gamma''(1) + 2\beta\gamma\mu + \mu^2 \\ &= \beta^2\{\psi'(1) + \gamma^2\} + 2\beta\gamma\mu + \mu^2,\end{aligned} \tag{2.3}$$

where $\Gamma''(1) = \psi'(1) + \gamma$. From (2.1) and (2.3), we can easily derive the following equation

$$\mu_2 - \mu_1^2 = \beta^2\psi'(1). \tag{2.4}$$

and solving this equation for β gives

$$\beta = \left\{\frac{\mu_2 - \mu_1^2}{\psi'(1)}\right\}^{1/2}. \tag{2.5}$$

To estimate μ_2 and μ_1, the sample moments obtained from a sample $X_1, X_2, \ldots, X_n$, $\bar{X^2} = \sum_{i=1}^n X_i^2/n$ and $\bar{X} = \sum_{i=1}^n X_i/n$, are used. By replacing μ_2 and μ_1 with the corresponding sample moment, β can be estimated by

$$\begin{aligned}\hat\beta_{MM} &= \left\{\frac{\bar{X^2} - \bar{X}}{\psi'(1)}\right\}^{1/2} \\ &= \left\{\frac{S_n^2}{\psi'(1)}\right\}^{1/2},\end{aligned} \tag{2.6}$$

where $S_n^2 = \sum_{i=1}^n \left(X_i - \bar{X}\right)^2/n = \bar{X^2} - \bar{X}$. The calculation of $\hat\beta_{MM}$ is simple, but underestimates β.

2.2. Method of maximum likelihood

The log-likelihood function based on a sample $X_1, X_2, \ldots, X_n$ drawn from the EV1 distribution can be written as

$$l(\mu,\beta) = -\sum_{i=1}^n \frac{x_i - \mu}{\beta} - n\log\ \beta - \sum_{i=1}^n \exp\left(-\frac{x_i-\mu}{\beta}\right). \tag{2.7}$$

Differentiating the function (2.7) with respect to μ and β yields the following likelihood equations:

$$\frac{\partial l\left(\mu,\beta\right)}{\partial\mu} = \frac{1}{\beta}\left\{n - \sum_{i=1}^{n}\exp\left(-\frac{x_i-\mu}{\beta}\right)\right\}, \tag{2.8}$$

$$\frac{\partial l\left(\mu,\beta\right)}{\partial\beta} = \frac{1}{\beta}\left\{\sum_{i=1}^{n}\frac{x_i-\mu}{\beta} - n - \frac{1}{\beta}\sum_{i=1}^{n}\frac{x_i-\mu}{\beta}\exp\left(-\frac{x_i-\mu}{\beta}\right)\right\}. \tag{2.9}$$

The solutions of satisfying $\partial l\left(\mu,\beta\right)/\partial\mu = 0$ and $\partial l\left(\mu,\beta\right)/\partial\beta = 0$ are the maximum likelihood estimator of μ and β.

From the likelihood equation for μ, $\partial l\left(\mu,\beta\right)/\partial\mu = 0$, it can be obtained that $\exp\left(\mu/\beta\right) = n/\sum_{i=1}^{n}\exp\left(-x_i/\beta\right)$. Substituting this quantity into (2.9) and solving $\partial l\left(\mu,\beta\right)/\partial\beta = 0$ give the following equation containing β only

$$\bar{x} = \beta + \frac{\sum_{i=1}^{n} x_i \exp\left(-x_i/\beta\right)}{\sum_{i=1}^{n}\exp\left(-x_i/\beta\right)}. \tag{2.10}$$

The solution satisfying the equation (2.10) becomes the maximum likelihood estimator $\hat{\beta}_{ML}$ of β. A shortcoming of the method of maximum likelihood is that $\hat{\beta}_{ML}$ can be obtained by employing iterative computational procedures such as the Newton-Raphson method. Also, $\hat{\beta}_{ML}$ is biased for β.

Fiorentino and Gabriele (1984) discussed how to reduce the bias of $\hat{\beta}_{ML}$. The resulting bias-corrected maximum likelihood estimator of β is given by

$$\hat{\beta}_{CML} = \frac{n\hat{\beta}_{ML}}{n-0.8}, \tag{2.11}$$

where n is sample size.

2.3. Method of maximum entropy

Jowitt (1979) discussed a method for estimating the parameters of the EV1 distribution based on the principle of maximum entropy. In this method, two parameters μ and β should be selected to produce $E\big\{\left(X-\mu\right)/\beta\big\} = \gamma$ and $E\left[\exp\big\{-\left(X-\mu\right)/\beta\big\}\right] = 1$, where γ is Euler's constant given by $\gamma = -\psi(1)$.

For estimation, the expectations are replaced by the corresponding estimators obtained from a sample. The resulting equations are given by

$$\frac{1}{n}\sum_{i=1}^{n}\frac{X_i-\mu}{\beta} = \gamma, \quad \frac{1}{n}\sum_{i=1}^{n}\exp\left(-\frac{X_i-\mu}{\beta}\right) = 1. \tag{2.12}$$

The estimator $\hat{\beta}_{ME}$ of β is obtained by solving the equation (2.12). As in the method of maximum likelihood, iterative computational procedures are needed to get the solution for β. It is a disadvantage of this method. $\hat{\beta}_{ME}$ is also biased.

2.4. Method of probability weighted moments

This method proposed by Greenwood *et al.* (1979) is widely used for estimating parameters of distributions. The probability weighted moments of a random variable X with a distribution function F are defined by

$$M_{r,s,t} = E\{X^r F^s (1-F)^t\}, \tag{2.13}$$

where r, s, and t are real numbers. If F can be expressed in inverse form, the expression (2.13) can be written as

$$M_{r,s,t} = \int_0^1 x(F)^r F^s (1-F)^t \, dF, \tag{2.14}$$

where $x(F)$ is the inverse function of F. As in the method of moments, the estimators for parameters are obtained by equating the analytical expressions for $M_{r,s,t}$ to sample moments. In practice, special cases of $M_{r,s,t}$ are commonly considered and the typical choice is to use $\eta_t = M_{1,0,t} = E\{X(1-F)^t\}$ or $\theta_s = M_{1,s,0} = E(XF^s)$.

The probability weighted moments for the EV1 distribution can be derived as follows. By using $F(x;\mu,\beta)$ given in (1.1) as F, the inverse function of F can be expressed as

$$x(F) = \mu - \beta\{\log(-\log F)\}. \tag{2.15}$$

The kth order probability weighted moment is obtained as

$$\theta_k = \int_0^1 x(F) F^k \, dF = \frac{1}{k+1}\left[\mu + \beta\{\gamma + \log(k+1)\}\right] \tag{2.16}$$

using subsequently the change of variables $y = -\log F$ and $z = (k+1)y$. In a similar manner, lth order probability weighted moment θ_l can be obtained. By taking the ratio of the resulting two moment equations, θ_k and θ_l, the equation for β can be derived as

$$\beta = \frac{(k+1)\theta_k - (l+1)\theta_l}{\log(k+1) - \log(l+1)}. \tag{2.17}$$

Therefore, an estimator of β is obtained by replacing θ_k and θ_l by their estimator.

The estimation of θ_k based on a sample has been discussed by Greenwood *et al.* (1979), Landwehr *et al.* (1979), Hosking *et al.* (1985), and Hosking and Wallis (1995). An estimator of θ_k proposed by Greenwood *et al.* (1979), when k is a nonnegative integer, is defined by

$$\hat{\theta}_k = \frac{1}{n}\sum_{i=1}^{n} X_{(i)} \binom{n-i}{k} \Big/ \binom{n-1}{k}, \tag{2.18}$$

where $X_{(i)}$, $i = 1,\ldots,n$, is the ordered sample of $X_1, X_2, \ldots, X_n$. Landwehr *et al.* (1979) showed that $\hat{\theta}_k$ is unbiased for θ_k. Hosking *et al.* (1985) proposed another estimators of θ_k and these estimators are given by

$$\hat{\theta}_{k,1} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i-a}{n}\right)^k X_{(i)}, \tag{2.19}$$

where $0 < a < 1$ and

$$\hat{\theta}_{k,2} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i-a}{n+1-2a}\right)^k X_{(i)}, \tag{2.20}$$

where $-1/2 < a < 1/2$. Hosking *et al.* (1985) proved that $\hat{\theta}_{k,1}$ and $\hat{\theta}_{k,2}$ are consistent for θ_k by showing that the estimators (2.19) and (2.20) are asymptotically equivalent to $\hat{\theta}_k$.

It is arbitrary to choose values of k and l used to estimate β based on the equation (2.17). The common practice is to set $k = 1$ and $l = 0$, and the equation of β is expressed as

$$\beta = \frac{2\theta_1 - \theta_0}{\log 2}. \tag{2.21}$$

The estimators of θ_0 and θ_1 obtained from (2.18) are given by

$$\hat{\theta}_0 = \bar{X}, \tag{2.22}$$

$$\hat{\theta}_1 = \frac{1}{n\,(n-1)} \sum_{i=1}^{n} (i-1)\, X_{(i)}. \tag{2.23}$$

Substituting these estimators into (2.21) produces the probability weighted moments estimator of β

$$\hat{\beta}_{PWM} = \frac{2\hat{\theta}_1 - \hat{\theta}_0}{\log 2}. \tag{2.24}$$

In case of using (2.19) or (2.20), the estimator of θ_0 can be obtained by $\hat{\theta}_{0,1}$ or $\hat{\theta}_{0,2}$. These two estimators are equivalent to sample mean $\bar{X}$. The estimator of θ_1, by putting $k = 1$ in (2.19) or (2.20), can be also obtained by $\hat{\theta}_{1,1}$ or $\hat{\theta}_{1,2}$. Using these estimators, the parameter β can be estimated as

$$\hat{\beta}^1_{PWM} = \frac{2\hat{\theta}_{1,1} - \hat{\theta}_{0,1}}{\log 2}, \tag{2.25}$$

$$\hat{\beta}^2_{PWM} = \frac{2\hat{\theta}_{1,2} - \hat{\theta}_{0,2}}{\log 2}. \tag{2.26}$$

Rasmussen and Gautam (2003) suggested the generalized method of probability weighted moments to estimate the scale parameter of the EV1 distribution. The generalized probability weighted moments estimator of β is obtained that for $-1/2 < a < 1/2$ and a real number m,

$$\begin{aligned} \hat{\beta}_{GPWM} &= \frac{1+m}{n} \sum_{i=1}^{n} \left[\left(\frac{i-a}{n+1-2a} \right)^{m} \left\{ 1 + (1+m) \log \frac{i-a}{n+1-2a} \right\} X_{(i)} \right] \\ &= (1+m)\, \hat{\theta}_{m,2} + (1+m)^2\, \hat{\theta}_{\log,m}, \end{aligned} \tag{2.27}$$

where

$$\hat{\theta}_{\log,m} = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{i-a}{n+1-2a} \right)^{m} \left(\log \frac{i-a}{n+1-2a} \right) X_{(i)}. \tag{2.28}$$

The optimal value of m is independent of the parent distribution, but depends on sample size. Based on simulation results that the values of m that yield minimum mean-squared error decrease with increasing sample size, they recommended the use of the following decision rule $m = 1.9/n$.

The estimators given in (2.24), (2.25), (2.26) and (2.27) have simplicity and robustness, and can be easily calculated without resort to iterative computational procedures; however, the remaining 3 estimators except for (2.24) are biased.

3. Simulation results

In this section, we carry out Monte Carlo simulation to compare the performance of the estimators in terms of mean-squared error and bias. In the simulation, sample size is selected as $n = 10, 20, 30, 50$. Values of μ and β are taken as $\mu = 0, 1, 2, 4$ and $\beta = 0.5, 1, 1.5, 2, 2.5, 3$. For each combination of values of n, μ and β, 10000 samples are generated from an EV1 distribution. For each sample size, 8 estimators of β given in the previous section are calculated from the samples. For the calculation of $\hat{\beta}^1_{PWM}$, $\hat{\beta}^2_{PWM}$ and $\hat{\beta}_{GPWM}$, a choice of a suitable value of a should be made. For $\hat{\beta}^1_{PWM}$, $a = 0.35$ was used according to Hosking *et al.* (1985)'s suggestion. In case of $\hat{\beta}^2_{PWM}$ and $\hat{\beta}_{GPWM}$, $a = 0.44$ was selected, which is commonly used as a value of a for the EV1 distribution by Guo (1990). The mean-squared error and bias of each estimator are estimated using calculated values of the estimators.

Table 3.1 Mean-squared error of the estimators calculated based on 10000 samples generated from the EV1 distribution with $\mu = 0, 1$

μ	β	n	$\hat{\beta}_{MM}$	$\hat{\beta}_{ML}$	$\hat{\beta}_{CML}$	$\hat{\beta}_{ME}$	$\hat{\beta}_{PWM}$	$\hat{\beta}^1_{PWM}$	$\hat{\beta}^2_{PWM}$	$\hat{\beta}_{GPWM}$
0	0.5	10	0.0229	0.0166	0.0180	0.0173	0.0220	0.0205	0.0202	0.0202
		20	0.0122	0.0080	0.0082	0.0083	0.0104	0.0101	0.0101	0.0095
		30	0.0086	0.0053	0.0054	0.0056	0.0070	0.0069	0.0069	0.0062
		50	0.0052	0.0032	0.0032	0.0033	0.0041	0.0041	0.0041	0.0036
	1	10	0.0923	0.0675	0.0728	0.0700	0.0881	0.0828	0.0819	0.0820
		20	0.0493	0.0322	0.0336	0.0338	0.0425	0.0410	0.0407	0.0379
		30	0.0341	0.0209	0.0213	0.0222	0.0279	0.0274	0.0273	0.0246
		50	0.0202	0.0121	0.0123	0.0128	0.0160	0.0158	0.0158	0.0139
	1.5	10	0.2046	0.1491	0.1603	0.1545	0.1938	0.1826	0.1814	0.1816
		20	0.1109	0.0734	0.0759	0.0766	0.0948	0.0922	0.0917	0.0864
		30	0.0749	0.0470	0.0481	0.0494	0.0616	0.0605	0.0603	0.0548
		50	0.0469	0.0279	0.0283	0.0295	0.0367	0.0364	0.0363	0.0319
	2	10	0.3773	0.2727	0.2940	0.2851	0.3613	0.3377	0.3334	0.3305
		20	0.1911	0.1235	0.1276	0.1300	0.1633	0.1584	0.1577	0.1478
		30	0.1357	0.0845	0.0863	0.0891	0.1110	0.1091	0.1088	0.0987
		50	0.0835	0.0488	0.0496	0.0516	0.0651	0.0642	0.0639	0.0553
	2.5	10	0.5763	0.4160	0.4502	0.4348	0.5530	0.5174	0.5109	0.5063
		20	0.2989	0.1990	0.2041	0.2064	0.2546	0.2489	0.2485	0.2363
		30	0.2161	0.1307	0.1337	0.1386	0.1748	0.1710	0.1701	0.1527
		50	0.1301	0.0781	0.0792	0.0822	0.1026	0.1014	0.1012	0.0885
	3	10	0.8053	0.5838	0.6262	0.6071	0.7662	0.7207	0.7163	0.7165
		20	0.4351	0.2838	0.2921	0.2971	0.3709	0.3611	0.3597	0.3386
		30	0.3074	0.1897	0.1942	0.2002	0.2510	0.2462	0.2451	0.2209
		50	0.1892	0.1146	0.1159	0.1204	0.1485	0.1470	0.1466	0.1296
1	0.5	10	0.0231	0.0166	0.0178	0.0173	0.0219	0.0192	0.0204	0.0149
		20	0.0123	0.0080	0.0083	0.0084	0.0105	0.0098	0.0101	0.0078
		30	0.0083	0.0053	0.0055	0.0056	0.0069	0.0067	0.0068	0.0053
		50	0.0053	0.0031	0.0032	0.0033	0.0042	0.0041	0.0041	0.0033
	1	10	0.0900	0.0661	0.0706	0.0683	0.0853	0.0758	0.0802	0.0676
		20	0.0499	0.0324	0.0334	0.0340	0.0425	0.0401	0.0411	0.0344
		30	0.0337	0.0212	0.0217	0.0223	0.0278	0.0268	0.0272	0.0226
		50	0.0206	0.0124	0.0125	0.0130	0.0161	0.0157	0.0159	0.0132
	1.5	10	0.2095	0.1522	0.1634	0.1588	0.2010	0.1813	0.1876	0.1655
		20	0.1092	0.0725	0.0756	0.0761	0.0957	0.0906	0.0914	0.0790
		30	0.0743	0.0466	0.0476	0.0489	0.0611	0.0591	0.0598	0.0513
		50	0.0470	0.0284	0.0288	0.0299	0.0374	0.0366	0.0368	0.0309
	2	10	0.3635	0.2671	0.2858	0.2764	0.3435	0.3132	0.3226	0.2966
		20	0.1918	0.1293	0.1334	0.1348	0.1670	0.1595	0.1614	0.1439
		30	0.1313	0.0818	0.0831	0.0860	0.1077	0.1045	0.1052	0.0914
		50	0.0825	0.0501	0.0508	0.0527	0.0653	0.0641	0.0643	0.0550
	2.5	10	0.5891	0.4157	0.4513	0.4378	0.5616	0.5080	0.5137	0.4704
		20	0.3078	0.2009	0.2091	0.2108	0.2654	0.2527	0.2537	0.2257
		30	0.2106	0.1326	0.1366	0.1392	0.1731	0.1678	0.1684	0.1473
		50	0.1301	0.0779	0.0791	0.0821	0.1025	0.1005	0.1007	0.0859
	3	10	0.8040	0.5787	0.6170	0.6022	0.7608	0.7015	0.7167	0.6748
		20	0.4391	0.2905	0.3029	0.3030	0.3788	0.3622	0.3637	0.3268
		30	0.3030	0.1868	0.1898	0.1968	0.2452	0.2393	0.2410	0.2122
		50	0.1871	0.1116	0.1138	0.1181	0.1482	0.1451	0.1449	0.1233

Table 3.2 Mean-squared error of the estimators calculated based on 10000 samples generated from the EV1 distribution with $\mu = 2, 4$

μ	β	n	$\hat{\beta}_{MM}$	$\hat{\beta}_{ML}$	$\hat{\beta}_{CML}$	$\hat{\beta}_{ME}$	$\hat{\beta}_{PWM}$	$\hat{\beta}^1_{PWM}$	$\hat{\beta}^2_{PWM}$	$\hat{\beta}_{GPWM}$
2	0.5	10	0.0233	0.0168	0.0180	0.0175	0.0220	0.0215	0.0206	0.0140
		20	0.0125	0.0081	0.0084	0.0085	0.0106	0.0106	0.0102	0.0076
		30	0.0081	0.0051	0.0052	0.0054	0.0066	0.0066	0.0066	0.0049
		50	0.0052	0.0031	0.0032	0.0033	0.0041	0.0041	0.0041	0.0031
	1	10	0.0895	0.0649	0.0697	0.0675	0.0850	0.0746	0.0796	0.0582
		20	0.0491	0.0318	0.0331	0.0333	0.0418	0.0394	0.0401	0.0311
		30	0.0335	0.0213	0.0217	0.0223	0.0277	0.0265	0.0271	0.0213
		50	0.0206	0.0125	0.0127	0.0132	0.0165	0.0161	0.0161	0.0128
	1.5	10	0.2129	0.1525	0.1646	0.1599	0.2040	0.1791	0.1884	0.1494
		20	0.1106	0.0726	0.0753	0.0762	0.0951	0.0893	0.0915	0.0744
		30	0.0734	0.0469	0.0476	0.0488	0.0598	0.0574	0.0592	0.0488
		50	0.0471	0.0271	0.0275	0.0289	0.0365	0.0356	0.0360	0.0290
	2	10	0.3666	0.2699	0.2892	0.2798	0.3505	0.3111	0.3268	0.2753
		20	0.1979	0.1278	0.1325	0.1343	0.1686	0.1590	0.1623	0.1355
		30	0.1342	0.0820	0.0840	0.0869	0.1095	0.1054	0.1070	0.0885
		50	0.0846	0.0494	0.0502	0.0525	0.0660	0.0644	0.0648	0.0532
	2.5	10	0.5834	0.4216	0.4540	0.4396	0.5556	0.4957	0.5150	0.4456
		20	0.3068	0.1984	0.2046	0.2087	0.2609	0.2474	0.2528	0.2157
		30	0.2117	0.1322	0.1356	0.1391	0.1733	0.1670	0.1684	0.1429
		50	0.1287	0.0775	0.0782	0.0817	0.1015	0.0994	0.1003	0.0842
	3	10	0.8286	0.5893	0.6313	0.6169	0.7839	0.7047	0.7295	0.6428
		20	0.4421	0.2906	0.2997	0.3047	0.3800	0.3612	0.3674	0.3196
		30	0.3053	0.1890	0.1930	0.1991	0.2490	0.2409	0.2436	0.2083
		50	0.1877	0.1125	0.1144	0.1188	0.1482	0.1450	0.1453	0.1221
4	0.5	10	0.0232	0.0166	0.0177	0.0173	0.0219	0.0375	0.0204	0.0252
		20	0.0121	0.0080	0.0083	0.0084	0.0104	0.0143	0.0101	0.0107
		30	0.0082	0.0051	0.0053	0.0054	0.0068	0.0086	0.0066	0.0067
		50	0.0051	0.0030	0.0031	0.0032	0.0040	0.0046	0.0039	0.0037
	1	10	0.0939	0.0671	0.0718	0.0702	0.0886	0.0870	0.0826	0.0564
		20	0.0493	0.0319	0.0329	0.0336	0.0422	0.0419	0.0408	0.0298
		30	0.0350	0.0214	0.0218	0.0227	0.0284	0.0282	0.0278	0.0209
		50	0.0206	0.0122	0.0124	0.0128	0.0160	0.0161	0.0158	0.0122
	1.5	10	0.2040	0.1487	0.1624	0.1548	0.1982	0.1780	0.1811	0.1268
		20	0.1089	0.0727	0.0750	0.0759	0.0940	0.0890	0.0910	0.0681
		30	0.0748	0.0469	0.0479	0.0492	0.0614	0.0591	0.0604	0.0459
		50	0.0480	0.0283	0.0286	0.0301	0.0378	0.0370	0.0374	0.0288
	2	10	0.3602	0.2625	0.2815	0.2722	0.3425	0.2997	0.3188	0.2335
		20	0.2020	0.1301	0.1344	0.1374	0.1722	0.1614	0.1665	0.1279
		30	0.1348	0.0842	0.0861	0.0887	0.1108	0.1062	0.1080	0.0847
		50	0.0842	0.0501	0.0507	0.0531	0.0663	0.0646	0.0652	0.0517
	2.5	10	0.5760	0.4085	0.4343	0.4264	0.5357	0.4686	0.5058	0.3847
		20	0.3072	0.1986	0.2059	0.2081	0.2607	0.2446	0.2514	0.1996
		30	0.2162	0.1282	0.1313	0.1367	0.1738	0.1665	0.1687	0.1331
		50	0.1288	0.0770	0.0777	0.0812	0.1011	0.0986	0.1000	0.0804
	3	10	0.8211	0.5983	0.6423	0.6229	0.7879	0.6915	0.7324	0.5824
		20	0.4417	0.2845	0.2940	0.2993	0.3745	0.3518	0.3626	0.2933
		30	0.3011	0.1910	0.1958	0.2008	0.2498	0.2395	0.2433	0.1988
		50	0.1873	0.1100	0.1113	0.1166	0.1467	0.1431	0.1449	0.1171

Simulated results are provided in Table 3.1-3.2, and the following observations can be drawn from the results:

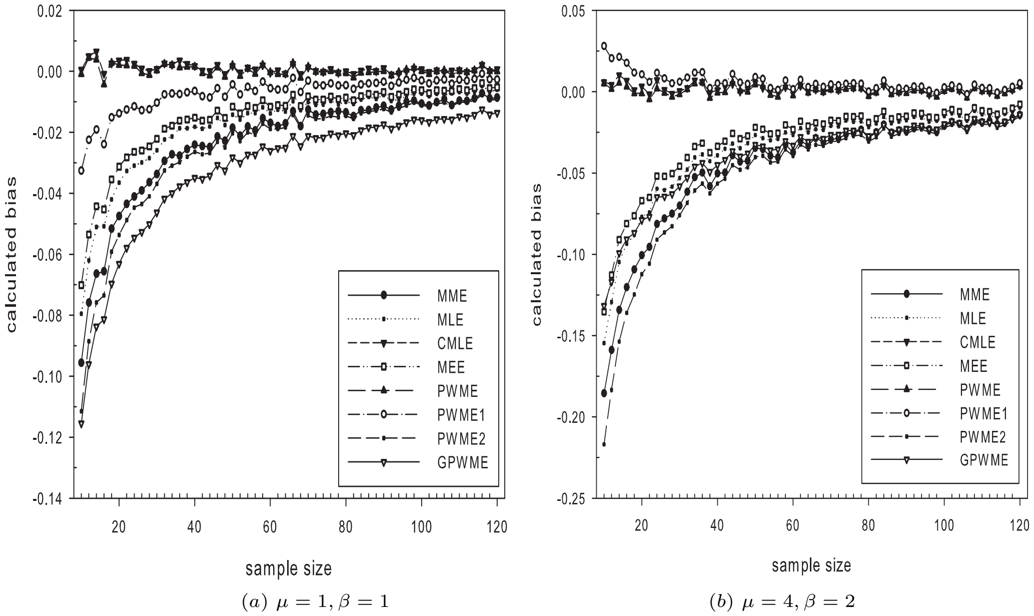
1) For given n and μ, the mean-squared error of the estimators shows a tendency to increase as β increases. Also, large values of both μ and β result in large mean-squared error of the estimators for all values of sample size. For given μ and β, the mean-squared error of the estimators decreases as n increases.

2) The mean-squared error of the moment estimator $\hat{\beta}_{MM}$ is observed to be larger than that of the other estimators for all sample sizes. From it, we can see that $\hat{\beta}_{MM}$ has very poor performance. It is noticeable that the difference of the mean-squared error between $\hat{\beta}_{MM}$ and its competitors diminishes as sample size increases, but on the contrary the performance of $\hat{\beta}_{MM}$ becomes worse since the mean-squared error efficiency of $\hat{\beta}_{MM}$ is observed to be lower for larger values of n.

3) The mean-squared error of the maximum likelihood estimator $\hat{\beta}_{ML}$ shows a tendency to be smaller than that of the other estimators in many cases. In case of $\hat{\beta}_{CML}$, its mean-squared error is nearly the same as that of $\hat{\beta}_{ML}$ and smaller than that of $\hat{\beta}_{ME}$ except for some cases. $\hat{\beta}_{ME}$ produces smaller mean-squared error than $\hat{\beta}_{CML}$ for rather small sample size ($n = 10$). It is noteworthy that $\hat{\beta}_{ML}$ takes larger mean-squared error than $\hat{\beta}_{GPWM}$ when $(\mu, \beta) = (1, 0.5), (2, 0.5), (2, 1), (4, 1), (4, 1.5), (4, 2)$. From the results obtained by the additional simulation works (not presented in this paper), we observed that the mean-squared error of $\hat{\beta}_{ML}$ is larger than that of $\hat{\beta}_{GPWM}$ when $0.25 \leq \beta/\mu < 0.75$. A similar observation is made for $\hat{\beta}_{CML}$ and $\hat{\beta}_{ME}$.

4) The probability weighted moments estimators and the generalized probability weighted moments estimator, generally, yield larger mean-squared error than the maximum likelihood, the bias-corrected maximum likelihood and the maximum entropy estimators. Among $\hat{\beta}_{PWM}$, $\hat{\beta}^1_{PWM}$, $\hat{\beta}^2_{PWM}$ and $\hat{\beta}_{GPWM}$, we can see that $\hat{\beta}_{GPWM}$ produces the smallest mean-squared error. In addition, it appears that $\hat{\beta}_{GPWM}$ shows better performance than $\hat{\beta}_{ML}$ in case where $0.25 \leq \beta/\mu < 0.75$. $\hat{\beta}^1_{PWM}$ has smaller mean-squared error than $\hat{\beta}^2_{PWM}$; however, two estimators show no great difference in mean-squared error, as sample size increases.

5) On the whole, $\hat{\beta}_{ML}$ and $\hat{\beta}_{CML}$ show good performance, and $\hat{\beta}_{ME}$ is comparable to these two estimators. The probability weighted moments estimators do not have better performance than $\hat{\beta}_{ML}$, $\hat{\beta}_{CML}$ and $\hat{\beta}_{ME}$. In case of $\hat{\beta}_{GPWM}$, its performance is observed to be the best of all estimators for $0.25 \leq \beta/\mu < 0.75$. It is also observed that $\hat{\beta}_{MM}$ does not outperform all estimators in all cases.



(a) $\mu = 1, \beta = 1$ (b) $\mu = 4, \beta = 2$

Figure 3.1 Calculated bias of the estimators based on 10000 samples generated from the EV1 distribution

Figure 3.1 displays the bias of 8 estimators obtained by Monte Carlo simulation with 10000 repetitions. The bias values of all estimators show a trend to decrease as sample size increases. For all sample sizes, the bias values of $\hat{\beta}_{CML}$ (CMLE) and $\hat{\beta}_{PWM}$ (PWME) show nearly the same, and the values are observed to be considerably smaller than those of the other estimators. $\hat{\beta}^1_{PWM}$ (PWME1) has larger bias than $\hat{\beta}_{CML}$ (CMLE) and $\hat{\beta}_{PWM}$ (PWME), but produces smaller bias than the remaining 5 estimators. It appears that the bias of $\hat{\beta}_{ME}$ (MEE) is slightly smaller than that of $\hat{\beta}_{ML}$ (MLE). Smaller values of sample size result in large bias of $\hat{\beta}_{ME}$ (MEE) and $\hat{\beta}_{ML}$ (MLE), and a similar observation is made for $\hat{\beta}_{MM}$ (MME), $\hat{\beta}^2_{PWM}$ (PWME2) and $\hat{\beta}_{GPWM}$ (GPWME).

4. Conclusion

In this paper, we introduced the estimators of the scale parameter of the type-I extreme value distribution and examined their performance in terms of mean-squared error and bias through simulation. From the results we have made the following observations:

The moment estimator showed very poor performance across all cases. It appeared that in many cases, the performance of the maximum likelihood and the bias-corrected maximum likelihood estimators is good and nearly the same. The performance of the maximum entropy estimator is compared favorably with that of these two estimators. In general, the probability weighted moments estimators did not outperform their competitors except for the moment estimator. The generalized probability weighted moments estimator by Rasmussen and Gautam (2003) was observed to be better performance than the other probability weighted moments estimators. Moreover, it surpassed all estimators in performance for the case where $0.25 \leq \beta/\mu < 0.75$. The bias-corrected maximum likelihood estimator and the probability weighted moments estimator base on Greenwood *et al.* (1979)'s estimator yielded nearly the same values of bias, and these values appeared considerably smaller than those of the other estimators. The remaining estimators except for these two estimators produced large bias for small sample size, but as sample size increases, their bias showed a trend to diminish to a great extent.

The method of maximum likelihood is the most popular and efficient one for estimating parameters; however, there is no explicit solution for getting the estimator of the scale parameter of the EV1 distribution by solving the likelihood equation. This leads to use iterative computational procedures such as the Newton-Raphson method for computation. Further, the maximum likelihood estimator does not always surpass all of the estimators of the scale parameter in performance. On the other hand, the generalized probability weighted moments estimator has advantages that it has simplicity and robustness, and that it can be easily calculated without resort to iterative computational procedures. Moreover, the estimator has the superiority in performance over the maximum likelihood estimator for considerable cases. Therefore, we expect that the generalized probability weighted moments estimator can be used in application as an alternative of the maximum likelihood estimator.

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