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ENTROPY, POSITIVELY CONTINUUM-WISE EXPANSIVENESS AND SHADOWING

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Abstract. Let (X,d) be a compact metric space, and let $f: X \to X$ be a continuous map. We consider that if a positively continuum-wise expansiveness continuous map f has the positively shadowing property in the nonwandering set, then the topological entropy is positive.

1. Introduction

Let $f: X \to X$ be a continuous map of a compact metric space (X,d). For the dynamic properties (shadowing, positively measure expansive) and entropy, Morales [3] proved that if a homeomorphism $f: X \to X$ has the shadowing property in the nonwandering set and it is positively measure expansive then the topological entropy is positive. In the paper, we consider that if a continuous map $f: X \to X$ has the positively shadowing property in the nonwandering set and it is another type of expansiveness then the topological entropy is positive.

We say that a point $x \in X$ is *nonwandering* if for any neighborhood U of x, there is $n > 0$ such that $f^n(U) \cap U \neq \emptyset$. Denote by $\Omega(f)$ the set of all nonwandering points of f. For given $x, y \in M$, we write $x \rightsquigarrow y$ if for any $\delta > 0$, there is a finite δ-pseudo orbit $\{x_i\}_{i=a}^{b} (a < b)$ of f such that $x_a = x$ and $x_b = y$. The set of points $\{x \in M : x \leftrightsquigarrow x\}$ is called the *chain recurrent set* of f and is denoted by $\mathcal{CR}(f)$. Then it is known that $\Omega(f) \subset \mathcal{CR}(f)$. For any $\delta > 0$, a sequence $\{x_i\}_{i\in\mathbb{Z}_+}$ is a *δ-pseudo orbit* of f if

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$d(f(x_i), x_{i+1}) < \delta$ for $i \geq 0$. Let $\Lambda \subset X$ be a closed f-invariant set. We say that f has the *positively shadowing property* on Λ if for any $\epsilon > 0$, there is $\delta > 0$ such that for any δ-pseudo orbit $\{x_i\}_{i\geq 0} \subset \Lambda$ there is $y \in X$ such that

$$d(f^i(y), x_i) < \epsilon \text{ for } i \geq 0.$$

Then the shadowing points can be in Λ or is in X. On the other hand, we say that f has the *positively shadowing property* in Λ if for any $\epsilon > 0$ there is $\delta > 0$ such that for any δ-pseudo orbit $\{x_i\}_{i\geq 0} \subset \Lambda$ there is $y \in \Lambda$ such that

$$d(f^i(y), x_i) < \epsilon \text{ for } i \geq 0.$$

If $\Lambda = X$ then we say that f has the positively shadowing property. Recall Bowen's definition of topological entropy on a closed set A. For $n \in \mathbb{N}$ and $\epsilon > 0$, a set $B \subset A$ is said to be (B, n, ϵ)*-separated* for f if for any distinct two points $a, b \in A$ there is $k \in \{0, 1, \ldots, n-1\}$ such that $d(f^k(a), f^k(b)) > \epsilon$. Let $\Phi_f(A, n, \epsilon)$ denote the maximal cardinality of a (A, n, ϵ)-separated set for f contained in A. Since X is compact, $\Phi_f(A, n, \epsilon)$ is finite. The topological entropy of f on A is defined by

$$h(f, A) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{\log \Phi_f(A, n, \epsilon)}{n}.$$

We can denote $h(f) = h(f, A)$ if there is no confusion.

Given $x \in X$ and $\delta > 0$, we define the *dynamical δ-ball* as follows, $\Gamma_\delta(x) = \{y \in X : d(f^i(x), f^i(y)) \leq \delta \text{ for all } i \geq 0\}$. Let $\mathcal{M}(X)$ be the set of all Borel probability measures on X and let $\mathcal{M}^*(X)$ be the set of nonatomic measures $\mu \in \mathcal{M}(X)$. We say that f is *positively measure expansive* (or, *positively μ-expansive*) if there is $\delta > 0$ (called an *expansive constant*) such that $\mu(\Gamma_\delta(x)) = 0$ for any $\mu \in \mathcal{M}^*(X)$. Note that in [3, Lemma 8], Morales proved that a homeomorphism $f : X \to X$ is positively measure expansive if and only if it is positively invariant measure expansive.

Kato [2] introduced positively continuum-wise expansiveness which is a more general notation of positively measure expansiveness (see [1, Lemma 2.3]). A continuous map f on X is said to be *positively continuum-wise expansive* if there is a constant $e > 0$ such that for any nondegenerate (is not singleton) continuum A there is an integer $n \geq 0$ such that $\mathrm{diam} f^n(A) \geq e$, where $\mathrm{diam} A = \sup\{d(x, y) : x, y \in A\}$ for any subset A of X. Here the constant e is called a *positively continuum-wise expansive constant* for f. In this paper, we prove that if f has the positively shadowing property in $\Omega(f)$ and f is positively continuum-wise expansive, then the topological entropy is positive.

We say that f is *equicontinuous* if for any $\epsilon > 0$ there is $\delta > 0$ such that for any $x, y \in X$ if $d(x,y) < \delta$ then $d(f^i(x), f^i(y)) < \epsilon$ for all $i \in \mathbb{Z}_+$. It is known that an equicontinuous map $f : X \to X$ has $h(f) = 0$.

LEMMA 1.1. [4, Corollary 6] *If f has the positively shadowing property and $h(f) = 0$ then $f|_{\Omega(f)} : \Omega(f) \to \Omega(f)$ is an equicontinuous homeomorphism.*

The following was proved in [3, Lemma 10] for a homeomorphism $f : X \to X$. However, in the paper, we consider for a continuous map $f : X \to X$.

LEMMA 1.2. *If a continuous map $f : X \to X$ is equicontinuous then f is not positively measure expansive.*

Proof. Suppose, by contradiction that an equicontinuous map f is positively measure μ-expansive for some $\mu \in \mathcal{M}^*(X)$. Let $e > 0$ be the number of the positively measure expansive constant of μ. Since f is equicontinuous, there is $\delta > 0$ such that $B[x,\delta] \subset \Gamma_e(x)$ for all $x \in X$, where $B[x,\delta] = \{y \in X : d(x,y) \leq \delta\}$ is a closed δ-ball centered at x. Note that f is positively measure expansive if and only if $f|_{\Omega(f)}$ is positively measure expansive. Thus we know $\mu(B[x,\delta]) = 0$ for $x \in \Omega(f)$. Since X is compact, we can take finite points $x_1, x_2, \ldots, x_n$ such that $X = \bigcup_{i=1}^n B[x_i,\delta]$. Then we know

$$\mu(X) \leq \sum_{i=1}^n \mu(B[x_i,\delta]).$$

Since f is positively measure expansive, $\mu(\Gamma_e(x)) = 0$ and so, $\mu(B[x,\delta]) = 0$. Thus $\sum_{i=1}^n \mu(B[x_i,\delta]) = 0$, which is a contradiction $\mu(X) \neq 0$. □

LEMMA 1.3. *A continuous map $f : X \to X$ is positively continuum-wise expansive if and only if there is $\delta > 0$ such that for $x \in X$, if a continuum $C \subset \Gamma_\delta(x)$ then C is a singleton.*

Proof. The proof is similar to [1, Lemma 2.2]. □

For a closed invariant set Λ, it is known that a continuous map $f : X \to X$ is continuum-wise expansive if and only if $f|_\Lambda$ is continuum-wise expansive.

LEMMA 1.4. *Let $\Lambda \subset X$ be a closed f-invariant set. If $f|_\Lambda$ is equicontinuous then f is not positively continuum-wise expansive.*

Proof. Let $f|_\Lambda$ be equicontinuous. To derive a contradiction, we may assume that f is positively continuum-wise expansive. Let $e > 0$ be a positively continuum-wise expansive constant of f. Since f is an equicontinuous, there is $\delta \in (0, e)$ such that if for any $x, y \in \Lambda$ with $d(x, y) < \delta$ then $d(f^i(x), f^i(y)) < e$ for all $i \in \mathbb{Z}_+$. We set $C_\delta(x) = \{y \in \Lambda \setminus \{x\} : d(x, y) \leq \delta\}$. Then it is clear $C_\delta(x) \subset \Gamma_e(x) = \{y \in \Lambda : d(f^i(x), f^i(y)) \leq e$ for all $i \geq 0\}$. Since f is positively continuum-wise expansive, $f|_\Lambda$ is positively continuum-wise expansive. By Lemma 1.3, $C_\delta(x)$ must be a singleton which is a contradiction since $C_\delta(x)$ is not singleton. Thus if $f|_\Lambda$ is equicontinuous, then f is not positively continuum-wise expansive. □

THEOREM 1.5. *Let $f : X \to X$ be a continuous map and f have the positively shadowing property in $\Omega(f)$. If f is positively continuum-wise expansive then the topological entropy $h(f)$ is positive.*

Proof. Let f have the shadowing property in $\Omega(f)$ and f be positively continuum-wise expansive. Suppose, by contradiction, that $h(f) = 0$. Since f has the positively shadowing property in $\Omega(f)$ and $h(f) = 0$, by Lemma 1.1 $f|_{\Omega(f)}$ is equicontinuous. Since $\Omega(f)$ is closed and invariant, by Lemma 1.4, f is not positively continuum-wise expansive which is a contradiction. □

References

[1] A. Artigue and D. Carrasco-Olivera, *A note on measure-expansive diffeomorphisms*, J. Math. Anal. Appl. **428** (2015), 713-716.

[2] H. Kato, *Continuum-wise expansive homeomorphisms*, Canad. J. Math. **45** (1993), 576-598.

[3] C. A. Morales, *Entropy, pseudo-orbit tracing property and positively expansive measure*, preprint, arxiV:1409.3470v1.

[4] T. K. S. Moothathu, *Implications of pseudo-orbit tracing property for continuous maps on compacta*, Topol. Appl, **158** (2011), 2232-2239.

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