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STABILITY OF QUARTIC SET-VALUED FUNCTIONAL EQUATIONS

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Abstract. We will show the general solution of the functional equation

$$\begin{aligned} & f(x+ay)+f(x-ay)+2(a^2-1)f(x) \\ = \ & a^2f(x+y)+a^2f(x-y)+2a^2(a^2-1)f(y) \end{aligned}$$

and investigate the Hyers-Ulam stability of the quartic set-valued functional equation.

1. Introduction

The theory of set-valued functions in Banach spaces is connected to the control theory and the mathematical economics. Aumann [4] and Debreu [9] wrote papers that were motivated from the topic. We refer the reader to the papers by [1], [18], [11], [3], [16], [7] and [10].

The stability problem of functional equations originated from a question of Ulam [25] concerning the stability of group homomorphisms. Hyers [12] gave a first affirmative partial answer to the question of Ulam. Afterwards, the result of Hyers was generalized by Aoki [2] for additive mapping and by Rassias [23] for linear mappings by considering a unbounded Cauchy difference. Later, the result of Rassias has provided a lot of influence in the development of what we call Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. For further information about the topic, we also refer the reader to [14], [13], [5] and [6]. Rassias [22] investigated stability properties of the following quartic functional equation

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$$f(x+2y)+f(x-2y)+6f(x)=4f(x+y)+4f(x-y)+24f(y)\,. \tag{1.1}$$

It is easy to see that $f(x)=x^4$ is a solution of (1.1) by virtue of the identity

$$(x+2y)^4+(x-2y)^4+6x^4=4(x+y)^4+4(x-y)^4+24y^4\,. \tag{1.2}$$

For this reason, (1.1) is called a quartic functional equation. Also Chung and Sahoo [8] determined the general solution of (1.1) without assuming any regularity conditions on the unknown function. In fact, they proved that the function $f:\mathbb{R}\to\mathbb{R}$ is a solution of (1.1) if and only if $f(x)=A(x,x,x,x)$, where the function $A:\mathbb{R}^4\to\mathbb{R}$ is symmetric and additive in each variable. Lee and Chung [17] introduced a quartic functional equation as follows:

$$\begin{aligned} &f(ax+y)+f(ax-y) \\ &= a^2f(x+y)+a^2f(x-y)+2a^2(a^2-1)f(x)-2(a^2-1)f(y)\,, \end{aligned} \tag{1.3}$$

for fixed integer a with $a\neq 0,\ \pm 1$.

In this paper, we deal with the following functional equation:

$$\begin{aligned} &f(x+ay)+f(x-ay)+2(a^2-1)f(x) \\ &= a^2f(x+y)+a^2f(x-y)+2a^2(a^2-1)f(y) \end{aligned} \tag{1.4}$$

for all $x\,,y\in X$ and an integer $a(a\neq 0\,,\pm 1)$. We will show the general solution of the functional equation (1.4) and investigate the Hyers-Ulam stability of the quartic set-valued functional equation.

2. A quartic functional equation

Let X, Y be real vector spaces. In this section, we will investigate the functional equation (1.4) is equivalent to the presented functional equation (1.3).

Theorem 2.1. *A mapping $f:X\to Y$ satisfies the functional equation (1.4) if and only if f satisfies the functional equation (1.3).*

Proof. Suppose that f satisfies the equation (1.4). Letting $x=y=0$ in the equation (1.4), we have $f(0)=0$. Letting $x=0$ in the equation (1.4), we get

$$f(ay)+f(-ay)=a^2f(y)+a^2f(-y)+2a^2(a^2-1)f(y) \tag{2.1}$$

for all $y\in X$. Replacing y by $-y$ in the equation (2.1),we obtain

$$f(ay)+f(-ay)=a^2f(y)+a^2f(-y)+2a^2(a^2-1)f(-y) \tag{2.2}$$

for all $y \in X$. Combining two equations (2.1) and (2.2), we have $f(y) = f(-y)$, for all $y \in X$. That is, f is even. Since f is even, the equation (2.1) implies that $f(ax) = a^4 f(y)$, for all $y \in X$ and an integer $a \neq 0, \pm 1$. Now, interchanging x and y in the equation (1.4), we have the desired equation (1.3). Conversely, suppose that f satisfies the equation (1.3). It is easy to show that $f(0) = 0, f(x) = f(-x)$ and $f(ax) = a^4 f(x)$, for all $x \in X$ and an integer $a\,(a \neq 0, \pm 1)$. Interchanging x and y in the equation (1.3), we have the equation (1.4). □

If f satisfies the equation (1.4), we call f a *quartic mapping*.

3. Stability of the quartic set-valued functional equation

In this section, we first introduce some definitions and notations which are needed to prove the main theorems. Let Y be a Banach space. Let A, B be nonempty subsets of a real vector space X and λ a real number. We define

$$\begin{aligned} A + B &= \{a + b \in X \mid a \in A, b \in B\} \\ \lambda A &= \{\lambda a \in X \mid a \in A\}. \end{aligned}$$

A subset $A \subseteq X$ is said to be a *cone* if $A + A \subseteq A$ and $\lambda A \subseteq A$ for all $\lambda > 0$. If the zero vector in X belongs to A, then we say that A is a *cone with zero*. Let $C_b(Y)$ be the set of all closed bounded subsets of Y, $C_c(Y)$ the set of all closed convex subsets of Y and $C_{cb}(Y)$ the set of all closed bounded convex subsets of Y. For elements A, B of $C_c(Y)$, we denote

$$A \oplus B = \overline{A + B}.$$

Lemma 3.1 ([20]). *Let λ and μ be real numbers. If A and B are nonempty subsets of a real vector space, then*

$$\begin{aligned} \lambda(A + B) &= \lambda A + \lambda B \\ (\lambda + \mu)A &\subseteq \lambda A + \mu A. \end{aligned}$$

Moreover, if A is a convex set and $\lambda\mu \geq 0$, then we have

$$(\lambda + \mu)A = \lambda A + \mu A.$$

For a subset A of Y, the distance function $d(\cdot, A)$ and the support function $s(\cdot, A)$ are defined by

$$\begin{aligned} d(x, A) &:= \inf\,\{\|x - y\| \mid y \in A\} \text{ for all } x \in Y \\ s(x^*, A) &:= \sup\,\{\langle x^*, x\rangle \mid x \in A\} \text{ for all } x^* \in Y^*. \end{aligned}$$

For $A, A' \in C_b(Y)$, the *Hausdorff distance* $h(A, A')$ between A and A' is defined by

$$h(A, A') := \inf\{\alpha > 0 \mid A \subseteq A' + \alpha B_Y, A' \subseteq A + \alpha B_Y\},$$

where B_Y is the closed unit ball in Y. Castaing and Valadier [7] proved that $(C_{cb}(Y), \oplus, h)$ is a complete metric semigroup. Rådström [21] showed that $(C_{cb}(Y), \oplus, h)$ is isometrically embedded in a Banach space. The following remark is directly obtained from the notion of the Hausdorff distance.

REMARK 3.2. Let $A, A', B, B', C \in C_{cb}(Y)$ and $\alpha > 0$. Then the following properties hold:
(1) $h(A \oplus A', B \oplus B') \le h(A, B) + h(A', B')$
(2) $h(\alpha A, \alpha B) = \alpha h(A, B)$
(3) $h(A, B) = h(A \oplus C, B \oplus C)$.

DEFINITION 3.3. Let $f : X \to C_{cb}(Y)$. The quartic set-valued functional equation is defined by

$$\begin{aligned} f(x+ay) \oplus f(x-ay) \oplus 2(a^2-1)f(x) \\ = a^2[f(x+y) \oplus f(x-y)] \oplus 2a^2(a^2-1)f(y) \end{aligned} \tag{3.1}$$

for all $x, y \in X$ and an integer $a\,(a \neq 0, \pm 1)$. Every solution of the quartic set-valued functional equation is called a *quartic set-valued mapping.*

THEOREM 3.4. *Let $\phi : X \times X \to [0, \infty)$ be a function such that*

$$\widetilde{\phi}(x, y) := \sum_{j=0}^{\infty} \frac{1}{a^{4j}} \phi(a^j x, a^j y) < \infty \tag{3.2}$$

for all $x, y \in X$ and an integer $a\,(a \neq 0, \pm 1)$. Suppose that $f : X \to (C_{cb}(Y), h)$ is an even mapping with $f(0) = \{0\}$ satisfying

$$\begin{aligned} h\Big(f(x+ay) \oplus f(x-ay) \oplus 2(a^2-1)f(x), \\ a^2 f(x+y) \oplus a^2 f(x-y) \oplus 2a^2(a^2-1)f(y)\Big) \le \phi(x, y) \end{aligned} \tag{3.3}$$

for all $x, y \in X$ and an integer $a\,(a \neq 0, \pm 1)$. Then there exists a unique quartic set-valued mapping $Q : X \to (C_{cb}(Y), h)$ such that

$$h(f(x), Q(x)) \le \frac{1}{2a^4} \widetilde{\phi}(0, x) \tag{3.4}$$

for all $x, y \in X$ and an integer $a\,(a \neq 0, \pm 1)$.

Proof. Let $x = 0$ in the inequality (3.3). Since $f(x)$ is convex and even, we have

$$h\Big(f(ay) \oplus f(ay),\, a^2 f(y) \oplus a^2 f(y) \oplus 2a^2(a^2-1)f(y)\Big) \le \phi(0,\, y)\,,$$

for all $y \in X$. By replacing y by x and dividing by $2a^4$ in the previous inequality, we obtain

$$h\Big(f(x),\, \frac{1}{a^4} f(ax)\Big) \le \frac{1}{2a^4}\phi(0,x) \tag{3.5}$$

for all $x \in X$ and an integer $a\,(a \ne 0\,, \pm 1)$. Letting $x = a^k x$ and dividing by a^{4k}, $k \in \mathbb{N}$ we get

$$h\Big(\frac{1}{a^{4k}} f(a^k x), \frac{1}{a^{4(k+1)}} f(a^{k+1}x)\Big) \le \frac{1}{2a^4}\,\frac{1}{a^{4k}}\phi(0,a^k x)$$

for all $x \in X$. Using the induction on k, we obtain

$$h\Big(f(x), \frac{1}{a^{4n}} f(a^n x)\Big) \le \frac{1}{2a^4}\sum_{k=0}^{n-1}\frac{1}{a^{4k}}\phi(0,a^k x) \tag{3.6}$$

for all $x \in X$ and $n \in \mathbb{N}$. Dividing the inequality (3.6) by a^{4m} and replacing x by $a^m x$, we have

(3.7)

$$h\Big(\frac{1}{a^{4m}} f(a^m x), \frac{1}{a^{4(n+m)}} f(a^{n+m}x)\Big) \le \frac{1}{2a^4}\,\frac{1}{a^{4m}}\sum_{k=0}^{n-1}\frac{1}{a^{4k}}\phi(0,a^{m+k}x)$$

for all $x \in X$ and $n\,, m \in \mathbb{N}$. Since the right-hand side of the inequality (3.7) tends to zero as $m \to \infty$, the sequence $\{\frac{1}{a^{4n}} f(a^n x)\}$ is a Cauchy sequence in $(C_{cb}(Y),\, h)$. By the completeness of $C_{cb}(Y)$, we can define

$$Q(x) := \lim_{n\to\infty}\frac{1}{a^{4n}} f(a^n x)$$

for all $x \in X$ and an integer $a\,(a \ne 0\,, \pm 1)$. We note that

$$\begin{aligned}
&h\Big(\frac{f(a^n(x+ay))}{a^{4n}} \oplus \frac{f(a^n(x-ay))}{a^{4n}} \oplus \frac{2(a^2-1)f(a^n y)}{a^{4n}},\\
&\frac{a^2 f(a^n(x+y))}{a^{4n}} \oplus \frac{a^2 f(a^n(x-y))}{a^{4n}} \oplus \frac{2a^2(a^2-1)f(a^n y)}{a^{4n}}\Big)\\
&\le \frac{1}{a^{4n}}\phi(a^n x, a^n y)
\end{aligned}$$

for all $x, y \in X$ and an integer $a\,(a \neq 0, \pm 1)$. By the definition of Q, we have

$$\begin{aligned} & h\Big(Q(x+ay) \oplus Q(x-ay) \oplus 2(a^2-1)Q(x), \\ & \qquad a^2Q(x+y) \oplus a^2Q(x-y) \oplus 2a^2(a^2-1)Q(y)\Big) \\ & \leq \lim_{n\to\infty} \frac{1}{a^{4n}}\phi(a^n x,\, a^n y) = 0\,. \end{aligned}$$

Hence Q is a quartic set-valued mapping. Now, by taking $n \to \infty$ in the inequality (3.6), we have the inequality (3.4). It remains to show the uniqueness of Q. Assume $Q' : X \to (C_{cb}(Y),\, h)$ is another quartic set-valued mapping satisfying the inequality (3.4). Then

$$\begin{aligned} h\Big(Q(x),\, Q'(x)\Big) &= \frac{1}{a^{4n}} h\Big(Q(a^n x),\, Q'(a^n x)\Big) \\ &\leq \frac{1}{a^{4n}} h\Big(Q(a^n x),\, f(a^n x)\Big) + \frac{1}{a^{4n}} h\Big(f(a^n x),\, Q'(a^n x)\Big) \\ &\leq \frac{2}{a^{4(n+1)}} \widetilde{\phi}(0,\, a^n x) \end{aligned}$$

for all $x \in X$. Since $\frac{2}{a^{4(n+1)}}\widetilde{\phi}(0,\, a^n x) \to 0$ as $n \to \infty$, we may conclude that the quartic set-valued mapping Q is unique. $\square$

Corollary 3.5. *Let $0 < p < 4, \theta \geq 0$ be real numbers and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y),\, h)$ is an even mapping with $f(0) = \{0\}$ satisfying*

$$\begin{aligned} & h\Big(f(x+ay) \oplus f(x-ay) \oplus 2(a^2-1)f(x), \\ & \qquad a^2 f(x+y) \oplus a^2 f(x-y) \oplus 2a^2(a^2-1)f(y)\Big) \leq \theta(||x||^p + ||y||^p) \end{aligned}$$

for all $x, y \in X$ and an integer $a\,(a \neq 0, \pm 1)$. Then there exists a unique quartic set-valued mapping $Q : X \to (C_{cb}(Y),\, h)$ satisfying

$$h(f(x),\, Q(x)) \leq \frac{\theta}{2(a^4 - a^p)}||x||^p$$

for all $x, y \in X$ and an integer $a\,(a \neq 0, \pm 1)$.

Proof. It follows from Theorem 3.4 by letting $\phi(x,\, y) = \theta(||x||^p + ||y||^p)$ for all $x, y \in X$. $\square$

4. Stability of quartic set-valued functional equation by the fixed point method

Now, we will investigate the stability of the given functional equation (3.1) using the alternative fixed point method. Before proceeding the proof, we will state the theorem, the alternative of fixed point; see [19] and [24].

Definition 4.1. Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies
(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 4.2 (The alternative of fixed point [19], [24]). *Suppose that we are given a complete generalized metric space (Ω, d) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then for each given $x \in \Omega$, either*

$$d(T^n x,\, T^{n+1}x) = \infty \text{ for all } n \geq 0,$$

or there exists a natural number n_0 such that
1. *$d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;*
2. *The sequence $(T^n x)$ is convergent to a fixed point y^* of T;*
3. *y^* is the unique fixed point of T in the set*

$$\triangle = \{y \in \Omega | d(T^{n_0}x, y) < \infty\};$$

4. *$d(y, y^*) \leq \frac{1}{1-L}\, d(y, Ty)$ for all $y \in \triangle$.*

Theorem 4.3. *Suppose that $f : X \to (C_{cb}(Y), h)$ is an even mapping with $f(0) = \{0\}$ satisfying*

$$h\Big(f(x+ay) \oplus f(x-ay) \oplus 2(a^2-1)f(x), \\ a^2 f(x+y) \oplus a^2 f(x-y) \oplus 2a^2(a^2-1)f(y)\Big) \leq \phi(x, y) \tag{4.1}$$

for all $x, y \in X$ and an integer a ($a \neq 0, \pm 1$) and there exists a constant L with $0 < L < 1$ for which the function $\phi : X^2 \to [0, \infty)$ satisfies

$$\phi(0, ax) \leq a^4 L \phi(0, x) \tag{4.2}$$

for all $x \in X$. Then there exists a unique quartic set-valued mapping $Q : X \to (C_{cb}(Y), h)$ given by $Q(x) = \lim_{n\to\infty} \frac{f(a^n x)}{a^{4n}}$ such that

$$h(f(x),\, Q(x)) \leq \frac{1}{2a^4(1-L)}\widetilde{\phi}(0, x) \tag{4.3}$$

for all $x,\, y \in X$ *and an integer* $a\,(a \neq 0\,, \pm 1)$.

Proof. Consider the set

$$\Omega = \{g \,|\, g : X \to C_{cb}(Y)\,, g(0) = \{0\}\}$$

and introduce the generalized metric on Ω defined by

$$d(g_1,\, g_2) = \inf\{\mu \in (0, \infty) \,|\, h(g_1(x)\,, g_2(x)) \leq \mu\phi(0, x)\,, \text{for all } x \in X\}\,.$$

We note that $\inf \emptyset := \infty$. It is easy to show that (Ω, d) is complete; see [15]. Now we define a function $T : \Omega \to \Omega$ by

$$T(g)(x) = \frac{1}{a^4} g(ax) \tag{4.4}$$

for all $x \in X$. Note that for all $g_1,\, g_2 \in \Omega$, let $\mu \in (0,\, \infty)$ be an arbitrary constant with $d(g_1,\, g_2) = \mu$. Then

$$h(\frac{1}{a^4} g_1(ax)\,, \frac{1}{a^4} g_2(ax)) \leq \frac{\mu}{a^4}\phi(0, ax) \tag{4.5}$$

for all $x \in X$. By using (4.2), we have

$$h(\frac{1}{a^4} g_1(ax)\,, \frac{1}{a^4} g_2(ax)) \leq \mu\, L\phi(0, x) \tag{4.6}$$

for all $x \in X$. Hence we obtain

$$d(Tg_1,\, Tg_2) \leq Ld(g_1,\, g_2)$$

for all $g_1\,, g_2 \in \Omega$, that is, T is a strictly self-mapping of Ω with the Lipschitz constant L . Letting $x = 0$ and replacing y by x in the inequality (4.1), we get

$$h(\frac{1}{a^4} f(ax)\,, f(x)) \leq \frac{1}{2a^4}\phi(0, x)$$

for all $x \in X$. This means that

$$d(Tf, f) \leq \frac{1}{2a^4}\,.$$

By Theroem 4.2, there exists a fixed point $Q : X \to (C_{cb}(Y),\, h)$ of T in $\{g \in \Omega \,|\, d(g_1\,, g_2) < \infty\}$ such that $\{T^k f\} \to 0$ ad $k \to \infty$. Hence we have

$$Q(x) = \lim_{n\to\infty} \frac{f(a^n x)}{a^{4n}}\,, \tag{4.7}$$

for all $x \in X$. Also, we have

$$d(f, Q) \leq \frac{1}{1-L} d(Tf, f) \leq \frac{1}{2a^4}\frac{1}{1-L}\,.$$

This implies that the inequality (4.3) holds for all $x \in X$. By the inequalities (4.1) and (4.2), we have

$$\begin{aligned} &h\Big(Q(x+ay) \oplus Q(x-ay) \oplus 2(a^2-1)Q(x), \\ &\qquad\qquad a^2Q(x+y) \oplus a^2Q(x-y) \oplus 2a^2(a^2-1)Q(y)\Big) \\ &\le \lim_{n\to\infty} L^n \phi(x,\, y) = 0 \end{aligned}$$

for all $x,\, y \in X$ and an integer $a\,(a \neq 0\,, \pm 1)$. Thus Q is a unique quartic set-valued mapping. □

COROLLARY 4.4. *Let $0 < p < 4$ and $\theta \geq 0$ be real numbers and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is an even mapping with $f(0) = \{0\}$ satisfying*

$$\begin{aligned} h\Big(&f(x+ay) \oplus f(x-ay) \oplus 2(a^2-1)f(x), \\ &a^2f(x+y) \oplus a^2f(x-y) \oplus 2a^2(a^2-1)f(y)\Big) \leq \theta(||x||^p + ||y||^p) \end{aligned} \tag{4.8}$$

for all $x,\, y \in X$ and an integer $a\,(a \neq 0\,, \pm 1)$. Then there exists a unique quartic set-valued mapping $Q : X \to (C_{cb}(Y),\, h)$ such that

$$h(f(x),\, Q(x)) \leq \frac{\theta}{2(a^4 - a^p)} ||x||^p \tag{4.9}$$

for all $x \in X$ and an integer $a\,(a \neq 0\,, \pm 1)$.

Proof. It follows from Theorem 4.3 by letting $\phi(x,\, y) = \theta(||x||^p + ||y||^p)$ for all $x,\, y \in X$. Then we can choose $L = a^{p-4}$ and hence we have the desired result. □

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