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# A change point estimator in monitoring the parameters of a multivariate IMA(1, 1) model

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### Abstract

Modern production process is a very complex structure combined observations which are correlated with several factors. When the error signal occurs in the process, it is very difficult to know the root causes of an out-of-control signal because of insufficient information. However, if we know the time of the change, the system can be controlled more easily. To know it, we derive a maximum likelihood estimator (MLE) of the change point in a process when observations are from a multivariate IMA(1,1) process by monitoring residual vectors of the model. In this paper, numerical results show that the MLE of change point is effective in detecting changes in a process.

*Keywords*: Change point, maximum likelihood estimator, multivariate exponentially weighted moving average control chart.

## 1. Introduction

Statistical process control (SPC) charts are utilized for the purpose of monitoring for process changes by finding the special causes of variation. A control chart statistic is compared to one or more control limits with data are gathered from a process. If the control chart statistic exceeds a control limit, then special causes exist in the process. Although the quality of products is characterized by several variables correlated in real industrial processes, most previous studies of process have been implemented in one variable process. If quality characteristics are influenced by several factors, one variable process control chart is useful no longer. The objective of multivariate SPC charts is finding the special causes of variation like those of univariate SPC. The charts send a signal, when the mean vector shifted, or when the covariance matrix perturbed, or when both the mean vector and the covariance matrix are shifted simultaneously.

Hotelling's T2 charts only take into account the present information of the process, thus being little powerful for detecting small changes. To improve efficiency in the case of small changes in the process, mulitivariate the exponentially weighted moving average (EWMA) control chart was developed. The advantage of the chart is that it can give consideration to the present and past information of the process. Therefore, it is more powerful to detect small changes than Hotelling's T2.

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Generally, the fundamental assumption in control charts is that process observations at distinct times are independent. However, this independent assumption is invalid in most cases. Ignoring autocorrelation in observations at distinct times generates a high false alarm rate. Thus, it is wrong that the observations apply to the usual control chart. To solve this problem, Montgomery and Mastrangelo (1991) proposed a method which fitted a proper time series model for original data and then constructed EWMA control chart for the residuals. Padgett *et al.* (1992) studied Shewhart charts when process observations can be modelled as an AR(1) process with random error. Lu and Reynolds (1999) considered the performance of EWMA charts of the residuals and observations.

Pignatiello and Samuel (2001) considered the use of the MLE for the change point instead of the built-in estimator when either a CUSUM chart or EWMA chart issued a signal. Timmer and Pignatiello (2003) argued MLE to determine when the parameters of the a first-order autoregressive (AR(1)) distribution have changed. Lee and Lee (2009) proposed the MLE for the process change point when a control chart is used in monitoring the parameters of a process in which the observations can be modeled as a intergrated first-order moving average IMA(1,1).

In this thesis, we assume that the readings follow independent common multivariate normal distributions with same mean vector ($\boldsymbol{\mu}$) and covariane matrix ($\boldsymbol{\Sigma}$) in control, but out of control the mean vector could change from $\boldsymbol{\mu}$ to $\boldsymbol{\mu}_1$ while the covariance structure remains unchanged. We derive a maximum likelihood estimator (MLE) for the process change point when observations are from a multivariate IMA(1,1) process by monitoring residual vectors of the model. Also, we study the performance of the proposed estimator.

## 2. Multivariate IMA(1,1) model

Box and Kramer (1992) argued that although IMA model is the simplest model among nonstationary models, the model is suitable for the noise model in the process adjustment because of well describing variations of process level.

When $\boldsymbol{X}_t$ is an observation vector at time $t$ in the process, a multivariate IMA(1,1) Model can be represented as

$$\boldsymbol{X}_t = \boldsymbol{X}_t - 1 + \boldsymbol{\epsilon}_t - \boldsymbol{\Theta}\epsilon_t - 1,\ or \tag{2.1}$$

$$(1 - B)\boldsymbol{X}_t = (\boldsymbol{I} - \boldsymbol{\Theta}B)\boldsymbol{\epsilon}_t, \tag{2.2}$$

where B (backshift operator) is $B\boldsymbol{X}_t = \boldsymbol{X}_{t-1}$ and assuming $\boldsymbol{\epsilon}_t \sim \boldsymbol{N}(\boldsymbol{0},\ \boldsymbol{\Sigma})$. Also, every elements of $\boldsymbol{\Theta}$ is in [0, 1] and $\widehat{\Theta}$ is called a smoothing parameter matrix.

Let $\boldsymbol{X}_0 = \boldsymbol{0}$ and $\boldsymbol{\epsilon}_0 = \boldsymbol{0}$ at the starting point of a process, in this case $\boldsymbol{X}_t$ can be represented as

$$\boldsymbol{X}_t = (\boldsymbol{I} - \boldsymbol{\Theta}) \sum_{j=1}^{t-1} \boldsymbol{\epsilon}_j + \boldsymbol{\epsilon}_t,\ t = 1, 2, 3, \cdot. \tag{2.3}$$

Also, $\widehat{\boldsymbol{X}_t}$ is the MMSE (minimum mean square error) estimate of all future values of the series and is therefore an estimate of location of the series at time $t$. In particular, the MMSE forecast estimator of $\boldsymbol{X}_t$ made at time $t-1$ can be represented as

$$\widehat{\boldsymbol{X}}_t = (\boldsymbol{I} - \boldsymbol{\Theta}) \sum_{j=1}^{t-1} \boldsymbol{\epsilon}_j, \ t = 1, 2, 3, \cdots . \tag{2.4}$$

By algebraic manipulations it is variously transformed as below

$$\widehat{\boldsymbol{X}}_t = (\boldsymbol{I} - \boldsymbol{\Theta})\boldsymbol{X}_{t-1} + \boldsymbol{\Theta}\widehat{\boldsymbol{X}}_{t-1}, \ or \tag{2.5}$$

$$\widehat{\boldsymbol{X}}_t = \frac{\boldsymbol{I} - \boldsymbol{\Theta}}{\boldsymbol{I} - \boldsymbol{\Theta}B}\boldsymbol{X}_{t-1} \tag{2.6}$$

In this thesis, so as to use as the process model of statistical process control, the $\boldsymbol{\Theta}$ of multivariate IMA(1, 1) model is a diagonal matrix with $\theta_1, \theta_2, \cdots, \theta_p$ on the diagonal, and $p$ is the number of variables; that is the number of elements in each vector. In particular, the process model would be more effected at observed values of current time than observations of past, fixed up as $\theta_1 = \theta_2 = \cdots = \theta_p = 0.1$ which is similar to white noise model.

## 3. Multivariate EWMA control charts

Lowry *et al.* (1992) has developed a multivariate version of EWMA control chart. Multivariate EWMA model is

$$\boldsymbol{E}_t = \boldsymbol{\Lambda}\boldsymbol{e}_t + (\boldsymbol{I} - \boldsymbol{\Lambda})\boldsymbol{E}_{t-1}, \tag{3.1}$$

where $\boldsymbol{E}_t$ is the $t$th EWMA vector, $\boldsymbol{e}_t$ is the $t$th observation vector $t = 1, 2, 3, \cdots$ and $\boldsymbol{E}_0 = \boldsymbol{0}$. $\boldsymbol{E}_t$ is the vector of variable values from the historical data, $\boldsymbol{\Lambda}$ is a diagonal matrix with $\lambda_1, \lambda_2, \cdots, \lambda_p$ on the diagonal, and $p$ is the number of variables; that is the number of elements in each vector.

The quantity to be plotted on the control chart is

$$T_i^2 = \boldsymbol{E}_i' \boldsymbol{\Sigma}_{\boldsymbol{E}_i}^{-1} \boldsymbol{E}_i \tag{3.2}$$

The $(k, l)$th element of the covariance matrix of the $i$th EWMA, is

$$\boldsymbol{\Sigma}_{\boldsymbol{E}_i}(k, l) = \lambda_k \lambda_l \frac{[1 - (1-\lambda_k)^i (1-\lambda_l)^i]}{[\lambda_k + \lambda_l - \lambda_k \lambda_l]} \sigma_{kl}, \tag{3.3}$$

where $\sigma_{kl}$ is the $(k, l)$th element of $\boldsymbol{\Sigma}$, the covariance matrix of $\boldsymbol{e}$'s.

If $\lambda_1 = \lambda_2 = \cdots = \lambda_p$,, then the above expression simplifies to

$$\boldsymbol{\Sigma}_{\boldsymbol{E}_i}(k, l) = \frac{\lambda}{2 - \lambda}\left[1 - (1-\lambda)^{2i}\right] \boldsymbol{\Sigma}, \tag{3.4}$$

where, $\boldsymbol{\Sigma}$ is the covariance matrix of the input data.

The shift size is reported in terms of a quantity

$$\eta = (\boldsymbol{\mu}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu})^{1/2} \tag{3.5}$$

usually called the noncentrality parameter. Basically, large values of $\eta$ correspond to bigger shifts in the mean.

## 4. Process change point in monitoring $\boldsymbol{\mu}$

We are concerned with monitoring mean vector for a multivariate normal quantity variable, and the objective of process monitoring is to detect the time at occurring special causes. Let $\tau$ be the process change point and $T$ $(> \tau)$ be the time that the chart signal which is defined as the last sample from in-control process. Also, we assume that special causes happen between $\tau$ and $\tau + 1$ which is unknown parameter.

### 4.1. Estimation of the process change point in monitoring

$\boldsymbol{\mu}$ We consider the problem of change in $\boldsymbol{\mu}$ and assume observed vectors from multivariate IMA(1,1). The change of mean vector is represented as $\boldsymbol{\delta}$ and assuming mean vector is $\mathbf{0}$ in the control process state, $\boldsymbol{\Sigma}_0$ is a known parameter through previous samples and $\boldsymbol{\delta}$ is an unknown parameter. In this case process model can be represented as

$$\begin{cases} \boldsymbol{X}_t = (\boldsymbol{I} - \boldsymbol{\Theta}) \sum_{j=1}^{\tau} \boldsymbol{\epsilon}_j + \boldsymbol{\epsilon}_t, & t \leq \tau \\ \boldsymbol{X}_t = (\boldsymbol{I} - \boldsymbol{\Theta}) \sum_{j=\tau+1}^{T} \boldsymbol{\epsilon}_j + \boldsymbol{\epsilon}_t + \boldsymbol{\delta}, & t \geq \tau + 1 \end{cases} \tag{4.1}$$

Because we don't know prior to occurring signal even after special causes occur, predicted values of MMSE use (2.4), then, the residual vector at each point is

$$\boldsymbol{e_t} = \begin{cases} \boldsymbol{\epsilon}_t, & t \leq \tau \\ \boldsymbol{\epsilon}_t + \boldsymbol{\delta}, & t \geq \tau + 1 \end{cases} \tag{4.2}$$

Thus, the expected value of the residual vector is

$$E(\boldsymbol{e_t}) = \begin{cases} \mathbf{0}, & t \leq \tau \\ \boldsymbol{\delta}, & t \geq \tau + 1 \end{cases} \tag{4.3}$$

and covariance matrix($\boldsymbol{\Sigma}$) is $\boldsymbol{\Sigma}_0$ for all $t$. That is, when $t \leq \tau$, the $\boldsymbol{e}_t$ is independent and identically distributed $\boldsymbol{N}_p(\mathbf{0}, \boldsymbol{\Sigma}_0)$ and when $t \geq \tau + 1$, $\boldsymbol{N}_p(\boldsymbol{\delta}, \boldsymbol{\Sigma}_0)$.

With these reasons, the monitoring observations from the multivariate IMA(1,1) model is able to be converted that of residual vectors under the assumption that the model is perfect. The basic idea behind residual-based charts is, namely, to directly monitor the residuals of multivariate IMA(1,1). So, detecting the change point in multivariate IMA(1,1) process is identical with monitoring the residual vector's variation.

In this case, let a point of time giving chart signal be $T$, then the MLE at the changing point of process is

$$\widehat{\tau_\delta} = \arg \max_{0 \leq t < T} \left\{ \frac{\left( \sum_{i=\tau+1}^{T} \boldsymbol{e}_i \right)' \boldsymbol{\Sigma}^{-1} \left( \sum_{i=\tau+1}^{T} \boldsymbol{e}_i \right)}{T - \tau} \right\}. \tag{4.4}$$

### 4.2. Simulation setting and procedure

As we mentioned earlier, fixing smoothing parameter matrix $\boldsymbol{\Theta}$ of multivariate IMA(1, 1) model as $0.1\boldsymbol{I}_p$ (all its diagonal entries equal as 0.1 in p-dimension), we use $\tau = 50$ which is change point in the process. For simplicity, we assume that the in-control process mean vector is $\boldsymbol{\mu} = \boldsymbol{0}$ and covariance $\boldsymbol{\Sigma}_0$ is symmetric matrix, whose diagonal entries are 1 and off-diagonal entries are 0.5 in p-dimension. Also, we get an observation vector at each time point with following steps.

| | |
|---|---|
| [Step 1] | Generate true value $\boldsymbol{X}_t$ of multivariate IMA(1,1) model through (2.2) to evaluate and implement the model, in here, $\boldsymbol{X}_0 = \boldsymbol{0}$, and generate $\boldsymbol{\epsilon}_t$ from $N(\boldsymbol{0}, \boldsymbol{\Sigma}_0)$, for $t \leq \tau$ and $N(\boldsymbol{\delta}, \boldsymbol{\Sigma}_0)$, for $t \geq \tau + 1$. |
| [Step 2] | Generating $\widehat{\boldsymbol{X}_t}$ from (2.6) and yielding $\boldsymbol{e}_t$. |
| [Step 3] | Do comparisons between the value of (3.2) at each time point of $\boldsymbol{e}_t$ through (3.1) and upper control limit (UCL), then find a chart signal ($T$) and produce bias by calculating the differences between (4.4) and the true value ($\tau = 50$). |
| [Step 4] | Do 100,000 iterations of length $n$=10,000 each the above steps. |

For reasons of consistency and fair comparison, we will detect false alarms. A false alarm occurs when chart signal at $T < 50$. If a false alarm occurs, then the result of the simulation is ignored and that is neglected at the counting the total frequencies.

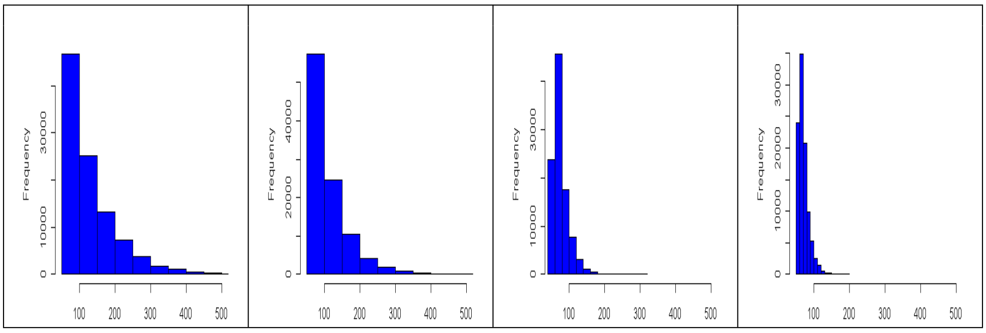
### 4.3. Performance of the estimator in monitoring $\boldsymbol{\mu}$

In this section, we study the performance of our proposed estimator using Monte Carlo simulation. Two performance measures, namely, the average change point estimate and the empirical distribution of the estimated change point around the actual change point are considered.

Control chart monitoring change of process time point uses multivariate exponentially weighted moving average (MEWMA) Control chart which has one sample size, that is, control chart statistic employs $\boldsymbol{E}_t = \boldsymbol{\Lambda}\boldsymbol{e}_t + (\boldsymbol{I} - \boldsymbol{\Lambda})\boldsymbol{E}_{t-1}$ and gives chart signal in the case of $\boldsymbol{E}_t \geq UCL$. In the condition of MEWMA Control chart suggested by Prabhu and Runger(1997), UCL fixing $ARL(average\ run\ length)_0 = 200$ is put to use. Let $\boldsymbol{\Lambda}$ be a diagonal matrix with $\lambda_1 = \lambda_2 = \cdots = \lambda_p = \lambda$ on the main diagonal for the purpose of simplicity and we will simulate cases of $\lambda$ such as 0.1, 0.2, 0.3, 0.4, 0.5, 0.6 and 0.8. The smaller $\lambda$ is, the more efficient for detecting a little change is, and it is equal with $T^2$ control chart when $\lambda = 1.0$.

At UCL values for various values of $\boldsymbol{\delta}$, $\lambda$, and $ARL_0 = 200$, we carry out simulation at 2 dimensions. We calculated $ARL_1$ and bias where $ARL_1$ is the time value of chart signal, and the bias is defined as the $\widehat{\tau}$ which presents the exactness of estimate. In the simulation, we carried out with 100,000 iterations of length $n = 10,000$ each, and $ARL_1$ and bias($|\widehat{\tau} - \tau|$) use mean value of iterative results. Ideally, the charts would signal immediately after $t = 50$ and change point estimate would be 50. The performance of our method can be judged graphically by the concentration of the bar graph immediately after the time of the shift.

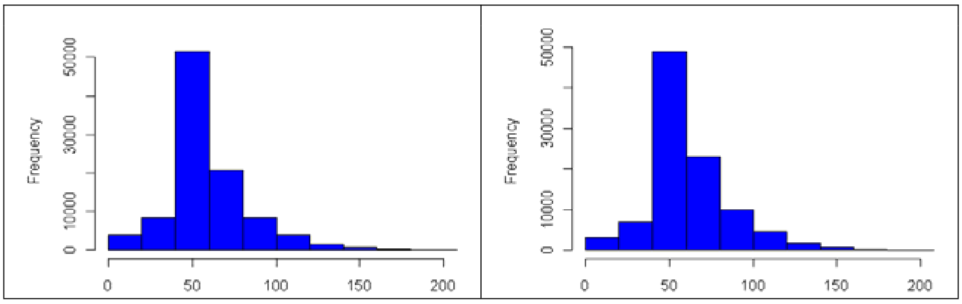
As an example, we change the values of $\lambda$ such as 0.8, 0.6, 0.2, and 0.1 at fixed $\boldsymbol{\delta} = \boldsymbol{0.5}$ in order to check the fact that smaller $\lambda$ is more sensitive. And we replace $\boldsymbol{\delta}$ as $\boldsymbol{2}$ in the same condition about $\lambda$ for the purpose of comparing the efficiency between $\boldsymbol{\delta} = \boldsymbol{0.5}$ and $\boldsymbol{\delta} = \boldsymbol{2}$.



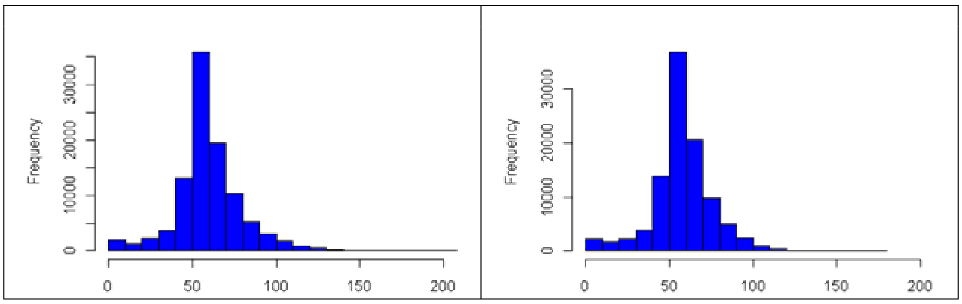

(a) $\lambda = 0.8$ (b) $\lambda = 0.6$ (c) $\lambda = 0.2$ (d) $\lambda = 0.1$

**Figure 4.1** Estimate of ARL $p = 2$ at $\boldsymbol{\delta} = \mathbf{0.5}$

Figure 4.1 shows the relative frequencies of the time of signal according to variation of $\lambda$. In this setting, signals of (d) are more concentrated in the neighborhood 50 than others. We are readily able to make sure that the smaller $\lambda$ is, the faster the speed of detecting chart signals is.



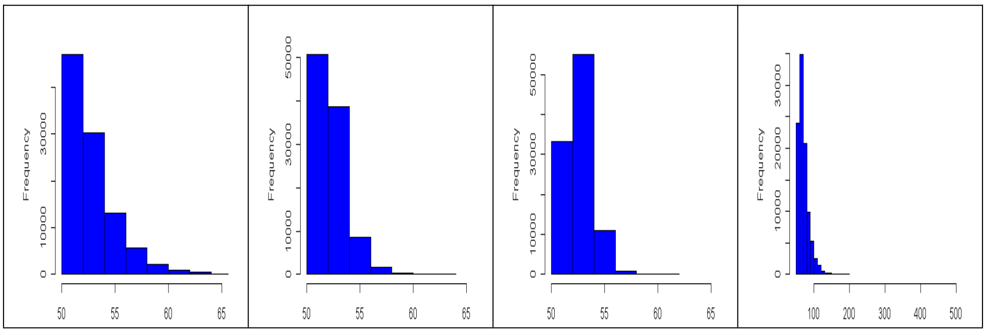

(a) $\lambda = 0.8$ (b) $\lambda = 0.6$




(c) $\lambda = 0.2$ (d) $\lambda = 0.1$

**Figure 4.2** Estimate of change point ($\hat{\tau}$) performance $p = 2$ at $\boldsymbol{\delta} = \mathbf{0.5}$

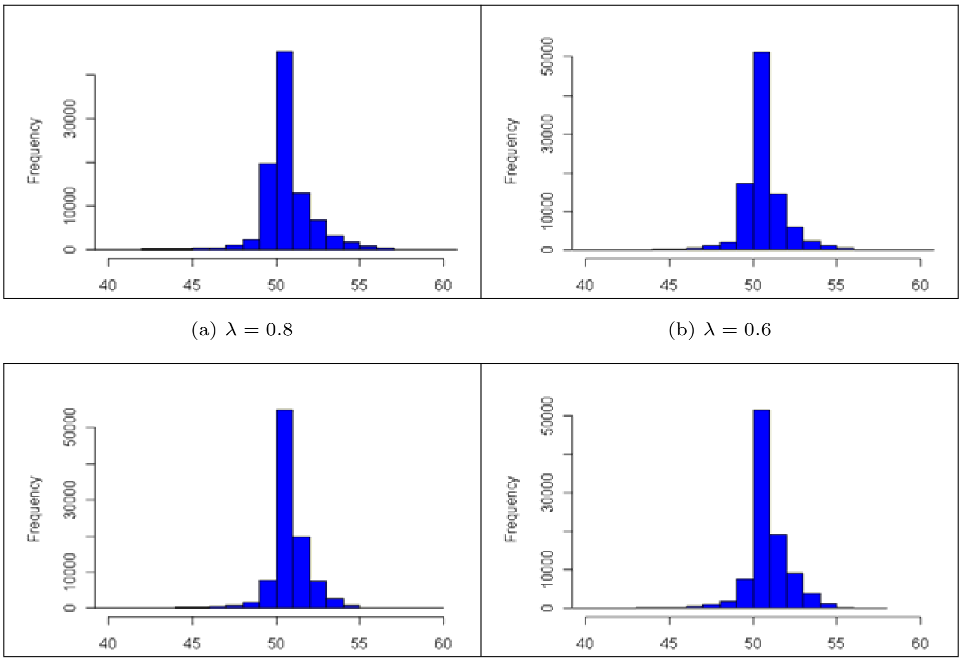
Figure 4.2 shows frequency of the $\hat{\tau}$ given following legitimate signals (those occurring after the actual change). Ideally, these distributions should peak strongly at 50. However, in this setting these are not somewhat concentrated at 50 because of small variation of mean vector as 0.5.



(a) $\lambda = 0.8$ (b) $\lambda = 0.6$ (c) $\lambda = 0.2$ (d) $\lambda = 0.1$

**Figure 4.3** Estimate of ARL $p = 2$ at $\boldsymbol{\delta} = \mathbf{2.0}$

Through comparing Figure 4.1 with Figure 4.3, we can easily perceive the fact that bigger variation of mean vector increases the number of signals detected near 50 in the simulation.

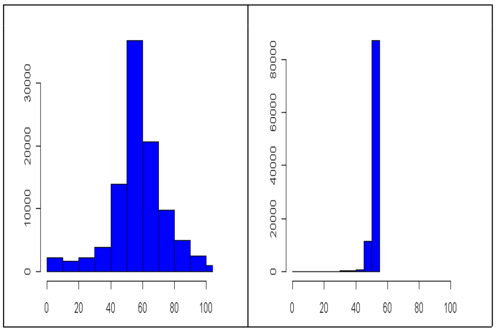


(a) $\lambda = 0.8$ (b) $\lambda = 0.6$

(c) $\lambda = 0.2$ (d) $\lambda = 0.1$

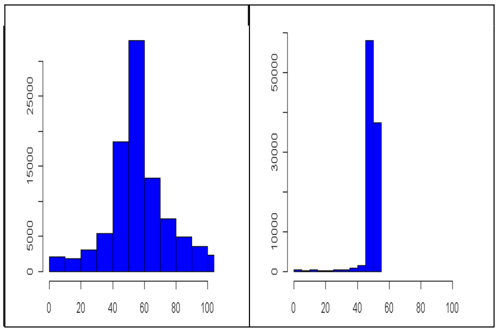
**Figure 4.4** Estimate of change point ($\widehat{\tau}$) performance $p = 2$ at $\boldsymbol{\delta} = \mathbf{2.0}$

Through comparing Figure 4.2 with Figure 4.4, we can know that estimates of change points tend to concentrate near 50 when variation is bigger. To give clear understanding, Figures 4.5 and 4.6 present the difference between $\boldsymbol{\delta} = \mathbf{0.5}$ and $\boldsymbol{\delta} = \mathbf{3.0}$ for each $\lambda$.



(a) $\delta = 0.5$  (b) $\lambda = 3.0$

**Figure 4.5** Estimate of change point ($\widehat{\tau}$) performance $p = 2$ at $\boldsymbol{\lambda} = \mathbf{0.1}$



(c) $\lambda = 0.5$  (d) $\lambda = 3.0$

**Figure 4.6** Estimate of change point ($\widehat{\tau}$) performance $p = 2$ at $\boldsymbol{\lambda} = \mathbf{0.8}$

Through Figures 4.5 and 4.6, we can recognize that the accuracy of estimates of change point increases according as $\boldsymbol{\delta}$ is bigger. Also, if $\boldsymbol{\delta}$ is big enough, estimates of change point are generally exact regardless of $\boldsymbol{\lambda}$ values. We can make firm the fact via Table 4.1.

**Table 4.1** Bias (standard error) in the change point estimate

| | $\boldsymbol{\delta} = 0.5$ | $\boldsymbol{\delta} = 1.0$ | $\boldsymbol{\delta} = 2.0$ | $\boldsymbol{\delta} = 3.0$ |
|---|---|---|---|---|
| $\boldsymbol{\lambda} = \mathbf{0.1}$ | 8.12 (17.96) | 1.49 (8.23) | 0.86 (4.02) | 0.81 (2.52) |
| $\boldsymbol{\lambda} = \mathbf{0.2}$ | 10.18 (20.04) | 1.95 (7.99) | 0.81 (3.71) | 0.55 (2.79) |
| $\boldsymbol{\lambda} = \mathbf{0.3}$ | 12.02 (21.62) | 2.35 (7.83) | 0.82 (3.24) | 0.42 (2.65) |
| $\boldsymbol{\lambda} = \mathbf{0.4}$ | 12.30 (23.70) | 2.69 (8.01) | 0.53 (3.54) | 0.15 (3.43) |
| $\boldsymbol{\lambda} = \mathbf{0.5}$ | 12.48 (24.49) | 2.82 (8.27) | 0.63 (3.71) | 0.00 (3.93) |
| $\boldsymbol{\lambda} = \mathbf{0.6}$ | 12.62 (25.51) | 2.97 (8.02) | 0.69 (3.96) | 0.02 (4.17) |
| $\boldsymbol{\lambda} = \mathbf{0.8}$ | 10.25 (25.34) | 3.01 (8.43) | 0.53 (4.74) | -0.02 (5.31) |

Since the efficiency of estimates at changing point of mean vector is slower than the other cases on which $\boldsymbol{\delta}$ is too small, that is, the exactness of $\widehat{\tau}$ is likely to fall down. But, we can confirm that when variation of mean vector ($\boldsymbol{\delta} \geq 1.0$) is big, estimator of change point is well estimated.

## 5. Conclusion

After chart signals, it is very essential procedures to restore a process in-control by correcting the causes of out of control and getting rid of them. In sophisticated industries with automatic process development, quality characteristics in many applications no longer meet a standard independent assumption. Many studies have recently investigated this characteristic. Lu and Reynolds (1999) argued that the EWMA control chart of autocorrelated observations is more efficient than Shewhart control chart in detecting small process mean shifts. In this thesis, assuming that process model is multivariate IMA(1,1), we check the performance of MLE of change point by monitoring residual vectors. How quickly control charts detect the change point is important in a process control, so if process managers in real fields use the MLE suggested by this article, then the estimates are very helpful for efficiently dealing with processes.

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